# LOWER BOUNDS FOR GENERA OF FIBER PRODUCTS

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ABSTRACT. We provide lower bounds for genera of components of fiber products of holomorphic maps between compact Riemann surfaces.

## 1. INTRODUCTION

In this paper, we extend results of the recent papers [47], [52], concerning lower bounds for genera of components of algebraic curves of the form

$$E_{A,B}:\ A(x)-B(y)=0, \tag{1}$$

where $A$ and $B$ are rational functions with complex coefficients, to the case of fiber products of arbitrary holomorphic maps between compact Riemann surfaces. Not less importantly, we simplify the approach used in [47], [52] directly relating the problem to the groups action on Riemann surfaces and to the classical Hurwitz inequality.

The problem of describing rational functions $A$, $B$ such that the algebraic curve (1) has a factor of genus zero or one, to which we refer below as "the low genus problem", naturally arises in several different branches of mathematics. First, since the equality $g(E_{A,B}) = 0$ holds if and only if there exist rational functions $X, Y$ satisfying

$$A\circ X = B\circ Y, \tag{2}$$

the low genus problem is central in the theory of functional decompositions of rational functions. In the polynomial case, this theory was developed by Ritt (see [54], [58]). The general case, however, is much less understood and known results are mostly concentrated on a study of either decompositions of special types of functions or functional equations of a special form (see e.g. [6], [10], [16], [23], [39], [40], [42], [46], [51], [55]). Notice also that, by the Picard theorem, any algebraic curve that can be parametrized by functions meromorphic on $\mathbb{C}$ has genus zero or one. Thus, the functional equation (2), where $X$, $Y$ are allowed to be entire or meromorphic functions, often studied in the context of Nevanlinna theory (see e.g. [4], [24], [32], [38], [64]), is also related to the low genus problem (see e. g. [7], [33], [44], [45]).

Second, algebraic curves (1) with factors of genus zero or one have special Diophantine properties. Indeed, by the Siegel theorem, if an irreducible algebraic curve $C$ with rational coefficients has infinitely many integer points, then $C$ is of genus zero with at most two points at infinity. More generally, by the Faltings theorem, if $C$ has infinitely many rational points, then $g(C) \leqslant 1$. Thus, since many interesting Diophantine equations have the form $A(x) = B(y)$, where $A, B$ are rational functions over $\mathbb{Q}$, the low genus problem is of importance in the number theory (see

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e.g. [5], [8], [9], [15], [17], [36], [37], [62]). The most notable result in this direction is the complete classification of polynomial curves $E_{A,B}$ having a factor of genus zero with at most two points at infinity obtained in the paper of Bilu and Tichy [9], which continued the line of researches started by Fried (see [17], [18], [19]).

Third, the low genus problem naturally arises in the new emerging field of arithmetic dynamics. For example, the problem of describing rational functions $A$ and $B$ such that all curves

$$A^{\circ n}(x) - B(y) = 0, \quad n \geqslant 1,$$

have a factor of genus zero or one is a geometric counterpart of the following problem of the arithmetic nature (see [11], [30], [49]): which rational functions $A$ defined over a number field $K$ have a $K$-orbit containing infinitely many points from the value set $B(\mathbb{P}^1(K))$ ? More generally, the problem of describing pairs of rational functions $A$ and $B$ such that all curves

$$A^{\circ n}(x) - B^{\circ m}(y) = 0, \quad n \geqslant 1, \quad m \geqslant 1, \tag{3}$$

have a factor of genus zero or one is a geometric counterpart of the problem of describing pairs of rational functions $A$ and $B$ having orbits with infinite intersection (see [25], [26], [52]). Finally, notice that the low genus problem is related to the study of amenable semigroups of rational functions under the operation of composition, since for a semigroup $S \subset \mathbb{C}(z)$ the amenability condition implies that for all $A, B \in S$ the curve (3) has a factor of genus zero (see [53]).

In case the curve $E_{A,B}$ is irreducible, the standard approach to the low genus problem, initiated by Fried ([17], [19]), relies on the use of an explicit formula for genus of $E_{A,B}$ in terms of the ramifications of $A$ and $B$ (see Section 2.1 below). However, the direct analysis of this formula is quite difficult, and obtaining a full classification of curves $E_{A,B}$ of genus zero or one on this way seems to be hardly possible. In addition, such an analysis results only in possible *patterns* of ramifications of $A$ and $B$. However, rational functions with such patterns may not exist. It is known that any "polynomial" pattern is realizable by some polynomial ([63]), but already for "Laurent polynomial" patterns there exists a number of exceptions ([43]). In general, the problem of existence of a rational function with a given ramification pattern, the so-called Hurwitz problem, is still widely open (see e. g. the recent papers [34], [41], [60], [65] and the bibliography therein).

A general lower bound for the genus of $E_{A,B}$, in case this curve is irreducible, was obtained in the paper [47]. To formulate it explicitly, let us recall that for a holomorphic map between compact Riemann surfaces $P : \mathcal{R} \to \mathcal{C}$ its *normalization* is defined as a compact Riemann surface $\mathcal{N}_P$ together with a Galois covering of the lowest possible degree $\widetilde{P} : \mathcal{N}_P \to \mathcal{C}$ such that $\widetilde{P} = P \circ H$ for some holomorphic map $H : \mathcal{N}_P \to \mathcal{R}$. From the algebraic point of view, the passage from $P$ to $\widetilde{P}$ corresponds to the passage from the field extension $\mathcal{M}(\mathcal{R})/P^*\mathcal{M}(\mathcal{C})$ to its Galois closure. In these terms, the main result of [47] may be formulated as follows: if $A$ and $B$ are rational functions of degree $n$ and $m$ correspondingly such that $E_{A,B}$ is irreducible and $g(\mathcal{N}_A) > 1$, then

$$g(E_{A,B}) > \frac{m - 84n + 168}{84}. \tag{4}$$

Thus, for fixed $A$ the genus of $\mathcal{E}_{A,B}$ grows linearly with respect to the degree of $B$, unless $A$ satisfies the condition $g(\mathcal{N}_A) \leqslant 1$. In particular, the curve (1) has genus greater than one whenever $m \geqslant 84(n-1)$. What is important is that the condition $g(\mathcal{N}_A) \leqslant 1$ is quite restrictive. Specifically, up to the change $X \to \alpha \circ X \circ \beta$, where $\alpha$ and $\beta$ are Möbius transformations, the list of rational functions $X$ with $g(\mathcal{N}_A) = 0$ consists of the series

$$z^n,\ n \geqslant 1, \qquad T_n,\ n \geqslant 2, \qquad \frac{1}{2}\left(z^n + \frac{1}{z^n}\right),\ n \geqslant 1,$$

and a finite number of functions, which can be calculated explicitly (see [48]). On the other hand, functions with $g(\mathcal{N}_A) = 1$ admit a simple geometric description. In particular, the simplest examples of such functions are provided by Lattès maps (see [48]).

