

# ON ITERATES OF RATIONAL FUNCTIONS WITH MAXIMAL NUMBER OF CRITICAL VALUES

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ABSTRACT. Let  $F$  be a rational function of one complex variable of degree  $m \geq 2$ . The function  $F$  is called simple if for each  $z \in \mathbb{CP}^1$  the preimage  $P^{-1}\{z\}$  contains at least  $m-1$  points. We show that if  $F$  is a simple rational function of degree  $m \geq 4$  and  $F^{\circ l} = G_r \circ G_{r-1} \circ \dots \circ G_1$ ,  $l \geq 1$ , is a decomposition of an iterate of  $F$  into a composition of indecomposable rational functions, then  $r = l$ , and there exist Möbius transformations  $\mu_i$ ,  $1 \leq i \leq r-1$ , such that  $G_r = F \circ \mu_{r-1}$ ,  $G_i = \mu_i^{-1} \circ F \circ \mu_{i-1}$ ,  $1 < i < r$ , and  $G_1 = \mu_1^{-1} \circ F$ . As an application, we provide explicit solutions of a number of problems in complex and arithmetic dynamics for “general” rational functions.

## 1. INTRODUCTION

Let  $F$  be a rational function of one complex variable of degree  $m \geq 2$ . We say that  $F$  is *simple* if for every  $z \in \mathbb{CP}^1$  the preimage  $F^{-1}\{z\}$  contains at least  $m-1$  points. Since, by the Riemann-Hurwitz formula,  $F$  has  $2m-2$  critical points counted with multiplicities, this is equivalent to say that  $F$  has  $2m-2$  distinct critical values. Writing a rational function  $F = F(z)$  of degree  $m$  as  $F = P/Q$ , where  $P$  and  $Q$  are polynomials of degree  $m$ , we can identify the space of rational functions of degree  $m$  with the algebraic variety  $\text{Rat}_m = \mathbb{CP}^{2m+1} \setminus \text{Res}_{m,m,z}(P, Q)$ . Having in mind this identification, it is easy to see that for every  $m \geq 2$  the property of a rational function to be simple is *general* in the following sense: there exists a proper Zariski closed subset  $Z$  of  $\mathbb{CP}^{2m+1}$  such that all  $F \in \text{Rat}_m \setminus Z$  are simple.

Let us recall that any representation of a rational function  $F$  of degree at least two in the form  $F = F_r \circ F_{r-1} \circ \dots \circ F_1$ , where  $F_1, F_2, \dots, F_r$  are rational functions, is called a *decomposition* of  $F$ . Two decompositions

$$F = F_r \circ F_{r-1} \circ \dots \circ F_1 \quad \text{and} \quad F = G_l \circ G_{l-1} \circ \dots \circ G_1$$

are called *equivalent* if  $l = r$  and either  $r = 1$  and  $F_1 = G_1$ , or  $r \geq 2$  and there exist Möbius transformations  $\mu_i$ ,  $1 \leq i \leq r-1$ , such that

$$F_r = G_r \circ \mu_{r-1}, \quad F_i = \mu_i^{-1} \circ G_i \circ \mu_{i-1}, \quad 1 < i < r, \quad \text{and} \quad F_1 = \mu_1^{-1} \circ G_1.$$

A rational function  $F$  of degree  $m \geq 2$  is called *indecomposable* if the equality  $F = F_2 \circ F_1$ , where  $F_1, F_2$ , are rational functions, implies that at least one of the functions  $F_1, F_2$  is of degree one. It is clear that any rational function  $F$  of degree  $m \geq 2$  can be decomposed into a composition of indecomposable functions, although in general not in a unique way. The problem of describing all such decompositions is quite delicate, and the general theory exists only if  $F$  is a polynomial or a Laurent polynomial (see [46], [30]).

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In dynamical applications, it is desirable to understand the structure of decompositions of the whole totality of iterates of a given rational function  $F$  (see e.g. [4], [18], [19], [26], [41]). Since however an explicit description of such decompositions usually is impossible, a “qualitative” description comes to the fore. An example of such a description is the result of Zieve and Müller ([55]), which states that if  $F$  is a *polynomial* of degree  $n \geq 2$ , not conjugate to  $z^n$  or to  $\pm T_n$ , where  $T_n$  stands for the Chebyshev polynomial, then decompositions of its iterates can be obtained from decompositions of a single iterate  $F^{\circ N}$  for  $N$  big enough. More precisely, there exists an integer  $N \geq 1$  such that any decomposition of  $F^{\circ d}$  with  $d \geq N$  has the form

$$X = F^{\circ k_1} \circ X', \quad Y = Y' \circ F^{\circ k_2},$$

where  $F^{\circ N} = X' \circ Y'$  and  $k_1, k_2 \geq 1$  (see [55] and also [33], [42] for different proofs of this fact).

Conjecturally, the result of [55] remains true for any *non-special* rational function  $F$ , where by a special function we mean a rational function  $F$  that is either a Lattès map or is conjugate to  $z^{\pm n}$  or  $\pm T_n$ . To the date, this conjecture is proved for “tame” rational functions, that is, for the functions  $F$  satisfying the following condition: the algebraic curve

$$F(x) - F(y) = 0$$

has no factors of genus zero or one distinct from the diagonal ([42]). In particular, this covers all simple rational function  $F$  of degree  $m \geq 4$ , since any such a function is tame ([31]). In this paper, we show however that for simple rational functions one can provide an *explicit* description of decompositions of their iterates, which turns out to be as simple as possible.

Namely, we prove the following result.

**Theorem 1.1.** *Let  $F$  be a simple rational function of degree  $m \geq 4$ . Then any decomposition of  $F^{\circ l}$ ,  $l \geq 1$ , into a composition of indecomposable rational functions is equivalent to  $F^{\circ l}$ .*

The significance of Theorem 1.1 is that it permits to solve explicitly a number of problems in complex and arithmetic dynamics for “general” rational functions. The first group of such problems is related to the functional equations

$$(1) \quad F^{\circ k} = G^{\circ l}$$

and

$$(2) \quad F^{\circ k_1} = F^{\circ k_2} \circ G^{\circ l}.$$

According to the results of Ritt ([47]) and Levin and Przytycki ([23], [24]), for a non-special rational function  $F$  of degree at least two the first of these equations describes rational functions  $G$  of degree at least two commuting with some iterate of  $F$ , while the second equation describes rational functions  $G$  with  $\mu_G = \mu_F$ , where  $\mu_F$  stands for the measure of maximal entropy of  $F$ .

Theorem 1.1 permits to describe solutions of (1) and (2) for simple  $F$  of degree  $m \geq 4$  explicitly. In more detail, for a rational function  $F$  of degree  $m \geq 2$  we denote by  $C(F)$  the semigroup of rational functions commuting with  $F$ , and by  $C_\infty(F)$  the semigroup of rational functions commuting with some iterate of  $F$ . We also set

$$\text{Aut}(F) = C(F) \cap \text{Aut}(\mathbb{CP}^1), \quad \text{Aut}_\infty(F) = C_\infty(F) \cap \text{Aut}(\mathbb{CP}^1).$$

Notice that  $C_\infty(F)$  obviously contains the semigroup  $\langle \text{Aut}_\infty(F), F \rangle$  generated by  $F$  and  $\text{Aut}_\infty(F)$ . Further, we denote by  $E_0(F)$  the subgroup of  $\text{Aut}(\mathbb{CP}^1)$  consisting of Möbius transformations preserving the measure of maximal entropy of  $F$ , and by  $E(F)$  the semigroup consisting of rational functions  $G$  of degree at least two with  $\mu_G = \mu_F$  completed by the group  $E_0(F)$ . Finally, we denote by  $G_0(F)$  the maximal subgroup of  $\text{Aut}(\mathbb{CP}^1)$  such that for every  $\sigma \in G_0(F)$  the equality

$$F \circ \sigma = \nu \circ F$$

holds for some  $\nu \in G_0(F)$ .

Theorem 1.1 yields that for simple rational functions the objects introduced above are related in a very easy way.

**Theorem 1.2.** *Let  $F$  be a simple rational function of degree  $m \geq 4$ . Then*

$$E_0(F) = \text{Aut}_\infty(F) = G_0(F) \quad \text{and} \quad E(F) = C_\infty(F) = \langle \text{Aut}_\infty(F), F \rangle.$$

In turn, Theorem 1.2 implies that for a general rational function  $F$  of degree  $m \geq 4$  the equality

$$(3) \quad E(F) = \langle F \rangle$$

holds (see Section 3.2). This provides an affirmative answer to the question of Ye, who proved that (3) holds after removing from  $\text{Rat}_m$  *countably* many algebraic sets, and asked whether (3) remains true if to remove from  $\text{Rat}_m$  only finitely many such sets ([52]).

Further applications of Theorem 1.1 concern problems that can be reformulated in terms of semiconjugacies between rational functions. Let  $A$  and  $B$  be rational functions of degree at least two. We recall that the function  $B$  is called *semiconjugate* to the function  $A$  if there exists a non-constant rational function  $X$  such that

$$(4) \quad A \circ X = X \circ B.$$

Semiconjugate rational functions appear naturally in complex and arithmetic dynamics (see, e.g., the recent papers [3], [8], [10], [22], [26], [35], [41]). A comprehensive description of solutions of (4) was obtained in the series of papers [32], [34], [36], [37]. Theorem 1.1 permits to simplify this description for simple  $A$  to the following very simple form suitable for applications.

