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On rational functions whose normalization has genus zero or one

by

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1. Introduction. Let $f : S \to \mathbb{CP}^1$ be a holomorphic function on a compact Riemann surface S. The normalization of f is defined as a holomorphic function $\widetilde{f} : \widetilde{S}_f \to \mathbb{CP}^1$ of the lowest possible degree between compact Riemann surfaces such that $\widetilde{f}$ is a Galois covering and

$$\widetilde{f} = f \circ h$$

for some holomorphic function $h : \widetilde{S}_f \to S$. In this paper we study rational functions $A : \mathbb{CP}^1 \to \mathbb{CP}^1$ for which the genus of the surface $\widetilde{S}_A$ equals zero or one. Equivalently, we study rational functions for which the genus of the Galois closure of the extension $\mathbb{C}(z)/\mathbb{C}(A)$ equals zero or one. Finally, the functions under consideration can be described as rational functions which are covering maps between orbifolds of non-negative Euler characteristic on the Riemann sphere.

Our main motivation for the study of rational functions A satisfying $g(\widetilde{S}_A) \le 1$ is the fact that these functions naturally appear in the description of "separate variable" algebraic curves of genus zero and, more generally, in the theory of functional decompositions of rational functions. Namely, it was shown in [25] that if an irreducible algebraic curve

$$A(x) - B(y) = 0, \tag{1.1}$$

where $A, B \in \mathbb{C}(z)$ and $\deg B \ge \deg A$, has genus zero, then whenever $\deg B \ge 84 \deg A$, the inequality $g(\widetilde{S}_A) \le 1$ holds. Moreover, for any fixed rational function A with $g(\widetilde{S}_A) \le 1$ one can find a rational function B of arbitrarily high degree such that corresponding curve (1.1) is irreducible and of genus zero.

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The algebraic curves (1.1) have been studied in number theory since many interesting Diophantine equations have the form $A(x) = B(y)$. By the Siegel theorem, an irreducible algebraic curve $\mathcal{C}$ with rational coefficients may have infinitely many integer points only if $\mathcal{C}$ is of genus zero with at most two points at infinity. More generally, by the Faltings theorem, $\mathcal{C}$ may have infinitely many rational points only if its genus is at most one. Therefore, the problem of describing curves (1.1) of genus zero is important for number theory (see e.g. [2], [9], [13]). On the other hand, since the curve (1.1) has genus zero if and only if there exist rational functions C and D such that

$$A \circ C = B \circ D, \tag{1.2}$$

the problem of describing curves (1.1) of genus zero is of importance also for the decomposition theory of rational functions (see e.g. [1], [20], [23], [24]).

Other results relating rational functions whose normalization has genus zero or one to the functional equation (1.2) were obtained in the paper [21] devoted to the functional equation

$$A \circ X = X \circ D, \tag{1.3}$$

especially important for complex and arithmetic dynamics (see e.g. [5], [6], [16], [22]). In particular, the results of [21] imply that for any solution A, X, D of (1.3) the function X admits a canonical representation

$$X = X_0 \circ W,$$

where X_0 satisfies $g(\widetilde{S}_{X_0}) \leq 1$, while W is a "compositional right factor" of some iterate $D^{\circ k}$, that is,

$$D^{\circ k} = U \circ W$$

for some rational function U (see [21], [27]).

In this paper we give a complete list of rational functions A satisfying the condition $g(\widetilde{S}_A) = 0$. Clearly, the definition implies that these functions are exactly all possible "compositional left factors" of Galois coverings of $\mathbb{CP}^1$ by $\mathbb{CP}^1$. Although all such coverings were described already by Klein, a practical calculation of their functional decompositions is not a trivial task, and to the best of our knowledge a complete list of functions with $g(\widetilde{S}_A) = 0$ has never been published, although some functions from this list, and possibly even all of them, appeared here and there.

In order to shorten the notation, we will say that rational functions A_1 and A_2 are *μ-equivalent* and write $A_1 \underset{\mu}{\sim} A_2$ if

$$A_1 = \mu_1 \circ A_2 \circ \mu_2,$$

for some Möbius transformations μ_1 and μ_2. Our main result is the following.

THEOREM 1.1. *Let A be a rational function. Then $g(\widetilde{S}_A) = 0$ if and only if A is μ-equivalent to one of the functions listed below.*

(I) *Cyclic functions:*

(a) $$z^n, \qquad n \geq 1.$$

(II) *Dihedral functions:*

(a) $$\frac{1}{2}\left(z^n + \frac{1}{z^n}\right), \qquad n \geq 2,$$ (b) $$T_n, \qquad n \geq 2.$$

(III) *Tetrahedral functions:*

(a) $$-\frac{1}{2^6}\,\frac{z^3(z^3-8)^3}{(z^3+1)^3},$$ (b) $$-\frac{1}{2^6}\,\frac{z(z-8)^3}{(z+1)^3},$$ (c) $$-\frac{1}{2^6}\left(\frac{z^2-4}{z-1}\right)^3.$$

(IV) *Octahedral functions:*

(a) $$\frac{1}{2^2 3^3}\,\frac{(x^8+14x^4+1)^3}{x^4(x^4-1)^4},$$

(b) $$\frac{1}{2^2 3^3}\,\frac{(z^2+14z+1)^3}{z(z-1)^4},$$ (c) $$-\frac{1}{3^3}\,\frac{(z^2-4)^3}{z^4},$$

(d) $$\frac{2^2}{3^3}\,\frac{(z^4-z^2+1)^3}{z^4(z^2-1)^2},$$ (e) $$-\frac{1}{27}\,\frac{(2z^2+1)^3(2z^2-3)^3}{(2z^2-1)^4},$$

(f) $$-\frac{2^8}{3^3}\,z^3(z-1),$$ (g) $$2^8\,\frac{z(z^2-7z-8)^3}{(z^2+20z-8)^4}.$$

(V) *Icosahedral functions:*

(a) $$\frac{1}{2^6 3^3}\,\frac{(z^{20}+228z^{15}+494z^{10}-228z^5+1)^3}{(z^{10}-11z^5-1)^5 z^5},$$

(b) $$-\frac{1}{2^{11}3}(3z+5)^3(z^2+15),$$ (c) $$\frac{1}{2^6 3^3}\,\frac{(z^2-20)^3}{(z-5)},$$

(d) $$\frac{2^9 5^4}{3^2}\,\frac{(20z^3-87z-95)^3}{(20z^2+140z+101)^5},$$

(e) $$\frac{1}{2^6 3^3}\,\frac{(z^4+228z^3+494z^2-228z+1)^3}{(z^2-11z-1)^5 z},$$

(f) $$\frac{5^4}{3^3}\,\frac{(-40z^2-20z-4)^3 z^3(5z^2+5z+1)^3}{(20z^2+10z+1)^5},$$

(g) $$\frac{5^3}{2^6}\,\frac{z(z^2+5z+40)^3(z^2-40z-5)^3(8z^2-5z+5)^3}{(z^4+55z^3-165z^2-275z+25)^5},$$

(h) $$\frac{1}{2^6 3^3}\,\frac{(z^2+3z+1)^3(z^4-4z^3+11z^2-14z+31)^3(z^4+z^3+11z^2-4z+16)^3}{(z-1)^5(z^4+z^3+6z^2+6z+11)^5}.$$

Notice that all the functions appearing in Theorem 1.1 are *Belyi functions*, that is, rational functions having only three critical values 0, 1, and ∞. Notice also that the theorem obviously implies that any rational function A of degree greater than 60 with $g(\widetilde{S}_A) = 0$ is either cyclic or dihedral. Without pretending to give a complete list of occurrences of rational functions μ-equivalent to the above functions in the literature, below we point out several such examples emerging in different contexts.

The polynomials z^n and T_n appear in papers devoted to number theory and functional decompositions very often (see e.g. [1], [2], [8], [9], [20], [28]). In particular, the central result of the decomposition theory of polynomials, the so-called second Ritt theorem (see [28]), is essentially equivalent to the statement that if (1.1) is an irreducible *polynomial* curve of genus zero with one point at infinity and $\deg B \geq \deg A$, then $A \underset{\mu}{\sim} z^n$ or $A \underset{\mu}{\sim} T_n$. Thus, the above mentioned result of [25] about algebraic curves (1.1) can be considered as an analogue of the Ritt theorem for rational functions.

The functions "(a)" from Theorem 1.1 form a complete list of Galois coverings of $\mathbb{CP}^1$ by $\mathbb{CP}^1$. They were calculated in the book [12], and nowadays can be interpreted in terms of the "dessins d'enfants" theory as Belyi functions of Platonic solids (see [3], [15]). The function (IV)(f) is μ-equivalent to the function $3z^4 - 4z^3$ appearing in the paper [2] providing a classification of polynomial curves (1.1) over $\mathbb{Q}$ having an infinite number of rational solutions with a bounded denominator. The function (V)(b) is μ-equivalent to the function P_2 from the paper [1] about rational solutions of the functional equation

$$A \circ C = A \circ B.$$

The functions (V)(b), (V)(c), and (IV)(f) appear in the paper [26] about the so-called Davenport–Zannier pairs defined over $\mathbb{Q}$. Namely, (V)(c) is a Belyi function corresponding to the "dessins D" from [26] with the parameters $s = 1$, $t = 1$, while (V)(b) and (IV)(f) are Belyi functions corresponding to the "dessins A" with the parameters $k = s = 2$, $t = 1$, and $s = 3$, $k = 1$, $t = 1$. A function which is μ-equivalent to (V)(c) appears also in the paper [4] devoted to the Hall conjecture about differences between cubes and squares of integers (see [26] for details).

Finally, in the paper [19], tetrahedral and octahedral functions are used to construct explicit examples of rational functions having decompositions into compositions of rational functions with a different number of indecomposable factors.

In contrast to rational functions A satisfying $g(\widetilde{S}_A) = 0$, functions A with $g(\widetilde{S}_A) = 1$ cannot be described in such an explicit way. Nevertheless, these functions admit quite a precise description in geometric terms:

THEOREM 1.2. *Let A be a rational function such that $g(\widetilde{S}_A) = 1$. Then there exist elliptic curves $\mathcal{C}_1$ and $\mathcal{C}_2$, subgroups $\Omega_1 \subseteq \mathrm{Aut}(\mathcal{C}_1)$ and $\Omega_2 \subseteq \mathrm{Aut}(\mathcal{C}_2)$, and a holomorphic map $\alpha : \mathcal{C}_1 \to \mathcal{C}_2$ such that the diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\alpha} & \mathcal{C}_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{1.4}$$

where $\pi_1 : \mathcal{C}_1 \to \mathcal{C}_1/\Omega_1$ and $\pi_2 : \mathcal{C}_2 \to \mathcal{C}_2/\Omega_2$ are quotient maps, commutes. Conversely, if A is a rational function which makes diagram (1.4) *commutative, then $g(\widetilde{S}_A) = 1$, unless A is μ-equivalent either to a cyclic function for some $n \leq 4$, or to a dihedral function for some $n \leq 4$, or to a tetrahedral function.*

The best known rational functions A with $g(\widetilde{S}_A) = 1$ are the so-called Lattès maps which are obtained from the above diagram for $\mathcal{C}_1 = \mathcal{C}_2$ and $\pi_1 = \pi_2$ (see [18] and Section 4 below). However, there exist Lattès maps A for which $g(\widetilde{S}_A) = 0$, as well as functions A with $g(\widetilde{S}_A) = 1$ which are not Lattès maps.