In case the curve $E_{A,B}$ is reducible, the above mentioned genus formula cannot be used for studying the low genus problem. On the other hand, the problem of reducibility of $E_{A,B}$, the so-called Davenport-Lewis-Schinzel problem, is very subtle and difficult (see [12], [13], [14], [18], [21], [22], [35], [57]). Thus, universal bounds for genera of components of $E_{A,B}$ are especially interesting. However, it is easy to see that it is not possible to provide such bounds for *all* components of $E_{A,B}$, since for arbitrary rational functions $A$ and $S$ setting $B = A \circ S$ we obtain a curve $E_{A,B}$ with an irreducible component of genus zero $x - S(y) = 0$. Nevertheless, it was shown in [52] that, excluding from the consideration components of the above form and changing the condition $g(\mathcal{N}_A) > 1$ to a stronger condition, a bound for genera of components of $E_{A,B}$ can be deduced from the bound (4).

To formulate the result of [52] explicitly, let us introduce the following definition. We say that a rational function $A$ is *tame* if the algebraic curve

$$A(x) - A(y) = 0$$

has no factors of genus zero or one distinct from the diagonal $x - y = 0$. Otherwise, we say that $A$ is *wild*. It can be shown that for every tame rational function $S$ the inequality $g(\mathcal{N}_A) > 1$ holds ([52]). Thus, the tameness condition is a strengthening of the condition $g(\mathcal{N}_A) > 1$. Notice that a generic rational function of degree at least four is tame ([45]), but a comprehensive classification of wild rational functions is not known (for some partial results see [3], [6], [45], [50], [56], [59]). In this notation, the result of [52] can be formulated as follows: if $A$ is a tame rational function of degree $n$ and $B$ a rational function of degree $m$, then for any irreducible component $C$ of the curve $E_{A,B}$ the inequality

$$g(C) \geqslant \frac{m/n! - 84n + 168}{84}$$

holds, unless $B = A \circ S$ for some rational function $S$ and $C$ is the graph $x - S(y) = 0$.

The algebraic curve $E_{A,B}$ can be interpreted as the *fiber product* of rational functions $A$ and $B$, and in this paper we generalize results of [47], [52] to the fiber products of arbitrary holomorphic maps between compact Riemann surfaces (see Section 2.1 for precise definitions). In practical terms, we consider commutative diagrams

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{U} & \mathcal{T} \\ \downarrow V & & \downarrow W \\ \mathcal{R} & \xrightarrow{P} & \mathcal{C} \end{array} \tag{5}$$

consisting of holomorphic maps between compact Riemann surfaces subject to the condition that the maps $V$ and $U$ have no *non-trivial common compositional right factor* in the following sense: the equalities

$$U = \widetilde{U} \circ T, \quad V = \widetilde{V} \circ T,$$

where

$$T : \mathcal{E} \to \widetilde{\mathcal{E}}, \quad \widetilde{U} : \widetilde{\mathcal{E}} \to \mathcal{R}, \quad \widetilde{V} : \widetilde{\mathcal{E}} \to \mathcal{T}$$

are holomorphic maps between compact Riemann surfaces, imply that $\deg T = 1$. For brevity, we will call such diagrams *reduced.* Notice that if (5) is reduced, then the inequalities $\deg W \geqslant \deg V$ and $\deg P \geqslant \deg U$ hold.

Our first main result is the following generalization of the bound (4) to the bound for the genus of the fiber product of arbitrary holomorphic maps between compact Riemann surfaces, in case this product consists of a unique component.

**Theorem 1.1.** *Let*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{U} & \mathcal{T} \\ \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle W} \\ \mathcal{R} & \xrightarrow{P} & \mathcal{C} \end{array} \tag{6}$$

*be a reduced commutative diagram of holomorphic maps between compact Riemann surfaces such that* $\deg W = \deg V > 1$ *and* $g(\mathcal{N}_W) > 1$*. Then*

$$g(\mathcal{E}) \geqslant (g(\mathcal{R}) - 1)(\deg V - 1) + 1 + \frac{\deg P}{84}. \tag{7}$$

We recall that for every holomorphic map $V : \mathcal{E} \to \mathcal{R}$ the inequality

$$g(\mathcal{E}) \geqslant (g(\mathcal{R}) - 1)\deg V + 1 \tag{8}$$

holds by the Riemann-Hurwitz formula. Thus, inequality (7) is a refinement of inequality (8) in case $V$ can be included to a diagram satisfying conditions of Theorem 1.1. Let us also observe that inequality (7) complements the classical Castelnuovo-Severi inequality which provides an *upper* bound

$$g(\mathcal{E}) \leqslant g(\mathcal{R})\deg V + g(\mathcal{F})\deg U + (\deg V - 1)(\deg U - 1)$$

for the genus of a compact Riemann surface $\mathcal{E}$ such that there exist holomorphic maps $V : \mathcal{E} \to \mathcal{R}$ and $U : \mathcal{E} \to \mathcal{F}$ having no non-trivial common compositional right factor (see [1], [2], [31]).