**Theorem 1.3.** *Let  $F$  be a simple rational function of degree  $m \geq 4$  and  $G, X$  rational functions of degree at least two such that the diagram*

$$(5) \quad \begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{G} & \mathbb{CP}^1 \\ X \downarrow & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{F^{\circ r}} & \mathbb{CP}^1 \end{array}$$

*commutes for some  $r \geq 1$ . Then there exist a Möbius transformation  $\nu$  and an integer  $l \geq 1$  such that the equalities*

$$X = F^{\circ l} \circ \nu, \quad G = \nu^{-1} \circ F^{\circ r} \circ \nu$$

*hold.*

As an example of an application of Theorem 1.3 we provide a description of periodic algebraic curves for endomorphisms of  $(\mathbb{P}^1(K))^2$  of the form

$$(6) \quad (z_1, z_2) \rightarrow (F_1(z_1), F_2(z_2)),$$

where  $F_1, F_2$  are simple rational functions of degree at least four. Dynamics of such endomorphisms, in the case where considered functions are *polynomials*, were studied by Medvedev and Scanlon in the paper [26]. In particular, they showed that for non-special polynomials  $F_1, F_2$  invariant curves for (6) have genus zero and can be parametrized by polynomials  $X_1, X_2$  satisfying the system of semiconjugacies

$$(7) \quad \begin{array}{ccc} (\mathbb{C}\mathbb{P}^1)^2 & \xrightarrow{(G,G)} & (\mathbb{C}\mathbb{P}^1)^2 \\ (X_1, X_2) \downarrow & & \downarrow (X_1, X_2) \\ (\mathbb{C}\mathbb{P}^1)^2 & \xrightarrow{(F_1, F_2)} & (\mathbb{C}\mathbb{P}^1)^2, \end{array}$$

where  $G$  is a polynomial. Medvedev and Scanlon also described solutions of (7) using the Ritt theory [46]. Notice that the results of [26] concerning invariant curves for (6) have numerous applications in complex and arithmetic dynamics (see e. g. [2], [10], [15], [16], [17], [20], [29]).

A description of invariant curves for endomorphisms (6), where  $F_1, F_2$  are arbitrary non-special rational functions, was obtained in the recent paper [41]. Instead of the Ritt theory, which cannot be generalized to rational functions, the approach of [41] relies on the results of the papers [32], [34], [37] about semiconjugate rational functions. Theorem 1.3 permits to simplify considerably this description for simple  $F_1$  and  $F_2$  leading to the following result, which can be used easily for different applications.

**Theorem 1.4.** *Let  $F_1$  and  $F_2$  be simple rational functions of degree  $m \geq 4$  and  $C$  an irreducible algebraic curve in  $(\mathbb{P}^1(\mathbb{C}))^2$  that is not a vertical or horizontal line. Then  $(F_1, F_2)^{\circ d}(C) = C$  for an integer  $d \geq 1$  if and only if*

$$F_2^{\circ d} = \alpha \circ F_1^{\circ d} \circ \alpha^{-1}$$

for some Möbius transformation  $\alpha$ , and  $C$  is one of the graphs

$$X_2 = (\alpha \circ \nu \circ F_1^{\circ s})(X_1), \quad X_1 = (\nu \circ F_1^{\circ s} \circ \alpha^{-1})(X_2),$$

where  $\nu \in \text{Aut}(F_1^{\circ d})$  and  $s \geq 0$ .

As an application of Theorem 1.4, we show that for *general pairs* of rational functions  $F_1, F_2$  of degree  $m \geq 4$  endomorphisms (6) have no periodic curves distinct from vertical or horizontal lines. In particular, the  $(F_1, F_2)$ -orbit of a point  $(x, y)$  in  $(\mathbb{P}^1(\mathbb{C}))^2$  is Zariski dense in  $(\mathbb{P}^1(\mathbb{C}))^2$ , unless  $x$  is a preperiodic point of  $F_1$ , or  $y$  is a preperiodic point of  $F_2$  (see Section 4.3).

For proving Theorem 1.1, we use the following strategy. First, we show that if  $F$  is a simple rational function of degree  $m \geq 4$  and  $G$  is an indecomposable rational function of degree at least two such that the algebraic curve

$$(8) \quad F(x) - G(y) = 0$$

is irreducible, then the genus of this curve is greater than zero. Then we show that if (8) is reducible, then either  $G = F \circ \mu$ , where  $\mu$  is a Möbius transformation, or  $\deg G$  is equal to the binomial coefficient  $\binom{m}{k}$  for some  $k$ ,  $1 < k < m - 1$ . Finally, using the theorem of Sylvester [50] and Schur [49] about prime divisors of binomial

coefficients, we show that there exists a prime number  $p$  such that  $p \mid \binom{m}{k}$  but  $p \nmid m$ . This implies that if the above function  $G$  is a compositional left factor of some iterate of  $F$ , that is,

$$F^{\circ l} = G \circ H$$

for some rational function  $H$ , then  $G$  necessarily has the form  $G = F \circ \mu$  for some  $\mu \in \text{Aut}(\mathbb{C}\mathbb{P}^1)$ . Since  $F$  is tame, the last statement implies inductively the conclusion of the theorem.

The paper is organized as follows. In the second section, using the above approach we prove Theorem 1.1. We also provides examples of simple rational functions of degree two and three for which Theorem 1.1 is not true. In the third section, we prove Theorem 1.2 and some related results. Finally, in the fourth section, we first prove Theorem 1.3. Then, using this theorem and results of the paper [41] describing invariant curves for (6) in terms of semiconjugacies, we prove Theorem 1.4. We also prove the above mentioned corollary of Theorem 1.4 concerning dynamics of endomorphisms (6) for general pairs of rational functions.

## 2. PROOF OF THEOREM 1.1

**2.1. Calculation of genus of  $F(x) - G(y) = 0$ .** Let  $F$  be a rational function of degree  $m \geq 2$ . We denote by  $\text{Mon}(F) \subseteq S_m$  the monodromy group of  $F$ .

**Theorem 2.1.** *Let  $F$  be a simple rational function of degree  $m \geq 2$ . Then  $F$  is indecomposable and  $\text{Mon}(F) = S_m$ .*

*Proof.* Assume that

$$F = F_1 \circ F_2,$$

where  $F_1$  and  $F_2$  are rational functions of degree  $m_1$  and  $m_2$ . It follows from the chain rule that the maximal number of critical values of the function in the right part of this equality is  $(2m_1 - 2) + (2m_2 - 2)$ . On the other hand, since  $F$  simple, the number of critical values of the function in the left part is  $2m_1m_2 - 2$ . Thus,

$$0 \geq 2m_1m_2 - 2 - (2m_1 - 2) - (2m_2 - 2) = 2(m_1 - 1)(m_2 - 1),$$

implying that at least one of the functions  $F_1$  and  $F_2$  has degree one.

Since  $F$  is indecomposable, the monodromy group  $\text{Mon}(F)$  of  $F$  is imprimitive. Moreover, for any critical value  $c$  of  $F$ , the permutation in  $\text{Mon}(P)$  corresponding to  $c$  is a transposition. Since a primitive permutation group containing a transposition is a full symmetric group (see [51], Theorem 13.3), we conclude that  $\text{Mon}(F) = S_m$ .  $\square$

For rational functions  $F$  and  $H$ , let us define algebraic curves  $h_{F,H}(x, y)$  and  $h_F(x, y)$  by the formulas

$$h_{H,F} : H_1(x)F_2(y) - H_2(x)F_1(y) = 0,$$

and

$$h_F : \frac{F_1(x)F_2(y) - F_2(x)F_1(y)}{x - y} = 0,$$

where  $H_1, H_2$  and  $F_1, F_2$  are pairs of polynomials without common roots such that  $H = H_1/H_2$  and  $F = F_1/F_2$ . In case the above curves are irreducible, their genera can be calculated explicitly in terms of ramification of  $H$  and  $F$  as follows.

Let us denote by  $S = \{z_1, z_2, \dots, z_r\}$  a union of critical values of  $H$  and  $F$ , and set  $n = \deg H$ ,  $m = \deg F$ . Let  $z_0$  be a fixed point from  $\mathbb{CP}^1 \setminus S$  and  $\gamma_i \in \pi_1(\mathbb{CP}^1 \setminus S, z_0)$  small loops around  $z_i$ ,  $1 \leq i \leq r$ . For  $i$ ,  $1 \leq i \leq r$ , we denote by  $\alpha_i \in S_n$  (resp.  $\beta_i \in S_m$ ) a permutation of points of  $H^{-1}\{z_0\}$  (resp. of  $F^{-1}\{z_0\}$ ) induced by the lifting of  $\gamma_i$  by  $H$  (resp.  $F$ ). Finally, for  $i$ ,  $1 \leq i \leq r$ , we denote by

$$(a_{i,1}, a_{i,2}, \dots, a_{i,p_i})$$

the collection of lengths of disjoint cycles in the permutation  $\alpha_i$ , and by

$$(b_{i,1}, b_{i,2}, \dots, b_{i,q_i})$$

the collection of lengths of disjoint cycles in the permutation  $\beta_i$ . In this notation, the following formulas hold (see [12] or [31]):

$$(9) \quad 2 - 2g(h_{H,F}) = \sum_{i=1}^r \sum_{j_2=1}^{q_i} \sum_{j_1=1}^{p_i} \text{GCD}(a_{i,j_1} b_{i,j_2}) - mn(r-2),$$

$$(10) \quad 4 - 2g(h_F) = \sum_{i=1}^r \sum_{j_2=1}^{p_i} \sum_{j_1=1}^{p_i} \text{GCD}(b_{i,j_1} b_{i,j_2}) - (r-2)m^2.$$