The paper is organized as follows. In the second section we provide some general definitions and results related to functions A with $g(\widetilde{S}_A) \leq 1$. In particular, we show that such functions can be described in terms of their ramifications. We also give a characterization of functions A with $g(\widetilde{S}_A) \leq 1$ as covering maps between orbifolds of non-negative characteristic on the Riemann sphere.

In the third and the fourth sections we establish some specific properties of rational functions A with $g(\widetilde{S}_A) = 0$, and prove Theorem 1.1. We also outline a practical way of calculating the functions from Theorem 1.1 using the "dessins d'enfants" theory.

Finally, in the fifth section we give a geometric characterization of covering maps between orbifolds of zero characteristic, and investigate interrelations between such coverings and rational functions A with $g(\widetilde{S}_A) \leq 1$. The results of the fifth section imply in particular Theorem 1.2. Another corollary of these results is that $g(\widetilde{S}_A) = 1$ for any Lattès map A of degree greater than four.

2. Preliminaries. Recall that a holomorphic map $f : R_1 \to R_2$ between compact Riemann surfaces is called a *Galois covering* if its group of deck transformations

$$\mathrm{Aut}(R_1, f) = \{h \in \mathrm{Aut}(R_1) \mid f \circ h = f\}$$

acts transitively on each fiber of f. Thus, a Galois covering can be thought of as a quotient map

$$R_1 \to R_1/\mathrm{Aut}(R_1, f) \cong R_2. \tag{2.1}$$

Equivalently, a holomorphic map $f : R_1 \to R_2$ is a Galois covering if the field extension $\mathcal{M}(R_1)/f^*(\mathcal{M}(R_2))$, where

$$f^* : \mathcal{M}(R_2) \to \mathcal{M}(R_1)$$

is the corresponding homomorphism of the fields of meromorphic functions, is a Galois extension. Moreover, if $f : R_1 \to R_2$ is a Galois covering, then

$$\mathrm{Gal}(\mathcal{M}(R_1)/f^*(\mathcal{M}(R_2)) \cong \mathrm{Aut}(R_1, f)$$

(see e.g. [11, Proposition 2.65]). Finally, a Galois covering can be defined as a holomorphic map $f : R_1 \to R_2$ such that

$$\deg f = |\mathrm{Mon}(f)|, \tag{2.2}$$

where $\mathrm{Mon}(f)$ is the monodromy group of f (see e.g. [11, Proposition 2.66]).

Let S be a compact Riemann surface and $f : S \to \mathbb{CP}^1$ a holomorphic function. The *normalization* of f is defined as a holomorphic function $\widetilde{f} : \widetilde{S}_f \to \mathbb{CP}^1$ of the lowest possible degree between compact Riemann surfaces such that $\widetilde{f}$ is a Galois covering and $\widetilde{f} = f \circ h$ for some holomorphic function $h : \widetilde{S}_f \to S$ (see e.g. [11, Section 2.9]).

In this paper we study rational functions $A : \mathbb{CP}^1 \to \mathbb{CP}^1$ for which the genus of the surface $\widetilde{S}_A$ equals zero or one, or equivalently for which the genus of the Galois closure of the extension $\mathbb{C}(z)/\mathbb{C}(A)$ equals zero or one. A convenient way for describing this class of functions in terms of their ramification uses the notion of Riemann surface orbifold (see e.g. [17, Appendix E] or [21]). By definition, a *Riemann surface orbifold* is a pair $\mathcal{O} = (R, \nu)$ consisting of a Riemann surface R and a ramification function $\nu : R \to \mathbb{N}$ which takes the value $\nu(z) = 1$ except at isolated points. The *Euler characteristic* of $\mathcal{O} = (R, \nu)$ is defined by

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z \in R} \left(\frac{1}{\nu(z)} - 1 \right), \tag{2.3}$$

where $\chi(R)$ is the Euler characteristic of R. For an orbifold $\mathcal{O} = (R, \nu)$ we set

$$c(\mathcal{O}) = \{z_1, z_2, \dots\} = \{z \in R \mid \nu(z) > 1\},$$
$$\nu(\mathcal{O}) = \{\nu(z_1), \nu(z_2), \dots\}.$$

For orbifolds $\mathcal{O} = (R, \nu)$ and $\mathcal{O}' = (R', \nu')$ we write

$$\mathcal{O} \preceq \mathcal{O}' \tag{2.4}$$

if $R = R'$ and $\nu(z) \mid \nu'(z)$ for all $z \in R$. Clearly, (2.4) implies that $\chi(\mathcal{O}) \geq \chi(\mathcal{O}')$.

If R_1, R_2 are Riemann surfaces provided with ramification functions ν_1, ν_2, and $f : R_1 \to R_2$ is a holomorphic branched covering map, then f is called a *covering map* $f : \mathcal{O}_1 \to \mathcal{O}_2$ *between the orbifolds* $\mathcal{O}_1 = (R_1, \nu_1)$ *and* $\mathcal{O}_2 = (R_2, \nu_2)$ if

$$\nu_2(f(z)) = \nu_1(z) \deg_z f \quad \text{for all } z \in R_1, \tag{2.5}$$

where $\deg_z f$ is the local degree of f at z. If R_1 and R_2 are compact and $\deg f = d$, then the Riemann–Hurwitz formula implies that

$$\chi(\mathcal{O}_1) = d\chi(\mathcal{O}_2). \tag{2.6}$$

A *universal covering* of an orbifold $\mathcal{O}$ is a covering map $\theta_{\mathcal{O}} : \widetilde{\mathcal{O}} \to \mathcal{O}$ between orbifolds such that $\widetilde{R}$ is simply connected and $\widetilde{\nu}(z) \equiv 1$. If $\theta_{\mathcal{O}}$ is such a map, then there exists a group $\Gamma_{\mathcal{O}}$ of conformal automorphisms of $\widetilde{R}$ such that for $z_1, z_2 \in \widetilde{R}$ the equality $\theta_{\mathcal{O}}(z_1) = \theta_{\mathcal{O}}(z_2)$ holds if and only if $z_1 = \sigma(z_2)$ for some $\sigma \in \Gamma_{\mathcal{O}}$. A universal covering exists and is unique up to a conformal isomorphism of $\widetilde{R}$, unless $\mathcal{O}$ is the Riemann sphere with one ramified point, or the Riemann sphere with two ramified points z_1, z_2 such that $\nu(z_1) \neq \nu(z_2)$. Furthermore, $\widetilde{R} = \mathbb{D}$ if and only if $\chi(\mathcal{O}) < 0$; $\widetilde{R} = \mathbb{C}$ if and only if $\chi(\mathcal{O}) = 0$; and $\widetilde{R} = \mathbb{CP}^1$ if and only if $\chi(\mathcal{O}) > 0$ (see [17, Appendix E] and [7, Section IV.9.12]). Abusing notation we will use the symbol $\widetilde{\mathcal{O}}$ both for the orbifold and for the Riemann surface $\widetilde{R}$.

For any covering map between orbifolds $A : \mathcal{O}_1 \to \mathcal{O}_2$ there exist an isomorphism $F : \widetilde{\mathcal{O}}_1 \to \widetilde{\mathcal{O}}_2$ and a homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \to \Gamma_{\mathcal{O}_2}$ such that the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{O}}_1 & \xrightarrow{F} & \widetilde{\mathcal{O}}_2 \\ {\scriptstyle \theta_{\mathcal{O}_1}}\downarrow & & \downarrow{\scriptstyle \theta_{\mathcal{O}_2}} \\ \mathcal{O}_1 & \xrightarrow{A} & \mathcal{O}_2 \end{array} \tag{2.7}$$

commutes and

$$F \circ \sigma = \varphi(\sigma) \circ F \quad \text{for any } \sigma \in \Gamma_{\mathcal{O}_1}. \tag{2.8}$$

Vice versa, any isomorphism $F : \widetilde{\mathcal{O}}_1 \to \widetilde{\mathcal{O}}_2$ satisfying (2.8) for some homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \to \Gamma_{\mathcal{O}_2}$ descends to a covering map $A : \mathcal{O}_1 \to \mathcal{O}_2$ which makes diagram (2.7) commutative (see e.g. [21, Proposition 3.1]).

With each holomorphic function $f : R_1 \to R_2$ between compact Riemann surfaces one can associate in a natural way two orbifolds $\mathcal{O}_1^f = (R_1, \nu_1^f)$ and $\mathcal{O}_2^f = (R_2, \nu_2^f)$, setting $\nu_2^f(z)$ equal to the least common multiple of the local degrees of f at the points of the preimage $f^{-1}\{z\}$, and

$$\nu_1^f(z) = \nu_2^f(f(z))/\deg_z f.$$

We will call $\mathcal{O}_2^f$ the *ramification orbifold* of f. By construction,

$$f : \mathcal{O}_1^f \to \mathcal{O}_2^f$$

is a covering map between orbifolds. Furthermore, it is easy to see that the covering map $f : \mathcal{O}_1^f \to \mathcal{O}_2^f$ is minimal in the following sense. For any covering map $f : \mathcal{O}_1 \to \mathcal{O}_2$ between orbifolds we have

$$\mathcal{O}_1^f \preceq \mathcal{O}_1, \quad \mathcal{O}_2^f \preceq \mathcal{O}_2. \tag{2.9}$$

The orbifolds $\mathcal{O}_1^f$ and $\mathcal{O}_2^f$ always have a universal covering (see [21, Lemma 4.2]).

Since any Galois covering $f : R_1 \to R_2$ is a quotient map (2.1), for any branch point z_i, $1 \le i \le r$, of f there exists a number d_i such that $f^{-1}\{z_i\}$ consists of $|\mathrm{Aut}(R_1, f)|/d_i$ points, and at each of these points the multiplicity of f equals d_i. Indeed, the points of $f^{-1}\{z_i\}$ form a single orbit of $\mathrm{Aut}(R_1, f)$. Thus, the corresponding stabilizers are conjugate and hence of the same order. In the above notation, we can formulate this property of Galois coverings as follows: for any Galois covering $f : R_1 \to R_2$ the orbifold $\mathcal{O}_1^f$ is non-ramified.

The following statement coincides with [25, Lemma 1]. For the reader's convenience we repeat the argument.