Notice that Theorem 1.1 implies that whenever the fiber product $(\mathcal{R}, P) \times_{\mathcal{C}} (\mathcal{T}, W)$ consists of a unique component $\mathcal{E}$ and $g(\mathcal{N}_W) > 1$, the inequality

$$g(\mathcal{E}) \geqslant \frac{\deg P - 84 \deg W + 168}{84}$$

holds. In particular, it holds if $g(\mathcal{T}) > 1$, since the last condition clearly implies that $g(\mathcal{N}_W) > 1$. Finally, let us mention that Theorem 1.1 is not true if $g(\mathcal{N}_W) \leqslant 1$ (see [47]). The simplest example is provided by the commutative diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{z^r R(z^n)} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle z^n} & & \downarrow{\scriptstyle z^n} \\ \mathbb{CP}^1 & \xrightarrow{z^r R^n(z)} & \mathbb{CP}^1, \end{array}$$

where $R$ is an arbitrary rational function and $r, n$ are integer positive numbers. In case $\mathrm{GCD}(r, n) = 1$, this diagram is obviously reduced. On the other hand, $g(\mathcal{E}) = 0$ even though, for fixed $n$, the degree of $z^r R^n(z)$ can be arbitrary large.

Our second main result is following.

**Theorem 1.2.** *Let*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{U} & \mathcal{T} \\ \downarrow V & & \downarrow W \\ \mathcal{R} & \xrightarrow{P} & \mathcal{C} \end{array} \tag{9}$$

*be a reduced commutative diagram of holomorphic maps between compact Riemann surfaces such that* $\deg V > 1$ *and the fiber product of* $(\mathcal{T}, W)$ *with itself* $\deg V$ *times contains no components of genus zero or one that do not belong to the big diagonal of* $\mathcal{T}^{\deg V}$. *Then*

$$g(\mathcal{E}) \geqslant (g(\mathcal{R}) - 1)(\deg V - 1) + 1 + \frac{\deg P}{(\deg W)(\deg W - 1)\dots(\deg W - \deg V + 1)}.$$

Notice that the condition $g(\mathcal{N}_W) > 1$ is equivalent to the condition that the fiber product of $(\mathcal{T}, W)$ with itself $\deg W$ times contains no components of genus zero or one that do not belong to the big diagonal of $\mathcal{T}^{\deg W}$ (see Section 2.2). Thus, in case $\deg V = \deg W$, conditions of Theorem 1.1 and Theorem 1.2 coincide. Nevertheless, the bound provided by Theorem 1.1 in this case is stronger.

Applying Theorem 1.2 to rational functions, we obtain the following statement improving the result of [52].

**Theorem 1.3.** *Let* $A$ *and* $B$ *rational functions of degree* $n$ *and* $m$ *correspondingly and* $C$ *an irreducible component of the curve* $E_{A,B}$ *of bi-degree* $(k, l)$ *such that* $k > 1$. *Then*

$$g(\mathcal{E}) > 2 - k + \frac{m}{n(n-1)\dots(n-k+1)}, \tag{10}$$

*unless the algebraic curve in* $(\mathbb{CP}^1)^k$ *defined by the equation*

$$A(x_1) = A(x_2) = \dots = A(x_k) \tag{11}$$

*has a component of genus zero or one that does not belong to the big diagonal of* $(\mathbb{CP}^1)^k$. *In particular, if* $A$ *is a tame, then for any irreducible component* $C$ *of the curve* $E_{A,B}$ *the inequality*

$$g(C) > 2 - n + \frac{m}{n!} \tag{12}$$

*holds, unless* $B = A \circ S$ *for some rational function* $S$ *and* $C$ *is the graph* $x - S(y) = 0$.

The paper is organized as follows. In the second section, we recall some basic facts about fiber products and normalizations. In the third section, we prove the main results and some of their reformulations and corollaries.

## 2. Fiber products and normalizations

2.1. **Fiber products.** Let $P : \mathcal{R} \to \mathcal{C}$ and $W : \mathcal{T} \to \mathcal{C}$ be holomorphic maps between compact Riemann surfaces. The collection

$$(\mathcal{R}, P) \times_{\mathcal{C}} (\mathcal{T}, W) = \bigcup_{j=1}^{n(P,W)} \{\mathcal{E}_j, V_j, U_j\},$$

where $n(P,W)$ is an integer positive number and $\mathcal{E}_j$, $1 \leqslant j \leqslant n(P,W)$, are compact Riemann surfaces provided with holomorphic maps

$$V_j : \mathcal{E}_j \to \mathcal{R}, \quad U_j : \mathcal{E}_j \to \mathcal{T}, \quad 1 \leqslant j \leqslant n(P,W),$$

is called the *fiber product* of $P$ and $W$ if

$$P \circ V_j = W \circ U_j, \quad 1 \leqslant j \leqslant n(P,W),$$

and for any holomorphic maps $V : \widetilde{\mathcal{E}} \to \mathcal{R}$, $U : \widetilde{\mathcal{E}} \to \mathcal{T}$ between compact Riemann surfaces satisfying

$$P \circ V = W \circ U \tag{13}$$

there exist a uniquely defined index $j$, $1 \leqslant j \leqslant n(P,W)$, and a holomorphic map $T : \widetilde{\mathcal{E}} \to \mathcal{E}_j$ such that

$$V = V_j \circ T, \quad U = U_j \circ T.$$

The fiber product is defined in a unique way up to natural isomorphisms and can be described by the following algebro-geometric construction. Let us consider the algebraic variety

$$\mathcal{L} = \{(x,y) \in \mathcal{R} \times \mathcal{T} \,|\, P(x) = W(y)\}. \tag{14}$$

Let us denote by $\mathcal{L}_j$, $1 \leqslant j \leqslant n(P,W)$, irreducible components of $\mathcal{L}$, by $\mathcal{E}_j$, $1 \leqslant j \leqslant n(P,W)$, their desingularizations, and by

$$\pi_j : \mathcal{E}_j \to \mathcal{L}_j, \quad 1 \leqslant j \leqslant n(P,W),$$

the desingularization maps. Then the compositions

$$x \circ \pi_j : \mathcal{E}_j \to \mathcal{R}, \quad y \circ \pi_j : \mathcal{E}_j \to \mathcal{T}, \quad 1 \leqslant j \leqslant n(P,W),$$

extend to holomorphic maps

$$V_j : \mathcal{E}_j \to \mathcal{R}, \quad U_j : \mathcal{E}_j \to \mathcal{T}, \quad 1 \leqslant j \leqslant n(P,W),$$

and the collection $\bigcup_{j=1}^{n(P,W)} \{\mathcal{E}_j, V_j, U_j\}$ is the fiber product of $P$ and $W$. Abusing notation we call the Riemann surfaces $\mathcal{E}_j$, $1 \leqslant j \leqslant n(P,W)$, irreducible components of the fiber product of $P$ and $W$.