**Theorem 2.2.** *Let  $F$  be a simple rational function of degree  $m \geq 4$  and  $H$  a rational function of degree  $n \geq 2$  such that the curve  $h_{H,F}$  is irreducible. Then  $g(h_{H,F}) > 0$ . In particular, the functional equation  $F \circ X = H \circ Y$  has no solutions in rational functions  $X, Y$ .*

*Proof.* If  $z_i$ ,  $1 \leq i \leq r$ , is not a critical value of  $F$ , then obviously

$$\sum_{j_1=1}^{p_i} \text{GCD}(a_{i,j_1} b_{i,j_2}) = p_i, \quad \sum_{j_2=1}^{q_i} \sum_{j_1=1}^{p_i} \text{GCD}(a_{i,j_1} b_{i,j_2}) = mp_i.$$

Assume now that  $z_i$ ,  $1 \leq i \leq r$ , is a critical value of  $F$ . Then, whenever  $b_{i,j_2} = 1$ ,  $1 \leq j_2 \leq q_i$ , we still have

$$\sum_{j_1=1}^{p_i} \text{GCD}(a_{i,j_1} b_{i,j_2}) = p_i.$$

On the other hand, if  $b_{i,j_2} = 2$ ,  $1 \leq j_2 \leq q_i$ , then

$$\sum_{j_1=1}^{p_i} \text{GCD}(a_{i,j_1} b_{i,j_2}) = p_i + l_i,$$

where  $l_i$  is the number of even numbers among the numbers  $a_{i,j_1}$ ,  $1 \leq j_1 \leq p_i$ . Therefore, since among the numbers  $b_{i,j_2}$ ,  $1 \leq j_2 \leq q_i$ , exactly one is equal to two and all the other are equal to one, we have:

$$\sum_{j_2=1}^{q_i} \sum_{j_1=1}^{p_i} \text{GCD}(a_{i,j_1} b_{i,j_2}) = (m-2)p_i + p_i + l_i = mp_i + (l_i - p_i).$$

Since the Riemann-Hurwitz formula implies that

$$\sum_{i=1}^r p_i = (r-2)n + 2,$$

this yields that

$$\sum_{i=1}^r \sum_{j_2=1}^{q_i} \sum_{j_1=1}^{p_i} \text{GCD}(a_{i,j_1} b_{i,j_2}) = \sum_{i=1}^r m p_i + \sum' (l_i - p_i) = m((r-2)n+2) + \sum' (l_i - p_i),$$

where the sum  $\sum'$  runs only over indices corresponding to critical values of  $F$ .

Substituting the last expression in (9), we conclude that  $g(h_{H,F}) = 0$  if and only if

$$2m - 2 + \sum' (l_i - p_i) = 0.$$

Stating differently,  $g(h_{H,F}) = 0$  if and only if the preimage  $H^{-1}\{c_1, c_2, \dots, c_{2m-2}\}$ , where  $c_1, c_2, \dots, c_{2m-2}$  are critical values of  $F$ , contains exactly  $2m-2$  points where the local multiplicity of  $H$  is odd. By the Riemann-Hurwitz formula,

$$H^{-1}\{c_1, c_2, \dots, c_{2m-2}\} \geq (2m-4)n + 2,$$

and the equality is attained if and only if any critical value of  $H$  is a critical value of  $F$ . On the other hand, the condition that  $H^{-1}\{c_1, c_2, \dots, c_{2m-2}\}$  contains  $2m-2$  points where the local multiplicity of  $H$  is odd implies that

$$H^{-1}\{c_1, c_2, \dots, c_{2m-2}\} \leq (2m-2) + \frac{n(2m-2) - (2m-2)}{2} = (n+1)(m-1),$$

and the equality is attained if and only if all  $2m-2$  points in  $H^{-1}\{c_1, c_2, \dots, c_{2m-2}\}$  with odd multiplicity have multiplicity one, while all points with even multiplicity have multiplicity two. Thus, if  $g(h_{H,F}) = 0$ , then

$$(2m-4)n + 2 \leq (n+1)(m-1),$$

implying that

$$(n-1)(m-3) \leq 0.$$

Since the last inequality is satisfied only for  $n = 1$  or for  $m = 2, 3$ , we conclude that  $g(h_{H,F}) > 0$ .  $\square$

**Theorem 2.3.** *Let  $F$  be a simple rational function of degree  $m \geq 3$ . Then the curve  $h_F$  is irreducible and  $g(h_F) > 0$ . In particular, the equality  $F \circ X = F \circ Y$ , where  $X$  and  $Y$  are rational functions, implies that  $X = Y$ .*

*Proof.* It is well-known (see e.g. [31], Corollary 2.3) that the curve  $h_F(x, y)$  is irreducible if and only if the monodromy group  $\text{Mon}(F)$  is doubly transitive. Therefore, since a symmetric group is doubly transitive, the irreducibility of  $h_F(x, y)$  follows from Theorem 2.1.

Further, applying formula (10) one can show (see [31], p. 210) that for a simple rational function  $F$  of degree  $m \geq 2$  the equality

$$g(h_F) = (m-2)^2$$

holds. Therefore,  $g(h_F) > 0$  whenever  $m \geq 3$ .  $\square$

**2.2. Conditions for reducibility of  $F(x) - G(y) = 0$ .** Let  $F$  be a rational function of degree  $m \geq 2$ , and  $U \subset \mathbb{CP}^1$  a simply connected domain containing no critical values of  $F$ . Then in  $U$  there exist  $m = \deg F$  different branches of the algebraic function  $F^{-1}(z)$ . We will denote these branches by small letters  $f_1, f_2, \dots, f_m$ .

**Lemma 2.4.** *Let  $F$  be a rational function of degree  $m \geq 2$  such that  $\text{Mon}(F) = S_m$ . Then the equality*

$$(11) \quad c_1(z)f_1(z) + c_2(z)f_2(z) + \cdots + c_m(z)f_m(z) = c_0(z),$$

where  $c_i(z) \in \mathbb{C}(z)$ ,  $0 \leq i \leq m$ , implies that

$$c_1(z) = c_2(z) = \cdots = c_m(z).$$

*Proof.* Assume, say, that  $c_1(z) \neq c_2(z)$ . Since  $\text{Mon}(F) = S_m$ , the transposition  $\sigma = (1, 2)$  is contained in  $\text{Mon}(F)$ , and acting by  $\sigma$  on equality (11) we obtain the equality

$$(12) \quad c_1(z)f_2(z) + c_2(z)f_1(z) + \cdots + c_m(z)f_m(z) = c_0(z).$$

It follows now from (11) and (12) that  $f_1(z) = f_2(z)$  in contradiction with the assumption that  $f_1, f_2, \dots, f_m$  are different branches of  $F^{-1}(z)$ .  $\square$

Let  $P(z)$  and  $Q(z)$  be non-constant rational functions, and  $p(z)$  a single-valued branch of  $P^{-1}(z)$  defined in some disk  $D \subset \mathbb{C}\mathbb{P}^1$ . We denote by  $Q(p(z))$  the algebraic function obtained by the analytic continuation of the germ  $\{D, Q(p(z))\}$ , and by  $d(Q(p(z)))$  the degree of  $Q(p(z))$ , that is, the number of its branches.

**Lemma 2.5.** *Let  $P(z)$  and  $Q(z)$  be non-constant rational functions. Then*

$$d(Q(p(z))) = \deg P(z) / [\mathbb{C}(z) : \mathbb{C}(P, Q)].$$

*In particular, the degree  $d(Q(p(z)))$  does not depend on the choice of the branch of  $P^{-1}(z)$ . Moreover, if  $P$  is indecomposable, then either  $d(Q(p(z))) = 1$ , or  $d(Q(p(z))) = \deg P$ .*

*Proof.* Since any algebraic relation between  $Q(p(z))$  and  $z$  over  $\mathbb{C}$  transforms to an algebraic relation between  $Q(z)$  and  $P(z)$  and vice versa we have:

$$(13) \quad d(Q(p(z))) = [\mathbb{C}(P, Q) : \mathbb{C}(P)] = [\mathbb{C}(z) : \mathbb{C}(P)] / [\mathbb{C}(z) : \mathbb{C}(P, Q)] = \deg P(z) / [\mathbb{C}(z) : \mathbb{C}(P, Q)].$$

Furthermore, by the Lüroth theorem,  $\mathbb{C}(P, Q) = \mathbb{C}(R)$  for some  $R \in \mathbb{C}(z)$ , implying that

$$P = \tilde{P} \circ R, \quad Q = \tilde{Q} \circ R$$

for some  $\tilde{P}, \tilde{Q} \in \mathbb{C}(z)$ . Therefore, if the function  $P$  is indecomposable, then either  $\mathbb{C}(P, Q) = \mathbb{C}(P)$ , or  $\mathbb{C}(P, Q) = \mathbb{C}(z)$ , implying by (13) that either  $d(Q(p(z))) = 1$ , or  $d(Q(p(z))) = \deg P$ .  $\square$

**Theorem 2.6.** *Let  $H$  and  $F$  be rational functions of degree  $n \geq 2$  and  $m \geq 2$  such that  $H$  is indecomposable,  $\text{Mon}(F) = S_m$ , and the curve  $h_{H,F}$  is reducible. Then either  $H = F \circ \mu$ , where  $\mu$  is a Möbius transformation, or  $n = \binom{m}{k}$  for some  $k$ ,  $1 < k < m - 1$ .*

*Proof.* Let  $H_1, H_2$  and  $F_1, F_2$  be pairs of polynomials without common roots such that  $H = H_1/H_2$  and  $F = F_1/F_2$ . Suppose that

$$(14) \quad H_1(x)F_2(y) - H_2(x)F_1(y) = M(x, y)N(x, y)$$

for some non-constant polynomials  $M(x, y), N(x, y)$ . Notice that since  $H$  and  $F$  are non-constants the numbers  $\deg_x M, \deg_y M, \deg_x N, \deg_y N$  are necessarily distinct from zero.