LEMMA 2.1. *Let A be a rational function. Then $g(\widetilde{S}_A) = 0$ if and only if $\chi(\mathcal{O}_2^A) > 0$, and $g(\widetilde{S}_A) = 1$ if and only if $\chi(\mathcal{O}_2^A) = 0$.*

Proof. Let $f : S \to \mathbb{CP}^1$ be a Galois covering. Applying the Riemann–Hurwitz formula, we see that

$$2g(S) - 2 = -2|\Gamma| + \sum_{i=1}^{r} \frac{|\Gamma|}{d_i}(d_i - 1),$$

where $\Gamma = \mathrm{Aut}(S, f)$, implying that

$$\chi(\mathcal{O}_2^f) = 2 + \sum_{i=1}^{r} \left(\frac{1}{d_i} - 1\right) = \frac{2 - 2g(S)}{|\Gamma|}.$$

Thus $g(S) = 0$ if and only if $\chi(\mathcal{O}_2^f) > 0$, while $g(S) = 1$ if and only if $\chi(\mathcal{O}_2^f) = 0$.

Let now $A : \mathbb{CP}^1 \to \mathbb{CP}^1$ be an arbitrary rational function of degree n. Since the normalization $\widetilde{A} : \widetilde{S}_A \to \mathbb{CP}^1$ of A can be described as any connected component of the n-fold fiber product of A distinct from the diagonal components (see [10, §I.G]), it follows from the construction of the fiber product (see e.g. [20, Sections 2 and 3]) that

$$\mathcal{O}_2^A = \mathcal{O}_2^{\widetilde{A}}. \tag{2.10}$$

Thus, $g(\widetilde{S}_A) = 0$ if and only if $\chi(\mathcal{O}_2^A) > 0$, and $g(\widetilde{S}_A) = 1$ if and only if $\chi(\mathcal{O}_2^A) = 0$. ∎

Lemma 2.1 gives a simple practical way of checking whether $g(\widetilde{S}_A) \le 1$ in terms of the ramification of A.

Corollary 2.2. *Let A be a rational function. Then $g(\widetilde{S}_A) = 0$ if and only if $\nu(\mathcal{O}_2^A)$ belongs to the list*

$$\{2,3,6\}, \quad \{2,4,4\}, \quad \{3,3,3\}, \quad \{2,2,2,2\}, \tag{2.11}$$

while $g(\widetilde{S}_A) > 0$ if and only if $\nu(\mathcal{O}_2^A)$ belongs to the list

$$\{n,n\}, \quad n \geq 1, \quad \{2,2,n\}, \quad n \geq 2, \quad \{2,3,3\}, \quad \{2,3,4\}, \quad \{2,3,5\}. \tag{2.12}$$

Proof. Indeed, it is well known and follows easily from (2.3) that if $\mathcal{O}$ is an orbifold on $\mathbb{CP}^1$ having a universal covering, then $\chi(\mathcal{O}) = 0$ if and only if $\nu(\mathcal{O})$ belongs to the list (2.11), and $\chi(\mathcal{O}) > 0$ if and only if $\nu(\mathcal{O})$ belongs to the list (2.12). ∎

Another corollary of Lemma 2.1 is the following statement.

Corollary 2.3. *Let A be a rational function. Then $g(\widetilde{S}_A) = 0$ if and only if there exist orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ on $\mathbb{CP}^1$ of positive Euler characteristic such that $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds. Similarly, if $g(\widetilde{S}_A) = 1$, then there exist orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ of zero Euler characteristic such that $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map. On the other hand, the fact that $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds of zero Euler characteristic implies only that $g(\widetilde{S}_A) \leq 1$.*

Proof. Indeed, if $\chi(\mathcal{O}_2^A) > 0$, then (2.6) implies that $\chi(\mathcal{O}_1^A) > 0$ and hence $A : \mathcal{O}_1^A \to \mathcal{O}_2^A$ is a covering map between orbifolds of positive Euler characteristic. Similarly, $\chi(\mathcal{O}_2^A) = 0$ implies that $\chi(\mathcal{O}_1^A) = 0$.

Conversely, if $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds of positive Euler characteristic, then (2.9) implies that $\mathcal{O}_1^A$ and $\mathcal{O}_2^A$ also have positive Euler characteristic. On the other hand, if $\mathcal{O}_1$ and $\mathcal{O}_2$ have zero Euler characteristic, then (2.9) implies only that $\mathcal{O}_1^A$ and $\mathcal{O}_2^A$ have non-negative Euler characteristic. □

3. Functions with $\chi(\mathcal{O}_2^A) > 0$. Let f, g be rational functions. We will call g a *compositional left factor* of f if $f = g \circ h$ for some rational function h. It is clear that the rational functions A with $g(\widetilde{S}_A) = 0$ are exactly the compositional left factors of Galois coverings $f : \mathbb{CP}^1 \to \mathbb{CP}^1$. Notice that since for a Galois covering $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ the orbifold $\mathcal{O}_1^f$ is non-ramified, any Galois covering $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ is a universal covering of the orbifold $\mathcal{O} = \mathcal{O}_2^f$ with $\chi(\mathcal{O}) > 0$, and vice versa for any orbifold $\mathcal{O}$ with $\chi(\mathcal{O}) > 0$ its universal covering

$$\theta_{\mathcal{O}} : \mathbb{CP}^1 \to \mathbb{CP}^1 / \Gamma_{\mathcal{O}} \cong \mathbb{CP}^1$$

is a Galois covering. It follows that we can identify Galois coverings $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ with universal coverings of orbifolds $\mathcal{O}$ of positive Euler characteristic on $\mathbb{CP}^1$.

Recall that any finite subgroup of $\mathrm{Aut}(\mathbb{CP}^1)$ is isomorphic to one of the following groups:

$$\mathbb{Z}/n\mathbb{Z}, \quad n \geq 1, \qquad D_{2n}, \quad n \geq 2, \qquad A_4, \qquad S_4, \qquad A_5. \tag{3.1}$$

Moreover, these isomorphism classes are also conjugacy classes. The groups (3.1) are the groups $\Gamma_{\mathcal{O}}$ for orbifolds $\mathcal{O}$ with $\chi(\mathcal{O}) > 0$ whose ramification collections are listed in (2.12). Let Γ be a finite subgroup of $\mathrm{Aut}(\mathbb{CP}^1)$. Abusing notation, we will denote by θ_Γ the universal covering $\theta_{\mathcal{O}}$, where $\mathcal{O}$ is an orbifold such that $\Gamma_{\mathcal{O}} = \Gamma$. Thus, θ_Γ is defined up to the transformation

$$\theta_\Gamma \mapsto \delta \circ \theta_\Gamma,$$

where δ is a Möbius transformation.

Let F be a rational function. Recall that two decompositions of F into compositions of rational functions

$$F = A \circ W \tag{3.2}$$

and

$$F = \widetilde{A} \circ \widetilde{W}$$

are called *equivalent* if

$$\widetilde{A} = A \circ \mu, \qquad \widetilde{W} = \mu^{-1} \circ W$$

for some Möbius transformation μ. Equivalence classes of decompositions of F are in one-to-one correspondence with imprimitivity systems of the monodromy group $\mathrm{Mon}(F)$ of F. Namely, if z_0 is a non-critical value of F and $\mathrm{Mon}(F)$ is realized as a permutation group acting on the fiber $F^{-1}\{z_0\}$, then to the equivalence class of the decomposition (3.2) corresponds the imprimitivity system consisting of $d = \deg A$ blocks $W^{-1}\{t_i\}$, $1 \leq i \leq d$, where $\{t_1, \dots, t_d\} = A^{-1}\{z_0\}$ (see e.g. [20, Section 2] for more details).

Imprimitivity systems of a transitive permutation group G acting on a set S are in one-to-one correspondence with subgroups of G containing the stabilizer G_α of some $\alpha \in S$ (see e.g. [29, Theorem 7.5]). On the other hand, if $f : S \to \mathbb{CP}^1$ is a Galois covering, then (2.2) implies that all stabilizer subgroups of $\mathrm{Mon}(f)$ are trivial. Thus, for any orbifold $\mathcal{O}$ with $\chi(\mathcal{O}) > 0$ equivalence classes of decompositions of $\theta_{\mathcal{O}} : \mathbb{CP}^1 \to \mathbb{CP}^1$ are in one-to-one correspondence with subgroups Γ' of $\Gamma_{\mathcal{O}}$, and any decomposition of $\theta_{\mathcal{O}}$ has the form

$$\theta_{\mathcal{O}} = A_{\Gamma'} \circ \theta_{\Gamma'},$$

where Γ' is a subgroup of $\Gamma_{\mathcal{O}}$, and $A_{\Gamma'}$ is some rational function depending on Γ'. However, the number of μ-equivalence classes of compositional left factors of $\theta_{\mathcal{O}}$ is in general less than the number of equivalence classes of decompositions of $\theta_{\mathcal{O}}$. In particular, to conjugate subgroups Γ_1, Γ_2 of $\Gamma_{\mathcal{O}}$

correspond μ-equivalent functions A_{Γ_1}, A_{Γ_2}. More precisely, the following statement holds.

LEMMA 3.1. *Let $\mathcal{O}$ be an orbifold of positive Euler characteristic on $\mathbb{CP}^1$, Γ_1, Γ_2 subgroups of $\Gamma = \Gamma_{\mathcal{O}}$, and*

$$\theta_{\mathcal{O}} = A_{\Gamma_i} \circ \theta_{\Gamma_i}, \qquad i = 1, 2,$$

the corresponding decompositions. Then $A_{\Gamma_2} = A_{\Gamma_1} \circ \delta$ for some Möbius transformation δ if and only if Γ_1, Γ_2 are conjugate in Γ.

Proof. The conjugacy condition

$$\Gamma_2 = \mu^{-1} \circ \Gamma_1 \circ \mu, \qquad \mu \in \Gamma,$$

is equivalent to the condition that for any choice of θ_{Γ_1} and θ_{Γ_2},

$$\delta \circ \theta_{\Gamma_2} = \theta_{\Gamma_1} \circ \mu \tag{3.3}$$

for some Möbius transformation δ. Assume that (3.3) holds. Then, since $\theta_{\mathcal{O}} = \theta_{\mathcal{O}} \circ \mu$ for any $\mu \in \Gamma$, we have

$$\theta_{\mathcal{O}} = A_{\Gamma_1} \circ \theta_{\Gamma_1} = A_{\Gamma_1} \circ \theta_{\Gamma_1} \circ \mu = A_{\Gamma_1} \circ \delta \circ \theta_{\Gamma_2}.$$