It follows from the definition that for every $j$, $1 \leqslant j \leqslant n(P,W)$, the functions $V_j, U_j$ have no *non-trivial common compositional right factor* in the following sense: the equalities

$$V_j = \widetilde{V} \circ T, \quad U_j = \widetilde{U} \circ T,$$

where

$$T : \mathcal{E}_j \to \widetilde{\mathcal{E}}, \quad \widetilde{V} : \widetilde{\mathcal{E}} \to \mathcal{R}, \quad \widetilde{U} : \widetilde{\mathcal{E}} \to \mathcal{T}$$

are holomorphic maps between compact Riemann surfaces, imply that $\deg T = 1$. Denoting by $\mathcal{M}(\mathcal{R})$ the field of meromorphic functions on a Riemann surface $\mathcal{R}$, we can restate this condition as the equality

$$V_j^* \mathcal{M}(\mathcal{R}) \cdot U_j^* \mathcal{M}(\mathcal{T}) = \mathcal{M}(\mathcal{E}_j),$$

meaning that the field $\mathcal{M}(\mathcal{E}_j)$ is a compositum of its subfields $V_j^* \mathcal{M}(\mathcal{R})$ and $U_j^* \mathcal{M}(\mathcal{T})$. In the other direction, if $U$ and $V$ satisfy (13) and have no non-trivial common compositional right factor, then

$$V = V_j \circ T, \quad U = U_j \circ T$$

for some $j$, $1 \leqslant j \leqslant n(P,W)$, and an isomorphism $T : \mathcal{E}_j \to \mathcal{E}_j$.

Notice that since $V_i, U_i$, $1 \leqslant j \leqslant n(P,W)$, parametrize components of (14), the equalities

$$\sum_j \deg V_j = \deg W, \quad \sum_j \deg U_j = \deg P$$

hold. In particular, if $(\mathcal{R}, P) \times_{\mathcal{C}} (\mathcal{T}, W)$ consists of a unique component $\{\mathcal{E}, V, U\}$, then

$$\deg V = \deg W, \quad \deg U = \deg P. \tag{15}$$

Vice versa, if holomorphic maps $U$ and $V$ satisfy (13) and (15), and have no non-trivial common compositional right factor, then $(\mathcal{R}, P) \times_{\mathcal{C}} (\mathcal{T}, W)$ consists of a unique component.

We recall that if $R : \mathcal{E} \to \mathcal{C}$ is a holomorpic map between compact Riemann surfaces, then by the Riemann–Hurwitz formula

$$\chi(\mathcal{E}) = \chi(\mathcal{C}) \deg R - \sum_{z \in \mathcal{E}} (e_R(z) - 1), \tag{16}$$

where $e_R(z)$ denotes the local multiplicity of $R$ at the point $z$. In case the fiber product $(\mathcal{R}, P) \times_{\mathcal{C}} (\mathcal{T}, W)$ consists of a unique component $\{\mathcal{E}, V, U\}$, we can calculate $\chi(\mathcal{E})$ applying (16) to the map

$$R = P \circ V = W \circ U$$

and using the Abhyankar lemma (see e. g. [61], Theorem 3.9.1). According to the last result, for every point $t_0$ of $\mathcal{E}$, the equality

$$e_R(t_0) = \operatorname{lcm}\big(e_P(V(t_0)), e_W(U(t_0))\big) \tag{17}$$

holds. In particular, $e_R(t_0) = 1$ whenever $R(t_0)$ is not a critical value of $P$ or $W$, implying that

$$\mathrm{c}(R) = \mathrm{c}(P) \cup \mathrm{c}(W),$$

where $\mathrm{c}(F)$ denotes the set of critical values of a holomorphic map $F$. Using (16) and (17), we obtain the following formula for $\chi(\mathcal{E})$ in terms of the ramification of $P$ and $W$:

$$\chi(\mathcal{E}) = \chi(\mathcal{C}) \deg P \deg W - \sum_{\substack{(x,y) \in \mathcal{R} \times \mathcal{T}, \\ P(x) = W(y)}} \big(\operatorname{lcm}\big(e_P(x), e_W(y)\big) - 1\big).$$

2.2. **Normalizations.** Let $F : \mathcal{N} \to \mathcal{R}$ be a holomorphic map between compact Riemann surfaces. Let us recall that $F$ is called a *Galois covering* if the covering group

$$\operatorname{Aut}(\mathcal{N}, F) = \{\sigma \in \operatorname{Aut}(\mathcal{N}) : F \circ \sigma = F\}$$

acts transitively on fibers of $F$. Equivalently, $F$ is a Galois covering if the field extension $\mathcal{M}(\mathcal{N})/F^*\mathcal{M}(\mathcal{R})$ is a Galois extension. In case $F$ is Galois covering, for the corresponding Galois group the isomorphism

$$\operatorname{Gal}(\mathcal{M}(\mathcal{N})/F^*\mathcal{M}(\mathcal{R})) \cong \operatorname{Aut}(\mathcal{N}, F) \tag{18}$$

holds. Notice that since the action of $\operatorname{Aut}(\mathcal{N}, F)$ on fibers of $F$ has no fixed points, any element of $\operatorname{Aut}(\mathcal{N}, F)$ is defined by its value on an arbitrary element of a fiber, implying that the action of $\mathrm{Aut}(\mathcal{N},F)$ on fibers of $F$ is transitive if and only the equality

$$|\mathrm{Aut}(\mathcal{N},F)| = \deg F \tag{19}$$

holds. Thus, the last equality is equivalent to the condition that $F$ is a Galois covering. Another equivalent condition for $F$ to be a Galois covering is the equality

$$|\mathrm{Mon}(F)| = \deg F, \tag{20}$$

where $\mathrm{Mon}(F)$ denotes the monodromy group of the holomorphic map $F$ (see e. g. [27], Proposition 2.66).