Let  $h_1, h_2, \dots, h_n$  be branches of  $H^{-1}(z)$  and  $f_1, f_2, \dots, f_m$  branches of  $F^{-1}(z)$  defined in a simply connected domain  $U \subset \mathbb{C}\mathbb{P}^1$  containing no critical values of  $F$  or  $H$ . Since  $\deg_y M > 0$ , it follows from (14) that for at least one  $i$ ,  $1 \leq i \leq m$ , the equality

$$(15) \quad M(h_1, f_i) = 0$$

holds. On the other hand, equality (15) cannot be satisfied for all  $i$ ,  $1 \leq i \leq m$ , since otherwise the equality  $\deg_y N = 0$  holds. Let  $i_1, i_2, \dots, i_k$  be indices for which (15) holds. Then (15) implies that

$$(16) \quad f_{i_1} + f_{i_2} + \dots + f_{i_k} = Q(h_1)$$

for some  $Q \in \mathbb{C}(z)$ . Furthermore, since the set  $\{i_1, i_2, \dots, i_k\}$  is a proper subset of  $\{1, 2, \dots, m\}$ , the function  $Q(h_1)$  is not a rational function by Lemma 2.4. Thus,  $d(Q(h_1(z))) \neq 1$ , implying by Lemma 2.5 that the functions  $Q(h_i(z))$ ,  $1 \leq i \leq n$ , are pairwise different.

Continuing equality (16) analytically along an arbitrary closed curve  $\gamma$  in  $\mathbb{C}\mathbb{P}^1$ , we obtain an equality where on the left side is a sum of branches of  $F^{-1}(z)$  over a subset of  $\{1, 2, \dots, m\}$  containing  $k$  elements, while on the right side is a branch  $Q(h_i)$ ,  $1 \leq i \leq n$ , of the function  $Q(h_1(z))$ . Furthermore, to different subsets of  $\{1, 2, \dots, m\}$  correspond different branches of  $Q(h_1(z))$ , for otherwise we obtain a contradiction with Lemma 2.4. Since the equality  $\text{Mon}(F) = S_m$  implies that for an appropriately chosen  $\gamma$  we can obtain on the left side a sum of branches of  $F^{-1}(z)$  over any  $k$ -element subset of  $\{1, 2, \dots, m\}$ , while the transitivity of  $\text{Mon}(H)$  implies that for an appropriately chosen  $\gamma$  we can obtain on the right side any branch of  $Q(h_1(z))$ , we conclude that the degree of  $Q(h_1(z))$  is equal to the number of  $k$ -element subsets of  $\{1, 2, \dots, m\}$ , for some  $k$ ,  $1 \leq k \leq n$ , that is,  $n = \binom{m}{k}$ .

Let us observe now that if  $k = 1$ , then  $n = m$  and equality (16) implies the equality  $z = (F \circ Q)(h_1)$ . Thus, the function  $F \circ Q$  is inverse to  $h_1$ , that is,  $H = F \circ Q$ . Moreover, since  $n = m$ , the function  $Q$  is a Möbius transformation. Finally, the same conclusion is true if  $k = m - 1$  since we can switch between  $M(x, y)$  and  $N(x, y)$ .  $\square$

**Remark 2.7.** Notice that reducible curves  $h_{H,F}$  satisfying  $n = \binom{m}{2}$  do indeed exist for every  $m > 3$  (see [13], Example 2.4). For further information regarding the reducibility problem for curves  $h_{H,F}$  we refer the reader to [14] and the references therein.

**2.3. Prime divisors of  $\binom{m}{k}$ .** The classical theorem of Sylvester [50] and Schur [49] states that in the set of integers  $a, a + 1, \dots, a + b - 1$ , where  $a > b$ , there is a number divisible by a prime greater than  $b$ . For a natural number  $x$ , let us denote by  $\mathcal{P}(x)$  the greatest prime factor of  $x$ . Then the theorem of Sylvester and Schur may be reformulated as follows ([6]): for any  $m \geq 2k$  the inequality  $\mathcal{P}\left(\binom{m}{k}\right) > k$  holds. Furthermore, the last inequality may be sharpened to the inequality

$$(17) \quad \mathcal{P}\left(\binom{m}{k}\right) \geq \frac{7}{5}k$$

(see [9], [21]). We will prove that this implies the following corollary.

**Theorem 2.8.** *Let  $m \geq 4$  be a natural number, and  $k$  a natural number such that  $1 < k < m - 1$ . Then there exists a prime number  $p$  such that  $p \mid \binom{m}{k}$  but  $p \nmid m$ .*

*Proof.* Since  $\binom{m}{k} = \binom{m}{m-k}$ , it is enough to prove the theorem under the assumption that  $m \geq 2k$ . Applying the Sylvester-Schur theorem, we conclude that there is a number  $s$ ,  $m - k + 1 \leq s \leq m$ , such that  $\mathcal{P}\left(\binom{m}{k}\right) = \mathcal{P}(s) = p > k$ . Moreover, if  $s$  is *strictly* less than  $m$ , then  $p$  cannot be a divisor of  $m$  for otherwise

$$(18) \quad p | (m - s)$$

in contradiction with

$$(19) \quad p > k.$$

Since however  $s$  can be equal to  $m$ , we will apply the Sylvester-Schur theorem in its strong form (17) to the binomial coefficient  $\binom{m-1}{k-1}$  related with  $\binom{m}{k}$  by the equality

$$(20) \quad \binom{m}{k} = \frac{m(m-1)\dots(m-k+1)}{k(k-1)\dots 1} = \frac{m}{k} \binom{m-1}{k-1}.$$

Notice that (20) implies that every prime factor  $p$  of  $\binom{m-1}{k-1}$  satisfying (19) remains a prime factor of  $\binom{m}{k}$ .

Since  $m \geq 2k$  implies  $m - 1 \geq 2(k - 1)$ , applying (17) to  $\binom{m-1}{k-1}$  we conclude that there is a number  $s$ ,  $m - k + 1 \leq s \leq m - 1$ , such that

$$\mathcal{P}\left(\binom{m-1}{k-1}\right) = \mathcal{P}(s) = p \geq \frac{7}{5}(k-1).$$

Furthermore, if  $k > 3$  then

$$p \geq \frac{7}{5}(k-1) > k,$$

implying that  $p | \binom{m}{k}$ . On the other hand,  $p \nmid m$  since otherwise (18) holds in contradiction with (19).

For  $k \leq 3$  the theorem can be proved by an elementary argument. If  $k = 2$  then

$$\binom{m}{k} = \frac{m(m-1)}{2}.$$

Therefore, since  $\text{GCD}(m, m-1) = 1$ , the statement of the theorem is true, whenever  $(m-1) \nmid 2$ , and the last condition is always satisfied if  $m > 3$ . Similarly, if  $k = 3$ , then

$$\binom{m}{k} = \frac{m(m-1)(m-2)}{2 \cdot 3},$$

and the statement of the theorem is true whenever  $(m-1) \nmid 6$ . The last condition fails to be true for  $m > 3$  only if  $m$  is equal to 4 or 7. However, the pair  $m = 4$ ,  $k = 3$  does not satisfy the condition  $1 < k < m - 1$ . On the other hand, for the pair  $m = 7$ ,  $k = 3$  we have  $\binom{m}{k} = \binom{7}{3} = 5 \cdot 7$ , and the statement of the theorem is satisfied for  $p = 5$ .  $\square$

*Proof of Theorem 1.1.* Let

$$(21) \quad F^{\circ l} = F_r \circ F_{r-1} \circ \dots \circ F_1$$

be a decomposition of  $F^{\circ l}$ ,  $l \geq 1$ , into a composition of indecomposable rational functions. Since  $F$  is indecomposable by Theorem 2.1, for  $l = 1$  the theorem is true. On the other hand, since by Theorem 2.3 the equality  $F \circ X = F \circ Y$  implies that  $X = Y$ , to prove the inductive step it is enough to show that equality (21) implies that

$$(22) \quad F_r = F \circ \mu$$

for some Möbius transformation  $\mu$ .