Since, on the other hand, $\theta_{\mathcal{O}} = A_{\Gamma_2} \circ \theta_{\Gamma_2}$, we conclude that

$$A_{\Gamma_2} = A_{\Gamma_1} \circ \delta. \tag{3.4}$$

Conversely, assume that (3.4) holds. Consider the algebraic curve obtained by equating the numerator of $\theta_{\mathcal{O}}(y) - \theta_{\mathcal{O}}(x)$ to zero. Abusing notation we will denote this curve simply by

$$\theta_{\mathcal{O}}(y) - \theta_{\mathcal{O}}(x) = 0. \tag{3.5}$$

Since the rational function θ_{θ} is a Galois covering, the curve (3.5) splits over $\mathbb{C}(x)$ into a product of factors of degree one,

$$y - \mu(x), \qquad \mu \in \Gamma.$$

On the other hand, since

$$\begin{aligned} \theta_{\mathcal{O}}(y) - \theta_{\mathcal{O}}(x) &= (A_{\Gamma_1} \circ \theta_{\Gamma_1})(y) - (A_{\Gamma_2} \circ \theta_{\Gamma_2})(y) \\ &= (A_{\Gamma_1} \circ \theta_{\Gamma_1})(y) - (A_{\Gamma_1} \circ \delta \circ \theta_{\Gamma_2})(y) \end{aligned}$$

and $y - x$ is a factor of the algebraic curve $A_{\Gamma_1}(y) - A_{\Gamma_1}(x) = 0$, the algebraic curve

$$\theta_{\Gamma_1}(y) - \delta \circ \theta_{\Gamma_2}(x) = 0 \tag{3.6}$$

is a factor of (3.5). Therefore, (3.6) also splits over $\mathbb{C}(x)$ into a product of factors of degree one. Since y in (3.6) can be locally represented in the form

$$y = (\theta_{\Gamma_1}^{-1} \circ \delta \circ \theta_{\Gamma_2})(x),$$

where $\theta_{\Gamma_1}^{-1}$ is a branch of the algebraic function inverse to θ_{Γ_1}, we conclude that

$$\theta_{\Gamma_1}^{-1} \circ \delta \circ \theta_{\Gamma_2} = \mu, \qquad \mu \in \Gamma,$$

implying (3.3). ∎

Lemma 2.1 and formula (2.10) ensure that any rational function A of degree greater than one with $\chi(\mathcal{O}_2^A) > 0$ is a compositional left factor of some $\theta_{\mathcal{O}}$ with $\nu(\mathcal{O}) = \nu(\mathcal{O}_2^A)$. However, the fact that A is a compositional left factor of $\theta_{\mathcal{O}}$ only implies that $\mathcal{O}_2^A \preceq \mathcal{O}$. More precisely, the following statement holds.

Lemma 3.2. *Let $\mathcal{O}$ be an orbifold of positive Euler characteristic on $\mathbb{CP}^1$ and A a compositional left factor of $\theta_{\mathcal{O}}$ of degree at least two. Then either $\nu(\mathcal{O}) = \nu(\mathcal{O}_2^A)$, or one of the following conditions holds:*

- $\nu(\mathcal{O}) = \{n, n\}$, $n \geq 2$, *and* $\nu(\mathcal{O}_2^A) = \{d, d\}$, *where* $d \,|\, n$, $d \geq 2$.
- $\nu(\mathcal{O}) = \{2, 2, n\}$, $n \geq 2$, *and* $\nu(\mathcal{O}_2^A) = \{2, 2, d\}$, *where* $d \,|\, n$, $d \geq 1$.
- $\nu(\mathcal{O}) = \{2, 3, 3\}$, *and* $\nu(\mathcal{O}_2^A) = \{3, 3\}$.
- $\nu(\mathcal{O}) = \{2, 3, 4\}$, *and* $\nu(\mathcal{O}_2^A) = \{2, 2, 3\}$ *or* $\nu(\mathcal{O}_2^A) = \{2, 2\}$.

Proof. Since $\mathcal{O}_2^{\theta_{\mathcal{O}}} = \mathcal{O}$, it follows from the definition of $\mathcal{O}_2^A$ and the chain rule that $\mathcal{O}_2^A \preceq \mathcal{O}$. In particular, $\chi(\mathcal{O}_2^A) \geq \chi(\mathcal{O}) > 0$. Since the orbifold $\mathcal{O}_2^A$ has a universal covering, it cannot have one ramified point, or two ramified points z_1, z_2 such that $\nu(z_1) \neq \nu(z_2)$. Furthermore, it cannot be non-ramified since any rational function of degree at least two has critical values. These observations easily yield the statements of the lemma. For example, if $\nu(\mathcal{O}) = \{2, 3, 3\}$ and $\nu(\mathcal{O}) \neq \nu(\mathcal{O}_2^A)$, then the condition $\mathcal{O}_2^A \preceq \mathcal{O}$ implies that either $\mathcal{O}_2^A$ is non-ramified, or $\nu(\mathcal{O}_2^A)$ is one of the collections $\{2\}$, $\{3\}$, $\{2, 3\}$, $\{3, 3\}$. Excluding orbifolds with no universal covering and the non-ramified sphere, we conclude that $\nu(\mathcal{O}_2^A) = \{3, 3\}$. ∎

4. Proof of Theorem 1.1. Basing on the results of Section 3, in this section we prove Theorem 1.1. In more detail, for each orbifold $\mathcal{O}$ on $\mathbb{CP}^1$ such that $\chi(\mathcal{O}) > 0$ we list all μ-equivalence classes of rational functions A with $\nu(\mathcal{O}_2^A) = \nu(\mathcal{O})$. We use the following strategy. First, for each conjugacy class of subgroups of $\Gamma = \Gamma_{\mathcal{O}}$ we find a compositional left factor $A_{\Gamma'}$ of $\theta_{\mathcal{O}}$ corresponding to a representative Γ' of this class. Then we reject $A_{\Gamma'}$ with $\nu(\mathcal{O}_2^{A_{\Gamma'}}) \neq \nu(\mathcal{O})$. Finally, we describe the μ-equivalence classes of the remaining functions. It is clear that any rational function A of degree one is μ-equivalent to z^n for $n = 1$, and is a Galois covering. So, we will only consider compositional left factors of $\theta_{\mathcal{O}}$ of degree greater than one.

The following elementary lemma is useful for proving that a concrete rational function A has only three critical values.

Lemma 4.1. *Let f be a rational function of degree d such that the set $f^{-1}\{0,1,\infty\}$ contains at most $d+2$ points. Then it contains exactly $d+2$ points, and f has no critical values distinct from* 0, 1, *and* ∞.

Proof. By the Riemann–Hurwitz formula,

$$2d-2=\sum_{z\in\mathbb{CP}^1}(\deg_z f-1),$$

implying that

$$\sum_{z\in f^{-1}\{0,1,\infty\}}(\deg_z f-1)\le 2d-2,$$

where equality is attained if and only f has no critical values distinct from $0,1$, and ∞. Therefore,

$$|f^{-1}\{0,1,\infty\}|\ge\sum_{z\in f^{-1}\{0,1,\infty\}}\deg_z f-2d+2=d+2,$$

where equality is attained if and only f has no critical values distinct from 0, 1, and ∞. ■

4.1. Functions with $\nu(\mathcal{O}_2^A)=\{n,n\}$. If $\nu(\mathcal{O})=\{n,n\}$, $n\ge 2$, then without loss of generality we may assume that

$$\nu(0)=n,\qquad \nu(\infty)=n,$$

the group $\Gamma_{\mathcal{O}}=\mathbb{Z}/n\mathbb{Z}$ is generated by the transformation

$$\alpha: z\mapsto e^{2\pi i/n}z,$$

and $\theta_{\mathcal{O}}$ equals

$$\theta_{\mathbb{Z}/n\mathbb{Z}}=z^n,\qquad n\ge 2.$$

Further, since any subgroup of $\mathbb{Z}/n\mathbb{Z}$ is cyclic, and for any $d\,|\,n$ the group $\mathbb{Z}/n\mathbb{Z}$ contains only one subgroup of order n/d, any decomposition of z^n into a composition of rational functions is equivalent to

$$z^n=z^d\circ z^{n/d},$$

where $d\,|\,n$. Thus, since $\mathcal{O}_2^{z^d}=\{d,d\}$ and $\{d,d\}\ne\{n,n\}$ for $d<n$, we see that $\nu(\mathcal{O}_2^A)=\{n,n\}$ if and only if $A\underset{\mu}{\sim}z^n$.

4.2. Functions with $\nu(\mathcal{O}_2^A)=\{2,2,n\}$. If $\nu(\mathcal{O})=\{2,2,n\}$, $n\ge 2$, then we may assume that

$$\nu(-1)=2,\qquad \nu(1)=2,\qquad \nu(\infty)=n,$$

the group $\Gamma_{\mathcal{O}}=D_{2n}$ is generated by the transformations

$$\alpha: z\mapsto e^{2\pi i/n}z,\qquad \beta: z\mapsto 1/z,$$

and $\theta_{\mathcal{O}}$ equals

$$\theta_{D_{2n}} = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right), \qquad n \geq 2. \tag{4.1}$$

Any subgroup G of D_{2n} is either cyclic or dihedral. More precisely, either $G = \langle \alpha^d \rangle$, where $d \,|\, n$, or $G = \langle \alpha^d, \alpha^r \beta \rangle$, where $d \,|\, n$ and $0 \leq r < d$. Thus, for any $d \,|\, n$ there exists one subgroup of the first type and d subgroups of the second. The subgroups of the second type form one conjugacy class if n is odd, and one or two conjugacy classes if n is even according as d is odd or even.

Correspondingly, any decomposition of (4.1) is equivalent either to the decomposition

$$\frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = \frac{1}{2}\left(z^d + \frac{1}{z^d}\right) \circ z^{n/d},$$

or to the decomposition

$$\frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = (\varepsilon^d T_d) \circ \frac{1}{2}\left(\varepsilon z^{n/d} + \frac{1}{\varepsilon z^{n/d}}\right), \tag{4.2}$$

where $\varepsilon^{2d} = 1$. For ε and $-\varepsilon$ the decompositions (4.2) are equivalent, since

$$T_d(-z) = (-1)^d T_d(z).$$

So, for odd d we can assume that $\varepsilon^d = 1$. In either case,

$$\varepsilon^d T_d \underset{\mu}{\sim} T_d.$$

Thus, any compositional left factor A of (4.2) is μ-equivalent either to $\frac{1}{2}(z^d + 1/z^d)$ or T_d, implying that $\nu(\mathcal{O}_2^A) = \{2, 2, n\}$ if and only if A is μ-equivalent either to (II)(a) or to (II)(b).

4.3. Functions with $\nu(\mathcal{O}_2^A) = \{2, 3, 3\}$. Any subgroup of A_4 distinct from A_4 is isomorphic to one of the groups $\{e\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, D_4. Furthermore, any two isomorphic subgroups are conjugate in A_4. Thus, the function

$$\theta_{A_4} = -\frac{1}{64}\,\frac{z^3(z^3-8)^3}{(z^3+1)^3}, \tag{4.3}$$

which is a universal covering of the orbifold $\mathcal{O}$ defined by the equalities

$$\nu(0) = 3, \qquad \nu(1) = 2, \qquad \nu(\infty) = 3, \tag{4.4}$$

has, up to the change $A \mapsto A \circ \mu$, where $\mu \in \mathrm{Aut}(\mathbb{CP}^1)$, three compositional left factors of respective degrees 6, 4, and 3. Moreover, these factors cannot be μ-equivalent since they have different degrees.