Let $V : \mathcal{E} \to \mathcal{R}$ be an arbitrary holomorphic map between compact Riemann surfaces. Then the *normalization* of $V$ is defined as a compact Riemann surface $\mathcal{N}_V$ together with a Galois covering of the lowest possible degree $\widetilde{V} : \mathcal{N}_V \to \mathcal{R}$ such that $\widetilde{V} = V \circ H$ for some holomorphic map $H : \mathcal{N}_V \to \mathcal{E}$. The map $\widetilde{V}$ is defined up to the change $\widetilde{V} \to \widetilde{V} \circ \alpha$, where $\alpha \in \mathrm{Aut}(\mathcal{N}_V)$, and is characterized by the property that the field extension $\mathcal{M}(\mathcal{N}_V)/\widetilde{V}^*\mathcal{M}(\mathcal{R})$ is isomorphic to the Galois closure $\widetilde{\mathcal{M}(\mathcal{E})}/V^*\mathcal{M}(\mathcal{R})$ of the extension $\mathcal{M}(\mathcal{E})/V^*\mathcal{M}(\mathcal{R})$. Notice that since

$$\mathrm{Mon}(V) \cong \mathrm{Gal}\big(\widetilde{\mathcal{M}(\mathcal{E})}/V^*\mathcal{M}(\mathcal{R})\big)$$

(see e. g. [28]), this implies by (18) and (19) that the normalization of $V$ can be characterized as a Galois covering $\widetilde{V}$ that factors through $V$ and satisfies the equality

$$|\mathrm{Mon}(V)| = \deg \widetilde{V}. \tag{21}$$

For a holomorphic map $V : \mathcal{E} \to \mathcal{R}$ of degree $n \geqslant 2$ its normalization can be described in terms of the fiber product of $V$ with itself $n$ times as follows (see [20], §I.G, or [29], Section 2.2). For $k$, $1 \leqslant k \leqslant n$, let $\mathcal{L}^{k,V}$ be an algebraic variety consisting of $k$-tuples of $\mathcal{E}^k$ with a common image under $V$, and $\widehat{\mathcal{L}}^{k,V}$ a variety obtained from $\mathcal{L}^{k,V}$ by removing points that belong to the big diagonal

$$\boldsymbol{\Delta}^{k,\mathcal{E}} := \{(x_i) \in \mathcal{E}^k \mid x_i = x_j \ \text{ for some } \ i \neq j\}$$

of $\mathcal{E}^k$. Further, let $\mathcal{L}$ be an arbitrary irreducible component of $\widehat{\mathcal{L}}^{n,V}$ and $\mathcal{N} \xrightarrow{\theta} \mathcal{L}$ the desingularization map. Finally, let $\psi : \mathcal{N} \to \mathcal{R}$ be a holomorphic map induced by the composition

$$\mathcal{N} \xrightarrow{\theta} \mathcal{L} \xrightarrow{\pi_i} \mathcal{E} \xrightarrow{V} \mathcal{R},$$

where $\pi_i$ is the projection to any coordinate. In this notation, the following statement holds.

**Theorem 2.1.** *The map $\psi : \mathcal{N} \to \mathcal{R}$ is the normalization of $V$.*

*Proof.* It follows easily from the construction that

$$\deg \psi = |\mathrm{Mon}(V)| \tag{22}$$

and that the action of $\mathrm{Mon}(\psi)$ on the fibers of $\psi$ has no fixed point. The last property yields that

$$\deg \psi = |\mathrm{Mon}(\psi)|,$$

implying that $\psi$ is a Galois covering according to the characterization (20). Moreover, since $\psi$ factors through $V$, equality (22) implies that $\psi$ is the normalization of $V$ according to the characterization (21). □

Notice that while any holomorphic map of degree one is a Galois covering, the above construction is meaningless in this case since the corresponding algebraic variety $\mathcal{L}^{1,V}$ coincides with the diagonal.

## 3. Lower bounds for genera of fiber products

3.1. **Ramification orbifold.** Let $\mathcal{R}$ be a compact Riemann surface. We recall that an orbifold $\mathcal{O}$ on $\mathcal{R}$ is a ramification function $\nu : \mathcal{R} \to \mathbb{N}$ which takes the value $\nu(z) = 1$ except at finitely many points. The Euler characteristic of an orbifold $\mathcal{O} = (\mathcal{R}, \nu)$ is defined by the formula

$$\chi(\mathcal{O}) = \chi(\mathcal{R}) + \sum_{z\in\mathcal{R}} \left(\frac{1}{\nu(z)} - 1\right),$$

where $\chi(\mathcal{R})$ is the Euler characteristic of $\mathcal{R}$. For a holomorphic map $V : \mathcal{E} \to \mathcal{R}$ between compact Riemann surfaces, we define its *ramification orbifold* $\mathcal{O}^V = (\mathcal{E}, \nu)$, setting $\nu$ equal to the least common multiple of local degrees of $V$ at the points of the preimage $V^{-1}\{z\}$. Notice that Theorem 2.1 combined with the Abhyankar lemma imply that

$$\mathcal{O}^{\widetilde{V}} = \mathcal{O}^V. \tag{23}$$

**Lemma 3.1.** *Let $V : \mathcal{E} \to \mathcal{R}$ be a holomorphic map between compact Riemann surfaces, and $\widetilde{V} : \mathcal{N}_V \to \mathcal{R}$ its normalization. Then*

$$\chi(\mathcal{N}_V) = \chi(\mathcal{O}^V)\deg \widetilde{V}. \tag{24}$$

*Proof.* Since $\widetilde{V} : \mathcal{N}_V \to \mathcal{R}$ is a Galois covering, the equality $|\mathrm{Aut}(\mathcal{N}_V, \widetilde{V})| = \deg \widetilde{V}$ holds and, up to the change $\widetilde{V} \to \alpha \circ \widetilde{V}$, where $\alpha \in \mathrm{Aut}(\mathcal{N}_V)$, the map $\widetilde{V}$ is the quotient map

$$\widetilde{V} : \mathcal{N}_V \to \mathcal{N}_V/\mathrm{Aut}(\mathcal{N}_V, \widetilde{V}).$$