Clearly, equality (21) implies that the algebraic curve

$$(23) \quad F(x) - F_r(y) = 0$$

has a factor of genus zero. Moreover, (23) is necessarily reducible by Theorem 2.2. Since  $\text{Mon}(F) = S_m$  by Theorem 2.1, it follows now from Theorem 2.6 that either (22) holds, or  $\deg F_r = \binom{m}{k}$  for some  $k$ ,  $1 < k < m - 1$ . However, the last case is impossible since (21) implies that any prime divisor of  $\deg F_r$  is a prime divisor of  $\deg F$  in contradiction with Theorem 2.8.  $\square$

**Corollary 2.9.** *Let  $F$  be a simple rational function of degree  $m \geq 4$ , and  $G_i$ ,  $1 \leq i \leq r$ , rational functions of degree at least two such that*

$$F^{\circ l} = G_r \circ G_{r-1} \circ \dots \circ G_1$$

for some  $l \geq 1$ . Then there exist Möbius transformations  $\nu_i$ ,  $1 \leq i < r$ , and integers  $s_i \geq 1$ ,  $1 \leq i \leq r$ , such that

$$G_r = F^{\circ s_r} \circ \nu_{r-1}, \quad G_i = \nu_i^{-1} \circ F^{\circ s_i} \circ \nu_{i-1}, \quad 1 < i < r, \quad \text{and} \quad G_1 = \nu_1^{-1} \circ F^{\circ s_1}.$$

*Proof.* To prove the corollary, it is enough to decompose each  $G_i$ ,  $1 \leq i \leq r$ , into a composition of indecomposable rational functions and to apply Theorem 1.1.  $\square$

**Remark 2.10.** Theorem 1.1 is not true if the degree  $m$  of  $F$  is equal to 2 or 3. Indeed, for example, for the rational functions

$$P = \frac{z^2 - 1}{z^2 + 1}, \quad Q = -\frac{1}{2z^2 - 1}, \quad R = \frac{1}{2} \left( z + \frac{1}{z} \right)$$

the equality

$$P \circ P = Q \circ R = -\frac{2z^2}{z^4 + 1}$$

holds. On the other hand, the equality

$$(24) \quad R = \mu \circ P$$

for a Möbius transformation  $\mu$  is impossible, since critical points of  $P$  and hence of  $\mu \circ P$  are 0 and  $\infty$ , while critical points of  $R$  are  $-1$  and  $1$ .

Further, setting

$$P = \frac{6x}{x^3 - 2}$$

and

$$Q = -\frac{23328x}{x^3 + 216 \sqrt[3]{2}x^2 + 3888 \cdot 2^{2/3}x - 93312}, \quad R = \frac{36x(2^{2/3}x^2 - 4x + 2 \sqrt[3]{2})}{2^{2/3}x^2 + 2x + 2 \sqrt[3]{2}},$$

one can check that  $P$  has critical points

$$-1, \quad \frac{1}{2} - \frac{i\sqrt{3}}{2}, \quad \frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \infty$$

with different critical values

$$2, \quad -1 + i\sqrt{3}, \quad -1 - i\sqrt{3}, \quad 0,$$

and that

$$P \circ P = Q \circ R = -\frac{18(x^3 - 2)^2 x}{x^9 - 6x^6 - 96x^3 - 8}.$$

On the other hand, since  $\infty$  is not a critical point of  $R$ , equality (24) cannot be satisfied for a Möbius transformation  $\mu$ .

## 3. PROOF OF THEOREM 1.2

**3.1. Groups and semigroups related to simple rational functions.** We start from recalling some basic facts concerning the groups and semigroups defined in the introduction.

It is obvious that  $C(F)$  is a semigroup, and it follows from the inclusions

$$C(F^{\circ k}), C(F^{\circ l}) \subseteq C(F^{\circ \text{LCM}(k,l)})$$

that  $C_\infty(F)$  is also a semigroup. Further, according to the Ritt theorem (see [47], and also [7], [39]), commuting rational functions of degree at least two are either special or have a common iterate. Thus, any element  $G$  of the semigroup  $C(F) \setminus \text{Aut}(F)$  for non-special  $F$  satisfies condition (1) for some  $k, l \geq 1$ . Since however a description of solutions of (1) is known only in the polynomial case (see [45], [47]), this condition provides very little information about the semigroup  $C(F)$  in the general case. A method for describing  $C(F)$  for an arbitrary non-special rational function  $F$  was given in the recent paper [39]. A satisfactory description of  $C_\infty(F)$  is still not known (see [44] for some particular results). Notice that since equality (1) implies that  $G$  commutes with  $F^{\circ k}$ , the Ritt theorem yields that the semigroup  $C_\infty(F) \setminus \text{Aut}_\infty(F)$  coincides with the set of rational functions sharing an iterate with  $F$ .

Let us recall that by the results of Freire, Lopes, Mañé ([11]) and Lyubich ([25]), for every rational function  $F$  of degree  $n \geq 2$  there exists a unique probability measure  $\mu_F$  on  $\mathbb{C}\mathbb{P}^1$ , which is invariant under  $F$ , has support equal to the Julia set  $J_F$ , and achieves maximal entropy  $\log n$  among all  $F$ -invariant probability measures. The measure  $\mu_F$  can be described as follows. For  $a \in \mathbb{C}\mathbb{P}^1$  let  $z_i^k(a)$ ,  $i = 1, \dots, n^k$ , be the roots of the equation  $F^{\circ k}(z) = a$  counted with multiplicity, and  $\mu_{F,k}(a)$  be the measure defined by

$$\mu_{F,k}(a) = \frac{1}{n^k} \sum_{i=1}^{n^k} \delta_{z_i^k(a)}.$$

Then for every  $a \in \mathbb{C}\mathbb{P}^1$  with two possible exceptions, the sequence  $\mu_{F,k}(a)$ ,  $k \geq 1$ , converges in the weak topology to  $\mu_F$ . In particular, this yields that

$$(25) \quad \text{Aut}_\infty(F) \subseteq E_0(F),$$

since the equality

$$F^{\circ n} = \alpha^{-1} \circ F^{\circ n} \circ \alpha$$

for some  $\alpha \in \text{Aut}(\mathbb{C}\mathbb{P}^1)$  and  $n \geq 1$  implies that for every set  $S \subset \mathbb{C}\mathbb{P}^1$  and  $k \geq 1$  the equality

$$|S \cap F^{-nk}(a)| = |\alpha(S) \cap F^{-nk}(\alpha(a))|$$

holds. Moreover, the above description of  $\mu_F$  implies that any  $G$  sharing an iterate with  $F$  belongs to  $E(F)$ . Thus,

$$(26) \quad C_\infty(F) \subseteq E(F).$$

The fact that  $E(F)$  is a semigroup can be established using the balancedness property of  $\mu_F$  or the Lyubich operator (see [30], [5]). Further, it follows from the results of Levin and Przytycki (see [23], [24], and also [52]) that if  $F$  is non-special, then  $G \in E(F) \setminus E_0(F)$  if and only if  $G$  satisfies equation (2), which generalizes equation (1). A complete description of  $E(F)$  is known only if  $F$  is a polynomial, in which case  $E(F) \setminus E_0(F)$  coincides with the set of polynomials sharing a Julia

set with  $F$  (see [1], [48] and also [40], [44]). Some partial results in the rational case can be found in [52], [38].

The group  $G_0(F)$  defined in the introduction is a subgroup of the group  $G(F)$  consisting of Möbius transformations  $\sigma$  such that

$$(27) \quad F \circ \sigma = \mu \circ F$$

for some Möbius transformations  $\nu$ . It is easy to see that  $G(F)$  is indeed a group and that the map

$$\gamma : \sigma \rightarrow \mu_\sigma$$

is a homomorphism from  $G(F)$  to the group  $\text{Aut}(\mathbb{CP}^1)$ . The group  $G(F)$  is finite and its order is bounded in terms of  $m = \deg F$ , unless

$$(28) \quad \alpha \circ F \circ \beta = z^m$$

for some  $\alpha, \beta \in \text{Aut}(\mathbb{CP}^1)$  (see [36], Section 4 or [43], Section 2). Thus, the group  $G_0(F)$  is also finite, unless (28) holds.

**Lemma 3.1.** *Let  $F$  be a simple rational function of degree  $m \geq 3$ . Then the group  $G_0(F)$  is finite and the restriction of  $\gamma$  to  $G_0(F)$  is an automorphism of  $G_0(F)$ .*

*Proof.* Since equality (28) is impossible for simple  $F$  of degree  $m \geq 3$ , the group  $G_0(F)$  is finite. Therefore, since Theorem 2.3 implies that the kernel of  $\gamma$  is trivial, it follows from  $\gamma(G_0(F)) \subseteq G_0(F)$  that the restriction of  $\gamma$  to  $G_0(F)$  is an automorphism  $G_0(F)$ .  $\square$

**Corollary 3.2.** *Let  $F$  be a simple rational function of degree  $m \geq 3$ . Then  $G_0(F) \subseteq \text{Aut}(F^{\circ s})$ , where  $s = |\text{Aut}(G_0(F))|$ .*

*Proof.* For  $s = |\text{Aut}(G_0(F))|$  the iterate  $\gamma^{\circ s}$  is the identical automorphism of  $G_0(F)$ . Therefore, since

$$F^{\circ s} \circ \sigma = \gamma^{\circ s}(\sigma) \circ F^{\circ s}, \quad \sigma \in G_0(F),$$

every element of  $G_0(F)$  commutes with  $F^{\circ s}$ .  $\square$

**Lemma 3.3.** *Let  $F$  be a rational function, and  $\nu$  a Möbius transformation such that*

$$(29) \quad (\nu \circ F)^{\circ l} = F^{\circ l}$$

for some  $l \geq 1$ . Then  $\nu \in \text{Aut}(F^{\circ l})$ .

*Proof.* Clearly, equality (29) implies the equality

$$(30) \quad (\nu \circ F)^{\circ(l-1)} \circ \nu = F^{\circ(l-1)}.$$

Composing now  $F$  with the both parts of equality (30), we obtain the equality

$$(31) \quad (F \circ \nu)^{\circ l} = F^{\circ l}.$$

It follows now from (29) and (31) that

$$F^{\circ l} \circ \nu = (\nu \circ F)^{\circ l} \circ \nu = \nu \circ (F \circ \nu)^{\circ l} = \nu \circ F^{\circ l}. \quad \square$$

**Lemma 3.4.** *Let  $F$  be a simple rational function of degree  $m \geq 3$ . Then  $F$  is not a special function.*

*Proof.* The proof follows easily from the analysis of ramifications of special functions. Since below we prove a more general result (Lemma 4.3), we omit the details.  $\square$

**Theorem 3.5.** *Let  $F$  be a simple rational function of degree  $m \geq 4$ . Then*

$$E_0(F) = G_0(F) = \text{Aut}_\infty(F) = \text{Aut}(A^{os}),$$

where  $s = |\text{Aut}(G_0(F))|$ .