Considering the obvious decomposition

$$-\frac{1}{64}\,\frac{z^3(z^3-8)^3}{(z^3+1)^3} = -\frac{1}{64}\,\frac{z(z-8)^3}{(z+1)^3} \circ z^3$$

and the decomposition

$$-\frac{1}{64}\,\frac{z^3(z^3-8)^3}{(z^3+1)^3} = -\frac{1}{64}z^3 \circ \frac{z^2-4}{z-1} \circ \frac{z^2+2}{z+1}$$

found in [19], we see that these factors are

$$(4.5)\qquad -\frac{1}{64}\left(\frac{z^2-4}{z-1}\right)^3, \qquad -\frac{1}{64}\,\frac{z(z-8)^3}{(z+1)^3}, \qquad -\frac{1}{64}z^3.$$

Since Lemma 3.2 implies that a compositional left factor A of $\theta_{\mathcal{O}}$ satisfies $\nu(\mathcal{O}) = \nu(\mathcal{O}_2^A)$ unless $A \sim z^3$, we conclude that a rational function A satisfies $\nu(\mathcal{O}_2^A) = \{2,3,3\}$ if and only if A is μ-equivalent to one of the functions in (III).

Notice that (4.3) does not coincide with the function

$$(4.6)\qquad \left(\frac{z^4+2i\sqrt{3}\,z^2+1}{z^4-2i\sqrt{3}\,z^2+1}\right)^3$$

found by Klein. However, it is easy to see that (4.3) along with (4.6) is a universal covering of $\mathcal{O}$ given by (4.4). Indeed, by the uniqueness of the universal covering, it is enough to show that (4.3) satisfies the following conditions: it has only three critical values $0, 1$ and ∞, the multiplicity of any critical point over 0 or ∞ is 3, and the multiplicity of any critical point over 1 is 2. Clearly, equalities (4.3) and

$$(4.7)\qquad f-1 = -\frac{1}{64}\,\frac{\left(z^6+20\,z^3-8\right)^2}{\left(z^3+1\right)^3}$$

imply that $f^{-1}\{0,1,\infty\}$ contains at most 14 points, implying by Lemma 4.1 that $f^{-1}\{0,1,\infty\}$ contains exactly 14 points, and f has no critical values distinct from $0, 1$, and ∞. It now follows from (4.3) and (4.7) that f has the required ramification over 0, 1, and ∞.

Although the above certainly *proves* that any rational function with $\nu(\mathcal{O}_2^A) = \{2,3,3\}$ has one of the forms in (III), the reader may ask how to *find* the function (4.3) and its compositional left factors. A convenient framework for this is provided by the "dessins d'enfants" theory which interprets the functions $\theta_{\mathcal{O}}$ for orbifolds $\mathcal{O}$ with $\chi(\mathcal{O}) > 0$ as Belyi functions of Platonic solids (see [3], [15]). Assuming that the reader is familiar with rudiments of this theory (see e.g. the books [14], [11]), below we sketch the corresponding calculations.

It follows from $\mathbb{Z}/3\mathbb{Z} < A_4$ that a universal covering of the orbifold given by (4.4) can be written in the form

$$\theta_{\mathcal{O}} = A \circ \theta_{\mathbb{Z}/3\mathbb{Z}} = A \circ z^3$$

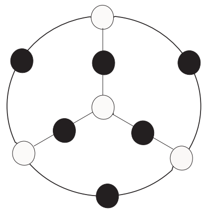

Fig. 1

for some rational function A, and the chain rule implies that A is a Belyi function. The dessin λ corresponding to the Belyi function $\theta_{\mathcal{O}}$ is the tetrahedron shown in Fig. 1, where as usual white vertices are preimages of 0, black vertices are preimages of 1, and the "centers" of faces are preimages of ∞. If the "interior" white vertex of λ is placed at the origin, while the center of the "exterior" face is placed at infinity, then the dessin λ_A corresponding to the Belyi function A is obtained from λ by factoring through the action of the group $\mathbb{Z}/3\mathbb{Z}$ viewed as a rotation group of order three around the origin. The corresponding dessin λ_A is shown in Fig. 2. By construction, the white vertex of valency 1 of λ_A is 0 and the center of the face of valency 1 is ∞. However, we can still place the center of the interior face of λ_A arbitrarily, say at the point -1. Then

$$A = \frac{az(z-b)^3}{(z+1)^3} \tag{4.8}$$

for some $a, b \in \mathbb{C}$. Finally, since the finite roots of the derivative of (4.8) are b and the roots α_1, α_2 of the polynomial $z^2 + (2b+4)z - b$, it follows from the conditions $A(\alpha_1) = A(\alpha_2) = 1$ and $\alpha_1 \neq \alpha_2$ that $a = -1/64$ and $b = 8$. Thus, we arrive at formula (4.3) and the second function in (4.5).

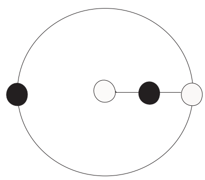

Fig. 2

Similarly, the inclusion $\mathbb{Z}/2\mathbb{Z} < A_4$ implies that a universal covering of the orbifold given by (4.4) can be written in the form

$$\theta_{\mathcal{O}} = B \circ z^2$$

for some Belyi function B, as in (4.6). However, in order to view the automorphism of order two of the dessin shown in Fig. 1 as a rotation of the second order about the origin, we must redraw it placing one of its black

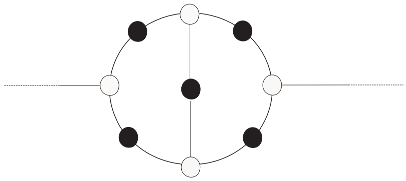

Fig. 3

vertices at infinity, as shown in Fig. 3. Factoring now through $\mathbb{Z}/2\mathbb{Z}$, we see that the dessin λ_B corresponding to B is the one depicted in Fig. 4.

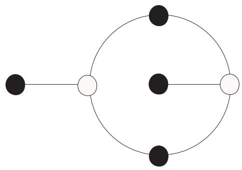

Fig. 4

We can find the function B using a reasoning similar to the one used for finding A. However, one can reduce calculations using the fact that the decomposition

$$\theta_{A_4} = -\frac{1}{64} z^3 \circ \frac{z(z^3-8)}{(z^3+1)}$$

corresponds to the subgroup D_4. Since any subgroup of A_4 isomorphic to $\mathbb{Z}/2\mathbb{Z}$ is contained in D_4, this implies that $B \underset{\mu}{\sim} \widetilde{B}$, where

$$\widetilde{B} = -\frac{1}{64} z^3 \circ f$$

for some rational function f of degree two. Since we can place the centers of the faces of λ_B at 1 and ∞, and assume that the sum of two white vertices of valency 3 of λ_B is zero, we have

$$\widetilde{B} = -\frac{1}{64}\left(\frac{z^2-c}{z-1}\right)^3, \quad c \in \mathbb{C}.$$

Finally, the finite roots of $\widetilde{B}'(z)$ are $\pm\sqrt{c}$ and $1 \pm \sqrt{1-c}$, and it follows from the condition

$$\widetilde{B}(1+\sqrt{1-c}) = \widetilde{B}(1-\sqrt{1-c}) = 1$$

that $c = 4$. This gives us the first function in (4.5).

4.4. Functions with $\nu(\mathcal{O}_2^A) = \{2,3,4\}$. Any subgroup of S_4 distinct from S_4 is isomorphic to one of the following groups: $\{e\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, D_4, S_3, D_8, A_4. Furthermore, S_4 has *two* conjugacy classes of subgroups isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and *two* conjugacy classes of subgroups isomorphic to D_4. Thus, a universal covering $\theta_{\mathcal{O}}$ of an orbifold $\mathcal{O}$ with $\nu(\mathcal{O}) = \{2, 3, 4\}$ has, up to the change $A \mapsto A \circ \mu$, where $\mu \in \mathrm{Aut}(\mathbb{CP}^1)$, ten compositional left factors. However, not all of them satisfy $\nu(\mathcal{O}_2^A) = \{2, 3, 4\}$. For example, for the factors A of degree two and three, corresponding to the subgroups A_4 and D_8, clearly $\nu(\mathcal{O}_2^A) \neq \{2, 3, 4\}$ since a rational function of degree less than 4 cannot have a critical point of multiplicity 4. Moreover, we will show that one of the factors corresponding to D_4 has the ramification orbifold $\{2, 2, 3\}$. As above, compositional left factors of θ_{S_4} can be found with the use of "dessins d'enfants" theory. We omit the details of calculations, restricting ourselves to a formal proof of Theorem 1.1.

First, it was already established by Klein that the function

$$\theta_{S_4} = \frac{1}{108} \frac{(x^8 + 14x^4 + 1)^3}{x^4(x^4 - 1)^4} \tag{4.9}$$

is a universal covering of the orbifold $\mathcal{O}$ defined by the equalities

$$\nu(0) = 3, \qquad \nu(1) = 2, \qquad \nu(\infty) = 4.$$

Clearly, the compositional left factor L in the decomposition

$$\theta_{S_4} = \frac{1}{54} \frac{(z+7)^3}{(z-1)^2} \circ \theta_{D_8},$$

where θ_{D_8} is given by (4.1), corresponds to the subgroup D_8. Using now the decomposition

$$\theta_{D_8} = \frac{1}{2}\left(z + \frac{1}{z}\right) \circ z^4,$$

we obtain the compositional left factor

$$R = L \circ \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{108} \frac{(z^2 + 14z + 1)^3}{z(z-1)^4}.$$

of θ_{S_4} corresponding to $\mathbb{Z}/4\mathbb{Z}$. Since R has a pole of order four, it follows from Lemma 3.2 that $\nu(\mathcal{O}_2^R) = \{2, 3, 4\}$.

Similarly, using the decompositions

$$\frac{1}{2}\left(z^4 + \frac{1}{z^4}\right) = T_2 \circ \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right), \qquad \frac{1}{2}\left(z^4 + \frac{1}{z^4}\right) = -T_2 \circ \frac{1}{2}\left(iz^2 + \frac{1}{iz^2}\right),$$

we obtain the compositional left factors

$$B_1 = L \circ T_2 = \frac{1}{27} \frac{(z^2+3)^3}{(z^2-1)^2}, \tag{4.10}$$

$$B_2 = L \circ (-T_2) = -\frac{1}{27} \frac{(z^2-4)^3}{z^4}$$

of θ_{S_4}, corresponding to subgroups isomorphic to D_4. Since B_2 has a pole of order four, $\nu(\mathcal{O}_2^{B_2}) = \{2, 3, 4\}$. On the other hand, it follows from (4.10) and

$$B_1 - 1 = \frac{1}{27}\,\frac{z^2(z^2-9)^2}{(z^2-1)^2}$$

that $\nu(\mathcal{O}_2^{B_1}) = \{2,2,3\}$. The subgroups corresponding to the functions B_1 and B_2 are not conjugate since otherwise Lemma 3.1 would imply that $B_1 = B_2 \circ \mu$ for some $\mu \in \mathrm{Aut}(\mathbb{CP}^1)$, in contradiction with $\nu(\mathcal{O}_2^{B_1}) \neq \nu(\mathcal{O}_2^{B_2})$.