Thus, for any critical value $z_i$, $1 \leqslant i \leqslant r$, of $\widetilde{V}$ there exists a number $d_i$ such that $\widetilde{V}^{-1}\{z_i\}$ consists of $\deg \widetilde{V}/d_i$ points, at each of which the local multiplicity of $\widetilde{V}$ equals $d_i$. Applying the Riemann-Hurwitz formula we see that

$$\chi(\mathcal{N}_V) = \chi(\mathcal{R})\deg \widetilde{V} - \sum_{i=1}^{r} \frac{\deg \widetilde{V}}{d_i}(d_i - 1) =$$

$$= \left(\chi(\mathcal{R}) + \sum_{i=1}^{r}\left(\frac{1}{d_i} - 1\right)\right)\deg \widetilde{V} = \chi(\mathcal{O}^{\widetilde{V}})\deg \widetilde{V}.$$

Therefore, (24) holds by (23). □

**Lemma 3.2.** *Let $V : \mathcal{E} \to \mathcal{R}$ be a holomorphic map between compact Riemann surfaces. Then*

$$\chi(\mathcal{O}^V) \geqslant \chi(\mathcal{E}) + \chi(\mathcal{R})(1 - \deg V). \tag{25}$$

*Proof.* It follows from the definition of $\mathcal{O}^V$ that

$$\chi(\mathcal{O}^V) \geqslant \chi(\mathcal{R}) - |\mathrm{c}(V)|. \tag{26}$$

On the other hand, it is clear that the number of critical values of $V$ does not exceed the number of critical points of $P$, which in turn does not exceed the number

$\sum_{z\in\mathcal{E}}(\deg_z V-1)$. Therefore, since

$$\chi(\mathcal{E})=\chi(\mathcal{R})\deg V-\sum_{z\in\mathcal{E}}(\deg_z V-1),$$

we have:

$$|\mathrm{c}(V)|\leqslant\sum_{z\in\mathcal{E}}(\deg_z V-1)=\chi(\mathcal{R})\deg V-\chi(\mathcal{E}). \tag{27}$$

Now (25) follows from (26) and (27). □

3.2. **Lifting lemma.** Let $W:\mathcal{T}\to\mathcal{C}$ be a holomorphic map between compact Riemann surfaces, and $\mathcal{D}$ a component of the fiber power of the map $W$ with itself $k$, $2\leqslant k\leqslant\deg W$, times. Then $\mathcal{D}$ is the desingularization of an irreducible component $D$ of the variety

$$\{(x_i)\in\mathcal{E}^k\,|\,W(x_1)=W(x_2)=\dots=W(x_k)\},$$

and abusing notation we will say that $\mathcal{D}$ does not belong to the big diagonal of $\mathcal{T}^k$, if $D$ does not belong to the big diagonal of $\mathcal{T}^k$.

Our proof of Theorem 1.1 and Theorem 1.2 is based on the following lemma of independent interest.

**Lemma 3.3.** *Let*

$$\begin{array}{ccc}\mathcal{E} & \xrightarrow{U} & \mathcal{T}\\ \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle W}\\ \mathcal{R} & \xrightarrow{P} & \mathcal{C}\end{array} \tag{28}$$

*be a reduced commutative diagram of holomorphic maps between compact Riemann surfaces such that* $\deg V>1$. *Then one can complete it to a diagram of holomorphic maps between compact Riemann surfaces*

$$\begin{array}{ccc}\mathcal{N}_V & \xrightarrow{L} & \mathcal{D}\\ \downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle F}\\ \mathcal{E} & \xrightarrow{U} & \mathcal{T}\\ \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle W}\\ \mathcal{R} & \xrightarrow{P} & \mathcal{C},\end{array} \tag{29}$$

*such that* $V\circ Q=\widetilde{V}$ *and* $\mathcal{D}$ *is an irreducible component of the fiber power of* $W:\mathcal{T}\to\mathcal{C}$ *with itself* $\deg V$ *times that does not belong to the big diagonal of* $\mathcal{T}^{\deg V}$. *Moreover, if* $\deg V=\deg W$, *then* $\mathcal{D}=\mathcal{N}_W$ *and* $W\circ F=\widetilde{W}$.

*Proof.* Let us set $k=\deg V$ and define the maps

$$V_k:\mathcal{E}^k\to\mathcal{R}^k,\quad U_k:\mathcal{E}^k\to\mathcal{T}^k,\quad W_k:\mathcal{T}^k\to\mathcal{C}^k,\quad P_k:\mathcal{R}^k\to\mathcal{C}^k$$

by the formulas

$$\begin{aligned}V_k:\,&(z_1,z_2,\dots,z_k)\to(V(z_1),V(z_2),\dots,V(z_k)),\\ U_k:\,&(z_1,z_2,\dots,z_k)\to(U(z_1),U(z_2),\dots,U(z_k)),\\ W_k:\,&(z_1,z_2,\dots,z_k)\to(W(z_1),W(z_2),\dots,W(z_k)),\\ P_k:\,&(z_1,z_2,\dots,z_k)\to(P(z_1),P(z_2),\dots,P(z_k)).\end{aligned}$$

Clearly, the diagram

$$\begin{array}{ccc} \mathcal{E}^k & \xrightarrow{U_k} & \mathcal{T}^k \\ \downarrow{\scriptstyle V_k} & & \downarrow{\scriptstyle W_k} \\ \mathcal{R}^k & \xrightarrow{P_k} & \mathcal{C}^k \end{array} \tag{30}$$

commutes. Furthermore, in the notation of Section 2.2,

$$\mathcal{L}^{k,V} = V_k^{-1}(\Delta^{k,\mathcal{R}}),$$

where $\Delta^{k,\mathcal{R}}$ is the usual diagonal in $\mathcal{R}^k$,

$$\Delta^{k,\mathcal{R}} := \{(x_i) \in \mathcal{R}^k \mid x_1 = x_2 = \cdots = x_k\}.$$

Since $P_k(\Delta^{k,\mathcal{R}}) = \Delta^{k,\mathcal{C}}$, it follows from the commutativity of (30) that

$$U_k(\mathcal{L}^{k,V}) \subseteq \mathcal{L}^{k,W}.$$

Let us show now that

$$U_k(\widehat{\mathcal{L}}^{k,V}) \subseteq \widehat{\mathcal{L}}^{k,W}. \tag{31}$$