*Proof.* By Corollary 3.2 and (25), we have:

$$G_0(F) \subseteq \text{Aut}(A^{os}) \subseteq \text{Aut}_\infty(F) \subseteq E_0(F).$$

Thus, to prove the theorem we only must show that  $E_0(F) \subseteq G_0(F)$ . In turn, for this purpose it is enough to establish that for every  $\nu \in E_0(F)$  there exists  $\mu \in E_0(F)$  such that (27) holds. Let  $\nu$  be an arbitrary element of  $E_0(F)$ . Then  $F \circ \nu \in E(F)$ , implying by Lemma 3.4 and the theorem of Levin and Przytycki that

$$F^{\circ k_1} = F^{\circ k_2} \circ (F \circ \nu)^{\circ l}$$

for some  $k_1, l \geq 1, k_2 \geq 0$ . Applying to the last equality recursively Theorem 2.3, we conclude that

$$F^{\circ(k_1-k_2)} = (F \circ \nu)^{\circ l}.$$

Therefore, by Theorem 1.1, there exist Möbius transformations  $\nu_i, 1 \leq i \leq l-1$ , such that

$$F \circ \nu = F \circ \nu_{r-1}, \quad F \circ \nu = \nu_i^{-1} \circ F \circ \nu_{i-1}, \quad 1 < i < l, \quad \text{and} \quad F \circ \nu = \nu_1^{-1} \circ F.$$

Thus, equality (27) holds for  $\mu = \nu_1^{-1}$ . Finally, since

$$F^{\circ l} = (F \circ \nu)^{\circ l} = (\nu_1^{-1} \circ F)^{\circ l},$$

it follows from Lemma 3.3 that  $\nu_1^{-1} \in \text{Aut}_\infty(F) \subseteq E_0(F)$ .  $\square$

Notice that Lemma 3.1 and Theorem 3.5 imply that every element of the semi-group  $\langle \text{Aut}_\infty(F), F \rangle$  can be represented in a unique way in the form  $\alpha \circ F^{os}$ , where  $\alpha \in \text{Aut}_\infty(F)$  and  $s \geq 1$ , or, in the form  $F^{os} \circ \alpha'$ , where  $\alpha' \in \text{Aut}_\infty(F)$  and  $s \geq 1$ .

*Proof of Theorem 1.2.* In view of Theorem 3.5, we only must show that

$$C_\infty(F) = E(F) = \langle \text{Aut}_\infty(F), F \rangle.$$

Moreover, in view of (26), the first equality follows from the theorem of Levin and Przytycki and Theorem 2.3, since the latter implies that any  $G$  satisfying (2) also satisfies (1). Since the semigroup  $\langle \text{Aut}_\infty(F), F \rangle$  is obviously a subsemigroup of  $C_\infty(F)$ , to finish the proof we only must show that if a rational function  $G$  satisfies (1), then it belongs to  $\langle \text{Aut}_\infty(F), F \rangle$ .

Applying Corollary 2.9 to equality (1), we see that there exist Möbius transformations  $\nu_i, 1 \leq i \leq l-1$ , such that

$$G = F^{os} \circ \nu_{l-1}, \quad G = \nu_i^{-1} \circ F^{os} \circ \nu_{i-1}, \quad 1 < i < l, \quad \text{and} \quad G = \nu_1^{-1} \circ F^{os},$$

where  $s = k/l$ . Moreover, since

$$F^{\circ sl} = G^{\circ l} = (\nu_1^{-1} \circ F^{os})^{\circ l},$$

Lemma 3.3 implies that

$$\nu_1^{-1} \in \text{Aut}(F^{\circ sl}) \subseteq \text{Aut}_\infty(F).$$

Thus,

$$G = \nu_1^{-1} \circ F^{os} \in \langle \text{Aut}_\infty(F), F \rangle. \quad \square$$

**3.2. Groups and semigroups related to general rational functions.** Let us recall that writing a rational function  $F = F(z)$  of degree  $m$  as  $F = P/Q$ , where  $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0$ ,  $Q(z) = b_m z^m + b_{m-1} z^{m-1} + \dots + b_1 z + b_0$  are polynomials of degree  $m$  without common roots, we can identify the space of rational functions of degree  $m$  with the algebraic variety

$$\text{Rat}_m = \mathbb{C}\mathbb{P}^{2m+1} \setminus \text{Res}_{m,m,z}(P, Q),$$

where  $\text{Res}_{m,m,z}(P, Q)$  denotes the resultant of  $P$  and  $Q$ . We will say that some property of rational functions is satisfied for *general* rational functions of degree  $m$  if there exists a proper Zariski closed subset  $Z$  of  $\mathbb{C}\mathbb{P}^{2m+1}$  such that this property is satisfied for all  $F \in \text{Rat}_m \setminus Z$ .

**Lemma 3.6.** *A general rational function  $F$  of degree  $m \geq 2$  is simple.*

*Proof.* First, let us observe that the Wronskian

$$W(z) = P'(z)Q(z) - P(z)Q'(z)$$

of  $F \in \text{Rat}_m$  has degree  $2m - 2$ , unless  $F$  belongs to the projective hypersurface

$$U : a_m b_{m-1} - b_m a_{m-1}$$

in  $\mathbb{C}\mathbb{P}^{2m+1}$ . Defining now a polynomial  $R(t)$  by the formula

$$R(t) = \text{Res}_{2m-2,m,z}(W(z), P(z) - Q(z)t)$$

and using a well-known property of the resultant, we see that for  $F \in \text{Rat}_m \setminus U$  the equality

$$(32) \quad R(t) = c \prod_{\zeta, W(\zeta)=0} (P(\zeta) - Q(\zeta)t)$$

holds for some  $c \in \mathbb{C}^*$ . Since the set of zeroes of  $W(z)$  coincides with the set of finite critical points of  $F$ , it follows from (32) that the set of finite critical values of  $F$  coincides with the set of zeroes of  $R(t)$ , and the inequality  $\deg R(t) < 2m - 2$  holds if and only if infinity is a critical value of  $F$ . Therefore, if  $Z$  is the projective hypersurface  $Z$  in  $\mathbb{C}\mathbb{P}^{2m+1}$  defined by the equality

$$Z = \text{Discr}_{2m-2,t} R(t) = \text{Res}_{2m-2,2m-3,t}(R(t), R'(t)),$$

then  $F \in \text{Rat}_m \setminus U$  belongs to  $Z$  if and only if either some of finite critical values of  $F$  coincide, or infinity is a critical value of  $F$ . Thus, every  $F \in \text{Rat}_m \setminus Z \cup U$  is simple.  $\square$

**Lemma 3.7.** *For a general rational function  $F$  of degree  $m \geq 3$  the group  $G(F)$  is trivial.*

*Proof.* Let us recall that the group  $G(F)$  is non-trivial if and only if there exist  $\alpha, \beta \in \text{Aut}(\mathbb{C}\mathbb{P}^1)$  such that either equality (28) holds (in which case the group  $G(P)$  is infinite), or

$$(33) \quad \alpha \circ F \circ \beta = z^r R(z^d),$$

for some  $R \in \mathbb{C}(z) \setminus \mathbb{C}$  and integers  $r$  and  $d$  satisfying  $2 \leq d \leq m$ ,  $0 \leq r \leq d - 1$  (see e.g. [36], Section 4 or [43], Section 2).

Equality (33) is equivalent to the vanishing of certain groups of coefficients in its left part, implying easily that (33) holds if and only if the coefficients of  $\alpha$ ,  $\beta$ , and  $F$  belong to some projective algebraic variety

$$W \subseteq \mathbb{C}\mathbb{P}^3 \times \mathbb{C}\mathbb{P}^{2m+1} \times \mathbb{C}\mathbb{P}^3.$$

The same is true for equality (28). Considering now a union of such varieties for all possible  $r$  and  $d$  and taking into account that the projection

$$p_k : (\mathbb{C}\mathbb{P}^1)^l \times (\mathbb{C}\mathbb{P}^1)^k \rightarrow (\mathbb{C}\mathbb{P}^1)^k, \quad k, l \geq 1,$$

is a closed map (see e.g. [28]), we conclude that the set of  $F \in \text{Rat}_m$  with  $G(F) \neq id$  is contained in some projective algebraic variety  $Z \subseteq \mathbb{C}\mathbb{P}^{2m+1}$ .

To finish the proof, we only must show that  $Z \neq \mathbb{C}\mathbb{P}^{2m+1}$ . For this purpose, it is enough to show that for every  $m \geq 3$  there exists a polynomial  $F$  of degree  $m$  such that  $G(F) = id$ . Let us recall that for any polynomial  $F$  of degree  $m \geq 2$  such that condition (28) does not hold the group  $G(F)$  is a finite cyclic group generated by a polynomial (see e.g. [43], Section 2). On the other hand, it is easy to see that if  $F$  has the form

$$(34) \quad F = z^m + a_{m-2}z^{m-2} + a_{m-3}z^{m-3} + \dots + a_0,$$

then (27) holds for polynomials  $\sigma = az + b$ ,  $\mu = cz + d$  only if  $b = 0$  and  $a$  is a root of unity. Therefore, for any polynomial of the form (34) with  $a_{m-2} \neq 0$ ,  $a_{m-3} \neq 0$  the group  $G(F)$  is trivial.  $\square$

Notice that for every rational function  $F$  of degree two there exists  $\mu \in \text{Aut}(\mathbb{C}\mathbb{P}^1)$  such that  $F \circ \mu = F$  (see [24]). Thus, Lemma 3.7 is not true for  $m = 2$ .