Further, composing L with the compositional left factors T_4, $-T_4$ of θ_{D_8} we obtain the compositional left factors

$$B_3 = L \circ T_4 = \frac{4}{27}\,\frac{(z^4-z^2+1)^3}{z^4(z^2-1)^2},$$

$$B_4 = L \circ (-T_4) = -\frac{1}{27}\,\frac{(2z^2+1)^3(2z^2-3)^3}{(2z^2-1)^4}$$

of θ_{S_4}, corresponding to subgroups isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Moreover, the subgroups corresponding to B_3 and B_4 are not conjugate, since B_3 has ramification $2,2,4,4$ over infinity, while B_4 has the ramification $4,4,4$.

The function

$$M = -\frac{256}{27} z^3(z-1)$$

corresponding to the subgroup $S_3 \cong D_6$ is obtained from the decomposition

$$(4.11) \qquad -\frac{1}{432}\,\frac{(16z^8-56z^4+1)^3}{z^4(4z^4+1)^4} = -\frac{256}{27} z^3(z-1) \circ \frac{1}{8}\,\frac{(2z^2+2z-1)(4z^4+8z^2+1)}{z(4z^4+1)}.$$

The function on the left side of (4.11) is obtained from (4.9) by the substitution $z = \omega z$, where $\omega^4 = -4$.

Finally, consider the decomposition

$$(4.12) \qquad 256\frac{z^3(z^6-7z^3-8)^3}{(z^6+20z^3-8)^4} = -\frac{4x}{x^2+1-2x} \circ \theta_{A_4},$$

where θ_{A_4} is given by (4.3). The function f on the left of (4.12) is μ-equivalent to (4.9). This can be checked as above using the formula

$$f - 1 = -\frac{(z^2+2)^2(z^4-2z^2+4)^2(z^2-4z-2)^2(z^4+4z^3+18z^2-8z+4)^2}{(z^6+20z^3-8)^4}$$

and Lemma 4.1. Composing now the left factor of f from (4.12) with the left factor of θ_{A_4} of degree 4 found above, we arrive at the function

$$\left(-\frac{4x}{x^2+1-2x}\right) \circ \left(-\frac{1}{64}\,\frac{z(z-8)^3}{(z+1)^3}\right) = 256\frac{z(z^2-7z-8)^3}{(z^2+20z-8)^4}$$

corresponding to the subgroup $\mathbb{Z}/3\mathbb{Z}$.

4.5. Functions with $\nu(\mathcal{O}_2^A)=\{2,3,5\}$. The subgroups of A_5 distinct from A_5 are $\{e\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, D_4, $\mathbb{Z}/5\mathbb{Z}$, D_6, D_{10}, and A_4. Since any two isomorphic subgroups in A_5 are conjugate, it follows from Lemma 3.1 that a universal covering $\theta_{\mathcal{O}}$ of an orbifold $\mathcal{O}$ with $\nu(\mathcal{O})=\{2,3,5\}$ has, up to the transformation $A\mapsto A\circ\mu$, where $\mu\in\mathrm{Aut}(\mathbb{CP}^1)$, eight compositional left factors A of respective degrees 60, 30, 20, 15, 12, 10, 6, and 5. Since these factors have different degrees, they cannot be μ-equivalent. Furthermore, by Lemma 3.2, all these factors satisfy $\nu(\mathcal{O}_2^A)=\{2,3,5\}$. Therefore, up to μ-equivalence, there exist exactly eight rational functions A with $\nu(\mathcal{O}_2^A)=\{2,3,5\}$, and to finish the proof of Theorem 1.1 we must only check that all functions A from (V) satisfy $\nu(\mathcal{O}_2^A)=\{2,3,5\}$. In turn, the last statement follows easily from Lemma 4.1 and the formulas for $A-1$ given below:

(a) $$\frac{1}{1728}\,\frac{(z^{30}-522z^{25}-10005z^{20}-10005z^{10}+522z^5+1)^2}{z^5(z^{10}-11z^5-1)^5},$$

(b) $$-\frac{1}{6144}(3z+11)(3z^2+2z+27)^2,$$

(c) $$\frac{1}{1728}\,\frac{(z^2+12z+40)(z^2-6z+4)^2}{z-5},$$

(d) $$-\frac{1}{9}\,\frac{(180z^2+380z+229)(20z^2+20z+41)^2(20z^2-580z-979)^2}{(20z^2+140z+101)^5},$$

(e) $$\frac{1}{1728}\,\frac{(z^6-522z^5-10005z^4-10005z^2+522z+1)^2}{z(z^2-11z-1)^5},$$

(f) $$-\frac{1}{27}\,\frac{(10z+3)(20z^2+20z+1)(10z^2+10z+3)^2\times(500z^4+300z^3+70z^2+10z+1)^2}{(20z^2+10z+1)^5},$$

(g) $$-\frac{1}{64}\,\frac{(z^2+5)^2(8z^4-100z^3+2055z^2+500z+200)^2\times(z^4-350z^3-2190z^2+1750z+25)^2}{(z^4+55z^3-165z^2-275z+25)^5},$$

(h) $$\frac{1}{1728}\,\frac{(z^2+4)(z^2-2z-4)^2(z^4+3z^2+1)^2\times(z^4+6z^3+21z^2+36z+61)^2(z^4-4z^3+21z^2-34z+41)^2}{(z-1)^5(z^4+z^3+6z^2+6z+11)^5}.$$

5. Functions with $\chi(\mathcal{O}_2^A)=0$. Let $\mathcal{O}$ be an orbifold on $\mathbb{CP}^1$ such that $\chi(\mathcal{O})=0$. Then the corresponding group $\Gamma_{\mathcal{O}}$ is generated by translations of $\mathbb{C}$ by elements of some lattice $L\subset\mathbb{C}$ of rank two and the transformation $z\mapsto\varepsilon z$, where ε is an nth root of unity with $n=2,3,4$, or 6, such that $\varepsilon L=L$. We will denote by $\Lambda_{\mathcal{O}}$ the subgroup of $\Gamma_{\mathcal{O}}$ generated by translations. The group $\Lambda_{\mathcal{O}}$ is normal in $\Gamma_{\mathcal{O}}$, and can be described as the kernel of the homomorphism $\psi:\Gamma_{\mathcal{O}}\to\mathbb{C}$ which sends $\sigma=az+b\in\Gamma_{\mathcal{O}}$ to $\psi(\sigma)=a\in\mathbb{C}$. For $\nu(\mathcal{O})=\{2,2,2,2\}$ the complex structure of $\mathbb{C}/\Lambda_{\mathcal{O}}$ may be arbitrary, and the function $\theta_{\mathcal{O}}$ is the corresponding Weierstrass function $\wp(z)$. On the other hand, for $\nu(\mathcal{O})$ equal to $\{2,4,4\}$, $\{3,3,3\}$, or $\{2,3,6\}$ the complex structure of $\mathbb{C}/\Lambda_{\mathcal{O}}$ is rigid and arises from the tiling of $\mathbb{C}$ by squares, equilateral triangles, or alternately colored equilateral triangles, respectively. Accordingly, the function $\theta_{\mathcal{O}}$ may be written in terms of the corresponding Weierstrass functions as $\wp^2(z)$, $\wp'(z)$, and $\wp'^2(z)$ (see [18] and [7, Section IV.9.12]).

The following statement provides a geometric description of covering maps $A : \mathcal{O}_1 \to \mathcal{O}_2$ between orbifolds of zero characteristic.

THEOREM 5.1. *Let A be a rational function. Then $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between some orbifolds on $\mathbb{CP}^1$ of zero Euler characteristic if and only if there exist elliptic curves $\mathcal{C}_1$ and $\mathcal{C}_2$, subgroups $\Omega_1 \subseteq \mathrm{Aut}(\mathcal{C}_1)$ and $\Omega_2 \subseteq \mathrm{Aut}(\mathcal{C}_2)$, and a holomorphic map $\alpha : \mathcal{C}_1 \to \mathcal{C}_2$ such that the diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\alpha} & \mathcal{C}_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{5.1}$$

where $\pi_1 : \mathcal{C}_1 \to \mathcal{C}_1/\Omega_1$ and $\pi_2 : \mathcal{C}_2 \to \mathcal{C}_2/\Omega_2$ are the quotient maps, commutes.

Proof. If A is a rational function such $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between some orbifolds of zero Euler characteristic, then there exists an isomorphism $F = az + b$, $a, b \in \mathbb{C}$, of the complex plane which makes the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F=az+b} & \mathbb{C} \\ {\scriptstyle \theta_{\mathcal{O}_1}}\downarrow & & \downarrow{\scriptstyle \theta_{\mathcal{O}_2}} \\ \mathcal{O}_1 & \xrightarrow{A} & \mathcal{O}_2 \end{array}$$

commutative and satisfies (2.8) for some homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \to \Gamma_{\mathcal{O}_2}$. Moreover, φ is a monomorphism since F is invertible, and hence the equality $F \circ \sigma = F$ implies that $\sigma = z$.

It is clear that $\mathcal{C}_1 = \mathbb{C}/\Lambda_{\mathcal{O}_1}$ and $\mathcal{C}_2 = \mathbb{C}/\Lambda_{\mathcal{O}_2}$ are Riemann surfaces of genus one, and the groups

$$\Omega_1 \cong \Gamma_{\mathcal{O}_1}/\Lambda_{\mathcal{O}_1}, \qquad \Omega_2 \cong \Gamma_{\mathcal{O}_2}/\Lambda_{\mathcal{O}_2}$$

are cyclic groups of order 2, 3, 4, or 6. Moreover, we can consider $\mathcal{C}_1$ and $\mathcal{C}_2$ as elliptic curves, whose marked points are projections of the origin, and Ω_1 and Ω_2 as the automorphism groups of $\mathcal{C}_1$ and $\mathcal{C}_2$. Further,

$$\theta_{\mathcal{O}_1} = \pi_1 \circ \psi_1, \qquad \theta_{\mathcal{O}_2} = \pi_2 \circ \psi_2,$$

where

$$\psi_1 : \mathbb{C} \to \mathbb{C}/\Lambda_{\mathcal{O}_1} \cong \mathcal{C}_1, \qquad \psi_2 : \mathbb{C} \to \mathbb{C}/\Lambda_{\mathcal{O}_2} \cong \mathcal{C}_2,$$

and

$$\pi_1 : \mathcal{C}_1 \to \mathcal{C}_1/\Omega_1 \cong \mathbb{CP}^1, \qquad \pi_2 : \mathcal{C}_2 \to \mathcal{C}_2/\Omega_2 \cong \mathbb{CP}^1,$$

are quotient maps. Finally, since φ is a monomorphism, it maps elements of infinite order of $\Gamma_{\mathcal{O}_1}$ to elements of infinite order of $\Gamma_{\mathcal{O}_2}$. Thus, $\varphi(\Lambda_{\mathcal{O}_1}) \subset \Lambda_{\mathcal{O}_2}$, implying that F descends to a holomorphic map $\alpha : \mathcal{C}_1 \to \mathcal{C}_2$ which makes the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F=ax+b} & \mathbb{C} \\ \psi_1 \big\downarrow & & \big\downarrow \psi_2 \\ \mathcal{C}_1 & \xrightarrow{\alpha} & \mathcal{C}_2 \\ \pi_1 \big\downarrow & & \big\downarrow \pi_2 \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{5.2}$$

commutative.