It follows from the commutativity of diagram (28) that the subfields $U^*\mathcal{M}(\mathcal{T})$ and $V^*\mathcal{M}(\mathcal{R})$ of $\mathcal{M}(\mathcal{E})$ contain a common subfield $k$ of the transcendence degree one over $\mathbb{C}$. By the primitive element theorem,

$$U^*\mathcal{M}(\mathcal{T}) = k[h]$$

for some $h \in U^*\mathcal{M}(\mathcal{T})$. Since

$$V^*\mathcal{M}(\mathcal{R}) \cdot U^*\mathcal{M}(\mathcal{T}) = \mathcal{M}(\mathcal{E}),$$

this implies that

$$\mathcal{M}(\mathcal{E}) = V^*\mathcal{M}(\mathcal{R})[h]. \tag{32}$$

As elements of $\mathcal{M}(\mathcal{E})$ separate points of $\mathcal{E}$, equality (32) implies that for every $z_0 \in \mathcal{R}$ that is not a critical value of $V$ the map $h$ takes $k$ distinct values on the set $V^{-1}\{z_0\}$. Since $h \in U^*\mathcal{M}(\mathcal{T})$, this implies in turn that the map $U$ takes $k$ distinct values on $V^{-1}\{z_0\}$. Therefore, (31) holds.

Let $\mathcal{L}$ be an arbitrary irreducible component of $\widehat{\mathcal{L}}^{k,V}$, and $\mathcal{D}$ the disingularization of $U_k(\mathcal{L})$. Then $\mathcal{D}$ does not belong to the big diagonal of $\mathcal{T}^k$ by (31). Furthermore, it follows from Theorem 2.1 that there exists a holomorphic map $L : \mathcal{N}_V \to \mathcal{D}$ such that the diagram

$$\begin{array}{ccc} \mathcal{N}_V & \xrightarrow{L} & \mathcal{D} \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \eta} \\ \mathcal{L} & \xrightarrow{P} & U_k(\mathcal{L}), \end{array}$$

where $\theta : \mathcal{N}_V \to \mathcal{L}$ and $\eta : \mathcal{D} \to U_k(\mathcal{L})$ are the desingularization maps, commutes. Therefore, (29) commutes for the holomorphic maps $Q$ and $F$ induced by the compositions $\mathcal{N}_V \xrightarrow{\theta} \mathcal{L} \xrightarrow{\pi_i} \mathcal{E}$ and $\mathcal{D} \xrightarrow{\eta} U_k(\mathcal{L}) \xrightarrow{\pi_i} \mathcal{T}$, where $\pi_i$ is the projection to any coordinate. Moreover, Theorem 2.1 implies that $V \circ Q = \widehat{V}$. Finally, if $\deg V = \deg W$, then Theorem 2.1 implies that $\mathcal{D} = \mathcal{N}_W$ and $W \circ F = \widetilde{W}$. $\square$

3.3. **Bounds for fiber products with one component.** *Proof of Theorem 1.1.* By Lemma 3.3, we can complete diagram (6) to the diagram

$$\begin{array}{ccc}
\mathcal{N}_V & \xrightarrow{L} & \mathcal{N}_W \\
\downarrow Q & & \downarrow F \\
\mathcal{E} & \xrightarrow{U} & \mathcal{T} \\
\downarrow V & & \downarrow W \\
\mathcal{R} & \xrightarrow{P} & \mathcal{C},
\end{array}$$

where $V \circ Q = \widetilde{V}$ and $W \circ F = \widetilde{W}$. By the Riemann-Hurwitz formula, we have:

$$\chi(\mathcal{N}_V) \leqslant \chi(\mathcal{N}_W) \deg L,$$

implying by Lemma 3.1 that

$$\chi(\mathscr{O}^V) \deg \widetilde{V} \leqslant \chi(\mathcal{N}_W) \deg L.$$

Since

$$\deg \widetilde{V} \deg P = \deg Q \deg V \deg P = \deg L \deg F \deg W = \deg L \deg \widetilde{W},$$

this yields that

$$\chi(\mathscr{O}^V) \leqslant \frac{\chi(\mathcal{N}_W) \deg L}{\deg \widetilde{V}} = \frac{\chi(\mathcal{N}_W) \deg P}{\deg \widetilde{W}},$$

implying by Lemma 3.2 that

$$\chi(\mathcal{E}) + \chi(\mathcal{R})(1 - \deg V) \leqslant \frac{\chi(\mathcal{N}_W) \deg P}{\deg \widetilde{W}}. \tag{33}$$

Let us recall that, by the Hurwitz inequality, for a compact Riemann surface of genus $g > 1$ the order of its automorphism group does not exceed $84(g-1)$. Thus, if $g(\mathcal{N}_W) > 1$, then

$$42\mathcal{N}_W \leqslant -|\mathrm{Aut}(\mathcal{N}_W)|,$$

implying by (33) that

$$42\Big(\chi(\mathcal{E}) + \chi(\mathcal{R})(1 - \deg V)\Big) \leqslant -\frac{|\mathrm{Aut}(\mathcal{N}_W)| \deg P}{\deg \widetilde{W}}. \tag{34}$$

On the other hand, since the map $\widetilde{W}$ can be thought of as the quotient map

$$\widetilde{W} : \mathcal{N}_W \to \mathcal{N}_W / \mathrm{Aut}(\mathcal{N}_W, \widetilde{W})$$

and $\mathrm{Aut}(\mathcal{N}_W, \widetilde{W}) \subseteq \mathrm{Aut}(\mathcal{N}_W)$ it follows from (19) that

$$\deg \widetilde{W} = |\mathrm{Aut}(\mathcal{N}_W, \widetilde{W})| \leqslant |\mathrm{Aut}(\mathcal{N}_W)|. \tag{35}$$

Combining now (34) and (35), we obtain the inequality

$$\chi(\mathcal{E}) + \chi(\mathcal{R})(1 - \deg V) \leqslant -\frac{\deg P}{42},$$

which is equivalent to (7). □

Notice that Theorem 1.1 can be reformulated as follows.