Theorem 1.2 combined with Lemma 3.6 and Lemma 3.7 implies the following result.

**Theorem 3.8.** *For a general rational function  $F$  of degree  $m \geq 4$ , the equalities*

$$E_0(F) = \text{Aut}_\infty(F) = G_0(F) = id \quad \text{and} \quad E(F) = C_\infty(F) = \langle F \rangle$$

*hold.*  $\square$

#### 4. PROOF OF THEOREM 1.3 AND THEOREM 1.4

**4.1. Generalized Lattès maps, semiconjugate rational functions, and invariant curves.** In this section, we recall basic definitions and results related to descriptions of semiconjugate rational functions and invariant curves for endomorphisms of  $(\mathbb{C}\mathbb{P}^1)^2$ .

An *orbifold*  $\mathcal{O}$  on  $\mathbb{C}\mathbb{P}^1$  is a ramification function  $\nu : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{N}$  which takes the value  $\nu(z) = 1$  except at a finite set of points. For an orbifold  $\mathcal{O}$ , the *Euler characteristic* of  $\mathcal{O}$  is the number

$$\chi(\mathcal{O}) = 2 + \sum_{z \in R} \left( \frac{1}{\nu(z)} - 1 \right),$$

the set of *singular points* of  $\mathcal{O}$  is the set

$$c(\mathcal{O}) = \{z_1, z_2, \dots, z_s, \dots\} = \{z \in \mathbb{C}\mathbb{P}^1 \mid \nu(z) > 1\},$$

and the *signature* of  $\mathcal{O}$  is the set

$$\nu(\mathcal{O}) = \{\nu(z_1), \nu(z_2), \dots, \nu(z_s), \dots\}.$$



Let  $f$  be a rational function and  $\mathcal{O}_1, \mathcal{O}_2$  orbifolds with ramifications functions  $\nu_1$  and  $\nu_2$ . We say that  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  is a *covering map* between orbifolds if for any  $z \in \mathbb{CP}^1$  the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f$$

holds. In case the weaker condition

$$\nu_2(f(z)) = \nu_1(z) \text{GCD}(\deg_z f, \nu_2(f(z)))$$

is satisfied, we say that  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  is a *minimal holomorphic map* between orbifolds.

In the above terms, a *Lattès map* can be defined as a rational function  $A$  such that  $A : \mathcal{O} \rightarrow \mathcal{O}$  is a covering self-map for some orbifold  $\mathcal{O}$  (see [27], [37]). Following [37], we say that a rational function  $A$  of degree at least two is a *generalized Lattès map* if there exists an orbifold  $\mathcal{O}$ , distinct from the non-ramified sphere, such that  $A : \mathcal{O} \rightarrow \mathcal{O}$  is a minimal holomorphic map. Thus,  $A$  is a Lattès map if there exists an orbifold  $\mathcal{O}$  such that

$$(35) \quad \nu(A(z)) = \nu(z) \deg_z A, \quad z \in \mathbb{CP}^1,$$

and a generalized Lattès map if there exists an orbifold  $\mathcal{O}$  such that

$$(36) \quad \nu(A(z)) = \nu(z) \text{GCD}(\deg_z A, \nu(A(z))), \quad z \in \mathbb{CP}^1.$$

Since (35) implies (36), any Lattès map is a generalized Lattès map. More generally, any special function is a generalized Lattès map (see [37]). Notice that a rational function  $A$  is a generalized Lattès map if and only if some iterate  $A^{\circ d}$ ,  $d \geq 1$ , is a generalized Lattès map (see [41], Section 2.3). Notice also that if  $\mathcal{O}$  is an orbifold such that (36) holds for some rational function  $A$ , then the Euler characteristic of  $\mathcal{O}$  is necessarily non-negative. This condition is quite restrictive, and signatures of orbifolds satisfying  $\chi(\mathcal{O}) \geq 0$  can be described explicitly (see e.g. [37] for more detail).

To prove Theorem 1.3 we need the following particular case of the classifications of semiconjugate rational functions (see [37], Theorem 3.2 or [41], Proposition 3.3).

**Theorem 4.1.** *If  $A, X, B$  is a solution of (4) and  $A$  is not a generalized Lattès map, then there exists a rational function  $Y$  such that the diagram*

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ X \downarrow & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \\ Y \downarrow & & \downarrow Y \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1, \end{array}$$

*commutes, and the equalities*

$$Y \circ X = B^{\circ d}, \quad X \circ Y = A^{\circ d}$$

*hold.* □

In turn, the proof of Theorem 1.4 uses the following description of  $(A_1, A_2)$ -invariant curves for rational functions  $A_1, A_2$  of degree at least two that are not generalized Lattès maps (see [41], Theorem 1.1).

**Theorem 4.2.** *Let  $A_1, A_2$  be rational functions of degree at least two that are not generalized Lattès maps, and  $C$  an irreducible algebraic curve in  $(\mathbb{CP}^1)^2$  that is not a vertical or horizontal line. Then  $C$  is  $(A_1, A_2)$ -invariant if and only if there exist rational functions  $X_1, X_2, Y_1, Y_2, B$  such that:*

1. *The diagram*

$$(37) \quad \begin{array}{ccc} (\mathbb{CP}^1)^2 & \xrightarrow{(B,B)} & (\mathbb{CP}^1)^2 \\ (X_1, X_2) \downarrow & & \downarrow (X_1, X_2) \\ (\mathbb{CP}^1)^2 & \xrightarrow{(A_1, A_2)} & (\mathbb{CP}^1)^2 \\ (Y_1, Y_2) \downarrow & & \downarrow (Y_1, Y_2) \\ (\mathbb{CP}^1)^2 & \xrightarrow{(B,B)} & (\mathbb{CP}^1)^2 \end{array}$$

*commutes,*

2. *The equalities*

$$\begin{aligned} X_1 \circ Y_1 &= A_1^{\circ d}, & X_2 \circ Y_2 &= A_2^{\circ d}, \\ Y_1 \circ X_1 &= Y_2 \circ X_2 &= B^{\circ d} \end{aligned}$$

*hold for some  $d \geq 1$ ,*

3. *The map  $t \rightarrow (X_1(t), X_2(t))$  is a parametrization of  $C$ .*  $\square$

Note that if  $A_1$  and  $A_2$  arbitrary rational functions that are not Lattès maps, and  $C$  is an  $(A_1, A_2)$ -invariant curve that is not a vertical or horizontal line, then  $C$  still has genus zero and we can find a rational parametrization  $t \rightarrow (X_1(t), X_2(t))$  of  $C$  such that the *top* square of (37) commutes for some rational function  $B$  (see [41], Theorem 4.1). In particular, the existence of an invariant curve implies that  $\deg A_1 = \deg A_2$ .

**4.2. Proof of Theorem 1.3 and Theorem 1.4.** We start by proving the following lemma.

**Lemma 4.3.** *Let  $F$  be a simple rational function of degree  $m \geq 3$ . Then  $F$  is not a generalized Lattès map.*

*Proof.* Let  $F$  be a simple rational function of degree  $m \geq 3$ . Then the preimage of any  $k$  distinct points of  $\mathbb{CP}^1$  under  $F$  contains at least  $k(m-1) \geq 2k$  distinct points  $z$  such that  $\deg_z F = 1$ . In turn, this implies that if the equality

$$\nu(F(z)) = \nu(z) \text{GCD}(\deg_z F, \nu(F(z))), \quad z \in \mathbb{CP}^1$$

holds for some orbifold  $\mathcal{O}$  distinct from the non-ramified sphere, then the preimage  $F^{-1}\{c(\mathcal{O})\}$  contains at least  $2|c(\mathcal{O})|$  points where  $\nu(z) > 1$ . However, this is impossible since any such a point belongs to  $c(\mathcal{O})$ .  $\square$

*Proof of Theorem 1.3.* Since  $F$  is not a generalized Lattès map by Lemma 4.3, it follows from Theorem 4.1 that there exists a rational function  $Y$  such that the equality

$$X \circ Y = F^{\circ d}$$

holds for some  $d \geq 1$ . By Corollary 2.9, this implies that

$$X = F^{\circ l} \circ \mu$$

for some Möbius transformation  $\mu$  and  $l \geq 1$ . Thus, (5) reduces to the equality

$$F^{\circ r} \circ F^{\circ l} \circ \mu = F^{\circ l} \circ \mu \circ G,$$

and applying to this equality Theorem 2.3, we conclude that

$$G = \mu^{-1} \circ F^{\circ r} \circ \mu. \quad \square$$

**Lemma 4.4.** *Let  $F$  be a simple rational function of degree  $m \geq 4$ . Then  $\gamma(\text{Aut}(F^{\circ k})) = \text{Aut}(F^{\circ k})$  for every  $k \geq 1$ .*