In the other direction, we can complete any diagram (5.1) to (5.2), setting ψ_1 and ψ_2 equal to the usual universal coverings of the Riemann surfaces $\mathcal{C}_1$ and $\mathcal{C}_2$. Since π_1 and π_2 are Galois coverings and the maps ψ_1 and ψ_2 are non-ramified, it is easy to see that the maps $\pi_1 \circ \psi_1 : \mathbb{C} \to \mathcal{O}_2^{\pi_1}$ and $\pi_2 \circ \psi_2 : \mathbb{C} \to \mathcal{O}_2^{\pi_2}$ are universal coverings of the orbifolds $\mathcal{O}_2^{\pi_1}$ and $\mathcal{O}_2^{\pi_2}$, implying that $A : \mathcal{O}_2^{\pi_1} \to \mathcal{O}_2^{\pi_2}$ is a covering map between orbifolds. Finally, since

$$\widetilde{\mathcal{O}}_2^{\pi_1} = \widetilde{\mathcal{O}}_2^{\pi_2} = \mathbb{C},$$

these orbifolds have zero Euler characteristic. ∎

Obviously, Corollary 2.3 and Theorem 5.1 imply the first part of Theorem 1.2. On the other hand, in order to prove the second part we must show that if $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds of zero Euler characteristic such that $\chi(\mathcal{O}_2^A) > 0$, then A is μ-equivalent either to a cyclic function for some $n \le 4$, or to a dihedral function for some $n \le 4$, or to a tetrahedral function. The theorem below provides a more precise version of the required statement.

THEOREM 5.2. *Let A be a rational function and $\mathcal{O}_1$, $\mathcal{O}_2$ orbifolds such that $\chi(\mathcal{O}_1) = \chi(\mathcal{O}_2) = 0$. Assume that $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds. Then either $\chi(\mathcal{O}_2^A) = 0$ and $\mathcal{O}_2 = \mathcal{O}_2^A$, $\mathcal{O}_1 = \mathcal{O}_1^A$, or $\chi(\mathcal{O}_2^A) > 0$ and one of the following conditions holds:*

(1) $\nu(\mathcal{O}_2^A) = \{2,2\}, \quad A \underset{\mu}{\sim} z^2, \quad \nu(\mathcal{O}_1) = \nu(\mathcal{O}_2) = \{2,2,2,2\},$

(2) $\nu(\mathcal{O}_2^A) = \{2,2\}, \quad A \underset{\mu}{\sim} z^2, \quad \nu(\mathcal{O}_1) = \nu(\mathcal{O}_2) = \{2,4,4\},$

(3) $\nu(\mathcal{O}_2^A) = \{2,2\}, \quad A \underset{\mu}{\sim} z^2, \quad \nu(\mathcal{O}_1) = \{2,2,2,2\}, \quad \nu(\mathcal{O}_2) = \{2,4,4\},$

(4) $\nu(\mathcal{O}_2^A) = \{2,2\}, \quad A \underset{\mu}{\sim} z^2, \quad \nu(\mathcal{O}_1) = \{3,3,3\}, \quad \nu(\mathcal{O}_2) = \{2,3,6\},$

(5) $\nu(\mathcal{O}_2^A)=\{3,3\}$, $A\underset{\mu}{\sim}z^3$, $\nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)=\{3,3,3\}$,

(6) $\nu(\mathcal{O}_2^A)=\{3,3\}$, $A\underset{\mu}{\sim}z^3$, $\nu(\mathcal{O}_1)=\{2,2,2,2\}$, $\nu(\mathcal{O}_2)=\{2,3,6\}$,

(7) $\nu(\mathcal{O}_2^A)=\{4,4\}$, $A\underset{\mu}{\sim}z^4$, $\nu(\mathcal{O}_1)=\{2,2,2,2\}$, $\nu(\mathcal{O}_2)=\{2,4,4\}$,

(8) $\nu(\mathcal{O}_2^A)=\{2,2,2\}$, $A\underset{\mu}{\sim}\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)$, $\nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)=\{2,2,2,2\}$,

(9) $\nu(\mathcal{O}_2^A)=\{2,2,2\}$, $A\underset{\mu}{\sim}\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)$,
$\nu(\mathcal{O}_1)=\{2,2,2,2\}$, $\nu(\mathcal{O}_2)=\{2,4,4\}$,

(10) $\nu(\mathcal{O}_2^A)=\{2,2,3\}$, $A\underset{\mu}{\sim}\frac{1}{2}\left(z^3+\frac{1}{z^3}\right)$,
$\nu(\mathcal{O}_1)=\{3,3,3\}$, $\nu(\mathcal{O}_2)=\{2,3,6\}$,

(11) $\nu(\mathcal{O}_2^A)=\{2,2,3\}$, $A\underset{\mu}{\sim}T_3$, $\nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)=\{2,3,6\}$,

(12) $\nu(\mathcal{O}_2^A)=\{2,2,4\}$, $A\underset{\mu}{\sim}\frac{1}{2}\left(z^4+\frac{1}{z^4}\right)$,
$\nu(\mathcal{O}_1)=\{2,2,2,2\}$, $\nu(\mathcal{O}_2)=\{2,4,4\}$,

(13) $\nu(\mathcal{O}_2^A)=\{2,2,4\}$, $A\underset{\mu}{\sim}T_4$,
$\nu(\mathcal{O}_1)=\{2,2,2,2\}$, $\nu(\mathcal{O}_2)=\{2,4,4\}$,

(14) $\nu(\mathcal{O}_2^A)=\{2,2,4\}$, $A\underset{\mu}{\sim}T_4$, $\nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)=\{2,4,4\}$,

(15) $\nu(\mathcal{O}_2^A)=\{2,3,3\}$, $A\underset{\mu}{\sim}-\frac{1}{64}\,\frac{z(z-8)^3}{(z+1)^3}$, $\nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)=\{2,3,6\}$,

(16) $\nu(\mathcal{O}_2^A)=\{2,3,3\}$, $A\underset{\mu}{\sim}-\frac{1}{64}\left(\frac{z^2-4}{z-1}\right)^3$,
$\nu(\mathcal{O}_1)=\{2,2,2,2\}$, $\nu(\mathcal{O}_2)=\{2,3,6\}$,

(17) $\nu(\mathcal{O}_2^A)=\{2,3,3\}$, $A\underset{\mu}{\sim}-\frac{1}{64}\,\frac{z^3(z^3-8)^3}{(z^3+1)^3}$,
$\nu(\mathcal{O}_1)=\{2,2,2,2\}$, $\nu(\mathcal{O}_2)=\{2,3,6\}$,

In particular, if $\deg A>12$, *then* $\mathcal{O}_2=\mathcal{O}_2^A$, $\mathcal{O}_1=\mathcal{O}_1^A$.

Proof. It follows from (2.9) that $\chi(\mathcal{O}_2^A)\geq\chi(\mathcal{O}_2)$ and equality is attained if and only if $\mathcal{O}_2^A=\mathcal{O}_2$. Therefore, if $\chi(\mathcal{O}_2^A)=0$, then $\mathcal{O}_2^A=\mathcal{O}_2$, and hence $\mathcal{O}_1^A=\mathcal{O}_1$ since (2.5) implies that for any covering map $A:\mathcal{O}_1\to\mathcal{O}_2$ the orbifold $\mathcal{O}_1$ is defined by $\mathcal{O}_2$ in a unique way. So, below we will assume that $\chi(\mathcal{O}_2^A)>0$.

We will denote by ν_A the ramification function of $\mathcal{O}_2^A$ and by ν the ramification function of $\mathcal{O}_2$. We will also use the notation

$$R(f) = \big(\{l_{11}, l_{12}, \dots, l_{1s_1}\}_{z_1}, \dots, \{l_{r1}, l_{r2}, \dots, l_{rs_r}\}_{z_r}\big)$$

to denote that a rational function f has r critical values $z_1, \dots, z_r$, and the collection of local degrees of f at points of the set $f^{-1}\{z_i\}$, $1 \leq i \leq r$, is $\{l_{i1}, \dots, l_{is_i}\}$.

As in Lemma 3.2, the conditions $\mathcal{O}_2^A \preceq \mathcal{O}_2$ and $\chi(\mathcal{O}_2) = 0$, $\chi(\mathcal{O}_2^A) > 0$ impose strong restrictions on possible collections $\nu(\mathcal{O}_2^A)$, and an easy analysis of the lists (2.11) and (2.12) shows that either $\nu(\mathcal{O}_2^A) = \{n, n\}$, $n \leq 4$, or $\nu(\mathcal{O}_2^A) = \{2, 2, n\}$, $n \leq 4$, or $\nu(\mathcal{O}_2^A) = \{2, 3, 3\}$.

Case 1: $\nu(\mathcal{O}_2^A) = \{n, n\}$. If $n = 2$, then $\mathcal{O}_2^A \preceq \mathcal{O}_2$ implies that $\nu(\mathcal{O}_2)$ is either $\{2, 2, 2, 2\}$, or $\{2, 4, 4\}$, or $\{2, 3, 6\}$. Assume that, say, $\nu(\mathcal{O}_2) = \{2, 4, 4\}$ and let $x_1, x_2, y_1, y_2, y_3 \in \mathbb{CP}^1$ be points such that

$$\nu_A(x_1) = 2, \quad \nu_A(x_2) = 2, \tag{5.3}$$

$$\nu(y_1) = 2, \quad \nu(y_2) = 4, \quad \nu(y_3) = 4. \tag{5.4}$$

Then either $\{x_1, x_2\} = \{y_1, y_2\}$, or $\{x_1, x_2\} = \{y_1, y_3\}$, or

$$\{x_1, x_2\} = \{y_2, y_3\}. \tag{5.5}$$

Further, since (5.3) implies that

$$\mathcal{R}(A) = (\{2\}_{x_1}, \{2\}_{x_2}),$$

it follows from (2.5) that in the first two cases

$$\nu(\mathcal{O}_1) = \{2, 4, 4\},$$

while in the third one

$$\nu(\mathcal{O}_1) = \{2, 2, 2, 2\}.$$

Thus, we arrive at cases (2) and (3) in the theorem.