**Theorem 3.4.** *Let $P : \mathcal{R} \to \mathcal{C}$ and $W : \mathcal{T} \to \mathcal{C}$ be holomorphic maps between compact Riemann surfaces such that $\deg W \geqslant 2$, the fiber product $(\mathcal{R}, P) \times_{\mathcal{C}} (\mathcal{T}, W)$ consists of a unique component $\mathcal{E}$, and $g(\mathcal{N}_W) > 1$. Then*

$$g(\mathcal{E}) \geqslant (g(\mathcal{R}) - 1)(\deg W - 1) + 1 + \frac{\deg P}{84}. \tag{36}$$

*In particular,*

$$g(\mathcal{E}) \geqslant \frac{\deg P - 84 \deg W + 168}{84}. \tag{37}$$

*Proof.* If $(\mathcal{R}, P) \times_{\mathcal{C}} (\mathcal{T}, W)$ consists of a unique component $\mathcal{E}$, then there exists a reduced diagram (6) such that

$$\deg V = \deg W \geqslant 2. \tag{38}$$

Thus, (36) follows from (7) and (38). Finally, (36) implies (37) since $g(\mathcal{R}) \geqslant 0$. □

In case $\mathcal{C} = \mathbb{CP}^1$, the assumption $\deg V \geqslant 2$ in Theorem 3.4 can be removed and the inequality in (36) can be made strict.

**Theorem 3.5.** *Let $P : \mathcal{R} \to \mathbb{CP}^1$ and $W : \mathcal{T} \to \mathbb{CP}^1$ be holomorphic maps between compact Riemann surfaces such that the fiber product $(\mathcal{R}, P) \times_{\mathbb{CP}^1} (\mathcal{T}, W)$ consists of a unique component $\mathcal{E}$ and $g(\mathcal{N}_W) > 1$. Then*

$$g(\mathcal{E}) > (g(\mathcal{R}) - 1)(\deg W - 1) + 1 + \frac{\deg P}{84}.$$

*In particular,*

$$g(\mathcal{E}) > \frac{\deg P - 84 \deg W + 168}{84}.$$

*Proof.* The assumption $\deg W > 1$ is not necessary anymore since the equality $\deg W = 1$ for a holomorphic map $W : \mathcal{T} \to \mathbb{CP}^1$ implies the equality $g(\mathcal{N}_W) = 0$. Further, any holomorphic map $W : \mathcal{T} \to \mathbb{CP}^1$ of degree greater than one has critical values. This yields that the inequality in (26) is strict, implying that the inequalities in (25), (33), (7), and (36) are also strict. □

**Remark 3.6.** Theorem 3.5 was also proved in the paper [49] by a modification of the method of [47] (see [49], Theorem 3.1). Unfortunately, by the mistake of the author, the formulation of the corresponding result in [49] was partly copied from an earlier version of the paper. As a result, it is stated in [49] that $P$ is a *rational function* but what is really meant is that $P : R \to \mathbb{CP}^1$ is a *holomorphic map* from a compact Riemann surface $R$ while $W : T \to \mathbb{CP}^1$ is a holomorphic map from another compact Riemann surface $T$.

3.4. **Bounds for fiber product with several components.** *Proof of Theorem 1.2.* Applying Lemma 3.3, we can complete diagram (9) to diagram (29), and arguing as in the proof of Theorem 1.1 we conclude that

$$\chi(\mathcal{N}_V) \leqslant \chi(\mathcal{D}) \deg L$$

and

$$\chi(\mathcal{E}) + \chi(\mathcal{R})(1 - \deg V) \leqslant \frac{\chi(\mathcal{D}) \deg P}{\deg W \deg F}. \tag{39}$$

By assumption, $g(\mathcal{D}) \geqslant 2$ and hence

$$\chi(\mathcal{D}) = 2 - 2g(D) \leqslant -2. \tag{40}$$

On the other hand, the degree of $W \circ F$ does not exceed the number of points in $\mathcal{T}^{\deg V} \setminus \mathbf{\Delta}^{\deg V, \mathcal{T}}$ with a common image under $W$, which is equal to the number of permutations of $\deg W$ things taken $\deg V$ at a time. Thus,

$$\deg W \deg F \leqslant (\deg W)(\deg W - 1) \dots (\deg W - \deg V + 1). \tag{41}$$

Now the statement of the theorem follows from (39), (40), and (41). □

*Proof of Theorem 3.4.* The first part of the theorem is simply a particular case of Theorem 1.2 for $\mathcal{R} = \mathcal{T} = \mathcal{C} = \mathbb{CP}^1$.

To prove the second, we observe that if $A$ is tame, then for any $k$, $2 \leqslant k \leqslant n$, curve (11) has no component of genus zero or one that does not belong to the big diagonal of $(\mathbb{CP}^1)^k$. Moreover, for any $k$, $2 \leqslant k \leqslant n$, the inequalities

$$-k \geqslant -n, \qquad \frac{m}{n(n-1)\dots(n-k+1)} \geqslant \frac{m}{n!}$$

hold. Thus, if $C$ has bi-degree $(k, l)$ with $k > 1$, then (12) follows from (10). On the other hand, if $k = 1$, then obviously $B = A \circ S$ for some rational function $S$, and $C$ is the graph $x - S(y) = 0$. □

Notice that applying Theorem 1.3 for $k = 2$ to the functional equation

$$A \circ C = B \circ D \tag{42}$$

we obtain the following statement: if $A$, $B$, $C$, $D$ are rational functions such that (42) holds and $\deg D = 2$, then either $C$ is a rational function in $D$ or $A$ is wild. This statement is rather easy since without loss of generality we can assume that $D = z^2$, implying by (42) that

$$A \circ C(z) = A \circ C(-z).$$

Therefore, either $A$ is wild, or $C(z) = C(-z)$ and hence $C$ is a rational function in $D$. Moreover, it is not hard to see that a similar statement holds whenever $D$ is a Galois covering (see [52], Theorem 2.1).

The application of Theorem 1.3 to equation (42) in the general case is more surprising. Specifically, Theorem 1.3 yields that for any fixed $k \geqslant 2$ the equality (42), where $D$ is an *arbitrary* rational function of degree $k$ such that $\mathbb{C}(C, D) = \mathbb{C}(z)$, implies that $A$ is wild and even satisfies the more restrictive condition that (11) has a factor of genus zero or one that does not belong to the big diagonal of $(\mathbb{CP}^1)^k$, whenever $\deg B$ is big enough with respect to $\deg A$ to the extent defined by $k$.

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