*Proof.* Since  $\text{Aut}_\infty(F) = G_0(F)$  by Theorem 3.5, for every  $\nu \in \text{Aut}(F^{\circ k})$  there exists  $\nu' \in \text{Aut}_\infty(F)$  such that

$$F \circ \nu = \nu' \circ F.$$

Moreover,

$$F^{\circ k} \circ \nu' \circ F = F^{\circ k} \circ F \circ \nu = F \circ F^{\circ k} \circ \nu = F \circ \nu \circ F^{\circ k} = \nu' \circ F \circ F^{\circ k} = \nu' \circ F^{\circ k} \circ F,$$

implying that  $\nu' \in \text{Aut}(F^{\circ k})$ .  $\square$

*Proof of Theorem 1.4.* Assume that

$$(38) \quad (F_1, F_2)^{\circ d}(C) = C, \quad d \geq 1.$$

Then Theorem 4.2 and Theorem 1.3 imply that  $C$  is parametrized by the functions

$$X_1 = (F_1^{\circ d_1} \circ \beta)(t), \quad X_2 = (F_2^{\circ d_2} \circ \alpha)(t),$$

where  $\beta$  and  $\alpha$  are Möbius transformations such that

$$\beta^{-1} \circ F_1^{\circ d} \circ \beta = \alpha^{-1} \circ F_2^{\circ d} \circ \alpha$$

and  $d_1$  and  $d_2$  are non-negative integers. It is clear that without loss of generality we assume that  $\beta = z$ , implying that

$$(39) \quad F_1^{\circ d} = \alpha^{-1} \circ F_2^{\circ d} \circ \alpha = (\alpha^{-1} \circ F_2 \circ \alpha)^{\circ d}.$$

Applying to this equality Theorem 1.2, we conclude that

$$(40) \quad \alpha^{-1} \circ F_2 \circ \alpha = \mu \circ F_1$$

for some  $\mu \in \text{Aut}_\infty(F_1)$ . Moreover, equalities (39) and (40) imply by Lemma 3.3 that  $\mu \in \text{Aut}(F_1^{\circ d})$ . Thus,

$$F_2 = \alpha \circ \mu \circ F_1 \circ \alpha^{-1}$$

for some  $\mu \in \text{Aut}(F_1^{\circ d})$ , and  $C$  is parametrized by the functions

$$(41) \quad X_1 = F_1^{\circ d_1}(t), \quad X_2 = \alpha \circ (\mu \circ F_1)^{\circ d_2}(t).$$

Moreover, it follows from (41) by Lemma 4.4 that there exists  $\mu' \in \text{Aut}(F_1^{\circ d})$  such that

$$(42) \quad X_1 = F_1^{\circ d_1}(t), \quad X_2 = \alpha \circ \mu' \circ F_1^{\circ d_2}(t).$$

If  $d_1 \leq d_2$ , then (42) implies that  $C$  can be parametrized by the functions

$$X_1 = t, \quad X_2 = (\alpha \circ \mu' \circ F_1^{\circ(d_2-d_1)})(t), \quad \mu' \in \text{Aut}(F_1^{\circ d}).$$

On the other hand, if  $d_1 > d_2$ , then  $C$  can be parametrized by the functions

$$X_1 = F_1^{\circ(d_1-d_2)}(t), \quad X_2 = (\alpha \circ \mu')(t).$$

Taking into account that by Lemma 4.4,

$$F_1^{\circ(d_1-d_2)} \circ \mu'^{-1} \circ \alpha^{-1} = \mu'' \circ F_1^{\circ(d_1-d_2)} \circ \alpha^{-1}$$

for some  $\mu'' \in \text{Aut}(F_1^{\circ d})$ , in this case  $C$  can be parametrized also by the functions

$$X_1 = (\mu'' \circ F_1^{\circ(d_1-d_2)} \circ \alpha^{-1})(t), \quad X_2 = t, \quad \mu'' \in \text{Aut}(F_1^{\circ d}).$$

In the other direction, assume that (39) holds and  $C$  is a curve parametrized by

$$X_1 = t, \quad X_2 = (\alpha \circ \nu \circ F_1^{\circ s})(t)$$

for some  $\alpha \in \text{Aut}(\mathbb{C}\mathbb{P}^1)$ ,  $\nu \in \text{Aut}(F_1^{\circ d})$ , and  $s \geq 1$ . Then

$$F_1^{\circ d} \circ X_1(t) = X_1 \circ F_1^{\circ d}(t)$$

and

$$F_2^{\circ d} \circ X_2(t) = \alpha \circ F_1^{\circ d} \circ \nu \circ F_1^{\circ s}(t) = \alpha \circ \nu \circ F_1^{\circ d} \circ F_1^{\circ s}(t) = X_2 \circ F_1^{\circ d}(t),$$

implying that (38) holds. Similarly, if  $C$  is parametrized by

$$X_1 = (\nu \circ F_1^{\circ s} \circ \alpha^{-1})(t), \quad X_2 = t,$$

then

$$F_2^{\circ d} \circ X_2(t) = X_2 \circ F_2^{\circ d}(t)$$

and

$$\begin{aligned} F_1^{\circ d} \circ X_1(t) &= F_1^{\circ d} \circ \nu \circ F_1^{\circ s} \circ \alpha^{-1}(t) = \nu \circ F_1^{\circ d} \circ F_1^{\circ s} \circ \alpha^{-1}(t) = \\ &= \nu \circ F_1^{\circ s} \circ \alpha^{-1} \circ \alpha \circ F_1^{\circ d}(t) \circ \alpha^{-1} = X_1 \circ F_2^{\circ d}(t). \end{aligned} \quad \square$$

**4.3. Invariant curves for general pairs of rational functions.** Identifying a pair of rational functions  $F_1, F_2$  of degree  $m$  with a point

$$\text{Rat}_m \times \text{Rat}_m = \mathbb{C}\mathbb{P}^{2m+1} \times \mathbb{C}\mathbb{P}^{2m+1},$$

we will say that some property of pairs of rational functions is satisfied for *general pairs* of rational functions of degree  $m$  if there exists a proper Zariski closed subset  $W$  of  $\mathbb{C}\mathbb{P}^{2m+1} \times \mathbb{C}\mathbb{P}^{2m+1}$  such that this property is satisfied for all

$$(F_1, F_2) \in \text{Rat}_m \times \text{Rat}_m \setminus W.$$

Let us recall that a variant of a conjecture of Zhang ([54]) on the existence of Zariski dense orbits for endomorphisms of varieties states that if  $K$  is a subfield of  $\mathbb{C}$  and  $F_1, F_2 \in K(z)$  are rational functions of degree at least two, then there is a point in  $(\mathbb{P}^1(K))^2$  whose  $(F_1, F_2)$ -forward orbit is Zariski dense in  $(\mathbb{P}^1(K))^2$ . For polynomials this conjecture was proved in [26], and for rational functions in [53] and [41].

The Zhang conjecture is closely related to the problem of describing periodic curves. Indeed, if the  $(F_1, F_2)$ -orbit  $\mathcal{O}$  of a point  $(x_0, y_0)$  is not dense, then it is easy to see (see e.g. [26], Lemma 7.20) that all but finitely many elements of  $\mathcal{O}$  are contained in some  $(F_1, F_2)$ -invariant algebraic set  $Z \subset (\mathbb{P}^1(\mathbb{C}))^2$ . Moreover, if  $x_0$  and  $y_0$  are not preperiodic points of  $A$ , then  $Z$  is a finite union of curves that are not vertical or horizontal lines. Therefore, there exist an irreducible  $(F_1, F_2)$ -periodic curve  $C \subset (\mathbb{P}^1(\mathbb{C}))^2$  that is not a vertical or horizontal line such that some iterate of  $(x_0, y_0)$  belongs to  $C$ .

Theorem 1.4 combined with Lemma 3.6 and Lemma 3.7 implies that for general pairs of rational functions the both above problems have the simplest possible solutions. Specifically, the following statement holds.

**Theorem 4.5.** *For a general pair of rational functions  $F_1, F_2$  of degree  $m \geq 4$  the endomorphism  $(z_1, z_2) \rightarrow (F_1(z_1), F_2(z_2))$  has no periodic curves distinct from vertical or horizontal lines. In particular, the  $(F_1, F_2)$ -orbit of a point  $(x, y)$  in  $(\mathbb{P}^1(\mathbb{C}))^2$  is Zariski dense in  $(\mathbb{P}^1(\mathbb{C}))^2$ , unless  $x$  is a preperiodic point of  $F_1$ , or  $y$  is a preperiodic point of  $F_2$ .*

*Proof.* By Lemma 3.7 and Lemma 3.7, we may find a proper algebraic subset  $Z$  of  $\mathbb{C}\mathbb{P}^{2m+1}$  such that every rational function  $F$  in  $\text{Rat}_m \setminus Z$  is simple with the trivial group  $G(F)$ . For such  $F$ , equality (39) is equivalent to the equality

$$(43) \quad F_2 = \alpha \circ F_1 \circ \alpha^{-1}, \quad \alpha \in \text{Aut}(\mathbb{C}\mathbb{P}^1),$$

by Theorem 1.2. Therefore, by Theorem 1.4, whenever

$$(F_1, F_2) \in (\text{Rat}_m \setminus Z) \times (\text{Rat}_m \setminus Z),$$

the endomorphism  $(z_1, z_2) \rightarrow (F_1(z_1), F_2(z_2))$  has no periodic curves distinct from vertical or horizontal lines, unless  $F_1$  and  $F_2$  are conjugated. Since equality (43) holds if and only if the coefficients of  $F_1$ ,  $F_2$ , and  $\alpha$  belong to some proper algebraic variety in

$$\mathbb{C}\mathbb{P}^{2m+1} \times \mathbb{C}\mathbb{P}^{2m+1} \times \mathbb{C}\mathbb{P}^3,$$

arguing as in Lemma 3.7 we conclude that for general pairs of rational functions  $F_1, F_2$  the endomorphism  $(z_1, z_2) \rightarrow (F_1(z_1), F_2(z_2))$  has no periodic curves distinct from vertical or horizontal lines.  $\square$

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