Similarly, if $\nu(\mathcal{O}_2) = \{2, 3, 6\}$ and $y_1, y_2, y_3 \in \mathbb{CP}^1$ are points such that

$$\nu(y_1) = 2, \quad \nu(y_2) = 3, \quad \nu(y_3) = 6,$$

then $\{x_1, x_2\} = \{y_1, y_3\}$ and

$$\nu(\mathcal{O}_1) = \{3, 3, 3\}.$$

Finally, if $\nu(\mathcal{O}_2) = \{2, 2, 2, 2\}$ we conclude that

$$\nu(\mathcal{O}_1) = \{2, 2, 2, 2\}.$$

The cases $n = 3$ and $n = 4$ are considered in the same way as above. Namely, if $\nu(\mathcal{O}_2^A) = \{3, 3\}$, then $\nu(\mathcal{O}_2)$ is either $\{3, 3, 3\}$ or $\{2, 3, 6\}$, and we arrive at cases (5) and (6), respectively, while if $\nu(\mathcal{O}_2^A) = \{4, 4\}$, then $\nu(\mathcal{O}_2) = \{2, 4, 4\}$, and we arrive at case (7).

Case 2: $\nu(\mathcal{O}_2^A) = \{2, 2, n\}$. The proof goes as above with some modifications. Let $x_1, x_2, x_3 \in \mathbb{CP}^1$ be points such that

$$\nu_A(x_1) = 2, \quad \nu_A(x_2) = 2, \quad \nu_A(x_3) = n.$$

Assume that, say, $n = 3$. Then $\nu(\mathcal{O}_2) = \{2, 3, 6\}$, and if $y_1, y_2, y_3 \in \mathbb{CP}^1$ are points such that

$$\nu(y_1) = 2, \qquad \nu(y_2) = 3, \qquad \nu(y_3) = 6,$$

then

$$\{x_1, x_2\} = \{y_1, y_3\}, \qquad x_3 = y_2.$$

Now however we must consider two types of branching of A corresponding to the possibilities $A \underset{\mu}{\sim} \frac{1}{2}(z^3 + z^{-3})$ and $A \underset{\mu}{\sim} T_3$. In the first case

$$\mathcal{R}(A) = \big(\{2, 2, 2\}_{x_1}, \{2, 2, 2\}_{x_2}, \{3, 3\}_{x_3}\big),$$

in the second

$$\mathcal{R}(A) = \big(\{1, 2\}_{x_1}, \{1, 2\}_{x_2}, \{3\}_{x_3}\big).$$

Correspondingly, either

$$\nu(\mathcal{O}_1) = \{3, 3, 3\} \quad \text{or} \quad \nu(\mathcal{O}_1) = \{2, 3, 6\}.$$

Similarly, if $n = 4$, then $\nu(\mathcal{O}_2) = \{2, 4, 4\}$, and either

$$\mathcal{R}(A) = \big(\{2, 2, 2, 2\}_{x_1}, \{2, 2, 2, 2\}_{x_2}, \{4, 4\}_{x_3}\big),$$

or

$$\mathcal{R}(A) = \big(\{1, 1, 2\}_{x_1}, \{2, 2\}_{x_2}, \{4\}_{x_3}\big), \tag{5.6}$$

or

$$\mathcal{R}(A) = \big(\{2, 2\}_{x_1}, \{1, 1, 2\}_{x_2}, \{4\}_{x_3}\big). \tag{5.7}$$

In the first case,

$$\nu(\mathcal{O}_1) = \{2, 2, 2, 2\},$$

while in each of cases (5.6) and (5.7), either

$$\nu(\mathcal{O}_1) = \{2, 2, 2, 2\}, \tag{5.8}$$

or

$$\nu(\mathcal{O}_1) = \{2, 4, 4\}. \tag{5.9}$$

Say, if (5.6) holds, and $y_1, y_2, y_3 \in \mathbb{CP}^1$ are the points such that

$$\nu(y_1) = 2, \qquad \nu(y_2) = 4, \qquad \nu(y_3) = 4,$$

then (5.8) holds if $x_1 = y_1$, while (5.9) holds if $x_1 = y_2$ or $x_1 = y_3$.

Finally, if $n = 2$, then $A \underset{\mu}{\sim} \frac{1}{2}(z^2 + z^{-2})$, and $\nu(\mathcal{O}_2)$ is either $\{2, 2, 2, 2\}$ or $\{2, 4, 4\}$. In both cases,

$$\nu(\mathcal{O}_1) = \{2, 2, 2, 2\}.$$

Case 3: $\nu(\mathcal{O}_2^A)=\{2,3,3\}$. In this case $\nu(\mathcal{O}_2)=\{2,3,6\}$, and considering three possible branching types for tetrahedral functions

$$\mathcal{R}(A)=\left(\{2,2\}_{z_1},\{1,3\}_{z_2},\{1,3\}_{z_3}\right),$$
$$\mathcal{R}(A)=\left(\{1,1,2,2\}_{z_1},\{3,3\}_{z_2},\{3,3\}_{z_3}\right),$$
$$\mathcal{R}(A)=\left(\{2,2,2,2,2,2\}_{z_1},\{3,3,3,3\}_{z_2},\{3,3,3,3\}_{z_3}\right),$$

we arrive at cases (15), (16), (17) respectively. ∎

Remark 5.3. Modifying the above proof one can see that all the possibilities listed in Theorem 5.2 actually occur. For example, for any rational function $A\sim z^2$, *there exist* orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $\nu(\mathcal{O}_1)=\{2,2,2,2\}$, $\nu(\mathcal{O}_2)=\{2,4,4\}$, and $A:\mathcal{O}_1\to\mathcal{O}_2$ is a covering map. Indeed, let x_1, x_2 be points such that (5.3) holds. Define $\mathcal{O}_2$ by (5.4), where y_2,y_3 satisfy (5.5) and y_1 is arbitrary, and then define $\mathcal{O}_1$ by (2.5). Since $\mathcal{O}_2^A\preceq\mathcal{O}_2$ implies that $\deg_z A$ divides $\nu(A(z))$ for any $z\in\mathbb{CP}^1$, the orbifold $\mathcal{O}_1$ is well-defined and $A:\mathcal{O}_1\to\mathcal{O}_2$ is a covering map. Thus, Theorem 5.2 gives a complete list of μ-equivalence classes of rational functions A which fit diagram (5.1) but satisfy $g(\widetilde{S}_A)=0$ instead of $g(\widetilde{S}_A)=1$.

Corollary 5.4. *Let A be a rational function and $\mathcal{O}_1$, $\mathcal{O}_2$ orbifolds such that $\nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)$ and $A:\mathcal{O}_1\to\mathcal{O}_2$ is a covering map between orbifolds. Then either $\chi(\mathcal{O}_2^A)=0$ and $\mathcal{O}_2=\mathcal{O}_2^A$, $\mathcal{O}_1=\mathcal{O}_1^A$, or $\chi(\mathcal{O}_2^A)>0$ and one of the following conditions holds:*

(1) $\nu(\mathcal{O}_2^A)=\{2,2\}$, $\quad A\underset{\mu}{\sim}z^2$, $\quad \nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)=\{2,2,2,2\}$,

(2) $\nu(\mathcal{O}_2^A)=\{2,2\}$, $\quad A\underset{\mu}{\sim}z^2$, $\quad \nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)=\{2,4,4\}$,

(3) $\nu(\mathcal{O}_2^A)=\{3,3\}$, $\quad A\underset{\mu}{\sim}z^3$, $\quad \nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)=\{3,3,3\}$,

(4) $\nu(\mathcal{O}_2^A)=\{2,2,2\}$, $\quad A\underset{\mu}{\sim}\dfrac{1}{2}\left(z^2+\dfrac{1}{z^2}\right)$, $\quad \nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)=\{2,2,2,2\}$,

(5) $\nu(\mathcal{O}_2^A)=\{2,2,3\}$, $\quad A\underset{\mu}{\sim}T_3$, $\quad \nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)=\{2,3,6\}$,

(6) $\nu(\mathcal{O}_2^A)=\{2,2,4\}$, $\quad A\underset{\mu}{\sim}T_4$, $\quad \nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)=\{2,4,4\}$,

(7) $\nu(\mathcal{O}_2^A)=\{2,3,3\}$, $\quad A\underset{\mu}{\sim}-\dfrac{1}{64}\,\dfrac{z(z-8)^3}{(z+1)^3}$, $\quad \nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)=\{2,3,6\}$.

In particular, if $\deg A>4$, *then* $\mathcal{O}_2=\mathcal{O}_2^A$, $\mathcal{O}_1=\mathcal{O}_1^A$.

Proof. The corollary follows from Theorem 5.2 since $\nu(\mathcal{O}_1)=\nu(\mathcal{O}_2)$ implies $\chi(\mathcal{O}_1)=\chi(\mathcal{O}_2)=0$ by (2.6). ∎

Recall that Lattès maps are rational functions which can be defined in one of the following ways (see [18], [27]). First, a Lattès map A may be defined by the condition that there exist a Riemann surface $\mathcal{C}$ of genus one and holomorphic maps $\alpha:\mathcal{C}\to\mathcal{C}$ and $\pi:\mathcal{C}\to\mathbb{CP}^1$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\alpha} & \mathcal{C} \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\
\mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1
\end{array}
\tag{5.10}
$$

commutes. This condition is equivalent to the apparently stronger condition that π in (5.10) is a quotient map $\pi : \mathcal{C} \to \mathcal{C}/\Omega$ for some finite subgroup $\Omega \subseteq \mathrm{Aut}(\mathcal{C})$. Finally, a Lattès map A may be defined by the condition that there exists an orbifold $\mathcal{O}$ in $\mathbb{CP}^1$ such that $\chi(\mathcal{O}) = 0$ and $A : \mathcal{O} \to \mathcal{O}$ is a covering map between orbifolds. Thus, Corollary 5.4 implies the following corollary.

COROLLARY 5.5. *For any Lattès map A of degree greater than four the equality $g(\widetilde{S}_A) = 1$ holds.* □

REMARK 5.6. It is easy to see that there exist rational functions A with $g(\widetilde{S}_A) = 1$ which are not Lattès maps. Indeed, let $A : \mathcal{O}_1 \to \mathcal{O}_2$ be any covering map between orbifolds of zero Euler characteristic such that $\mathcal{O}_1 \neq \mathcal{O}_2$ and $\deg A > 12$. Then it follows from Theorem 5.1 that $\chi(\mathcal{O}_2^A) = 0$ and $\mathcal{O}_2 = \mathcal{O}_2^A$, $\mathcal{O}_1 = \mathcal{O}_1^A$. Thus, $g(\widetilde{S}_A) = 1$ by Lemma 2.1. On the other hand, if $\mathcal{O}$ is an orbifold such that $A : \mathcal{O} \to \mathcal{O}$ is a covering map between orbifolds, then (2.9) yields

$$\chi(\mathcal{O}_2^A) \geq \chi(\mathcal{O}), \qquad \chi(\mathcal{O}_1^A) \geq \chi(\mathcal{O}),$$

and equality is attained if and only if $\mathcal{O}_2^A = \mathcal{O}$, $\mathcal{O}_1^A = \mathcal{O}$. Since

$$\chi(\mathcal{O}_2^A) = \chi(\mathcal{O}_1^A) = \chi(\mathcal{O}) = 0,$$

this implies that $\mathcal{O}_2 = \mathcal{O}_1 = \mathcal{O}$, in contradiction with $\mathcal{O}_1 \neq \mathcal{O}_2$.

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