ON RATIONAL FUNCTIONS SHARING THE MEASURE OF MAXIMAL ENTROPY

F. PAKOVICH

Abstract. We show that describing rational functions \( f_1, f_2, \ldots, f_n \) sharing the measure of maximal entropy reduces to describing solutions of the functional equation \( A \circ X_1 = A \circ X_2 = \cdots = A \circ X_n \) in rational functions. We also provide some results about solutions of this equation.

1. Introduction

Let \( f \) be a rational function of degree \( d \geq 2 \) on \( \mathbb{CP}^1 \). It was proved by Friere, Lopes, and Mañé ([11]), and independently by Lyubich ([15]) that there exists a unique probability measure \( \mu_f \) on \( \mathbb{CP}^1 \), which is invariant under \( f \), has support equal to the Julia set \( J(f) \) of \( f \), and achieves maximal entropy \( \text{deg} \, d \) among all \( f \)-invariant probability measures. In this note, we study rational functions sharing the measure of maximal entropy, that is rational functions \( f \) and \( g \) such that \( \mu_f = \mu_g \), and more generally rational functions \( f_1, f_2, \ldots, f_n \) such that \( \mu_{f_1} = \mu_{f_2} = \cdots = \mu_{f_n} \). We assume that considered functions are non-special in the following sense: they are neither Lattès maps nor conjugate to \( z \pm n \) or \( \pm T_n \).

In case if \( f \) and \( g \) are polynomials, the condition \( \mu_f = \mu_g \) is equivalent to the condition \( J(f) = J(g) \). In turn, for non-special polynomials \( f \) and \( g \) the equality \( J(f) = J(g) = J \) holds if and only if there exists a polynomial \( h \) such that \( J(h) = J \) and

\[
(1) \quad f = \mu_1 \circ h^{\circ s}, \quad g = \mu_2 \circ h^{\circ t}
\]

for some integers \( s, t \geq 1 \) and polynomials of degree one \( \mu_1, \mu_2 \) such that \( \mu_1(J) = \mu_2(J) = J \) (see [2], [29], and also [3], [6]-[7] for other related results). Note that a similar conclusion remains true if instead of the condition \( J(f) = J(g) \) one were to assume only that \( f \) and \( g \) share a completely invariant compact set in \( \mathbb{C} \) (see [18]). Note also that in the polynomial case any of the conditions \( J(f) = J(g) \) and (1) is equivalent to the condition that

\[
(2) \quad f^{\circ k} = \mu \circ g^{\circ l},
\]

for some integers \( k, l \geq 1 \) and Möbius transformation \( \mu \) such that \( \mu(J(g)) = J(g) \).

Since \( \mu_f = \mu_{f^{\circ k}} \), the equality \( \mu_f = \mu_g \) holds whenever \( f \) and \( g \) share an iterate, that is satisfy

\[
(3) \quad f^{\circ k} = g^{\circ l}
\]

for some integers \( k, l \geq 1 \). Moreover, \( \mu_f = \mu_g \) whenever \( f \) and \( g \) commute. However, the latter condition in fact is a particular case of the former one, since non-special commuting \( f \) and \( g \) always satisfy (3) by the result of Ritt ([27]). Note that in distinction with the polynomial case rational solutions of (3) not necessarily have the form (1) (see [27], [25]).

1
The problem of describing rational functions \( f \) and \( g \) with \( \mu_f = \mu_g \) can be expressed in algebraic terms. Namely, by the results of Levin ([13]) and Levin and Przytycki ([14]), for non-special \( f \) and \( g \) the equality \( \mu_f = \mu_g \) holds if and only if some iterates \( F = f^{\circ k} \) and \( G = g^{\circ l} \) satisfy the system of equations
\[
F \circ F = F \circ G, \quad G \circ G = G \circ F \tag{4}
\]
(see [30] for more detail).

Examples of rational functions \( f, g \) with \( \mu_f = \mu_g \) which do not have the form (2) were constructed by Ye in the paper [30]. These examples are based on the following remarkable observation: if \( X, Y, \) and \( A \) are rational functions such that
\[
A \circ X = A \circ Y, \tag{5}
\]
then the functions
\[
F = X \circ A, \quad G = Y \circ A
\]
satisfy (4). The simplest examples of solutions of (5) are obtained from rational functions satisfying \( A \circ \mu = A \) for some Möbius transformation \( \mu \), by setting
\[
X = \mu \circ Y. \tag{6}
\]

For such solutions of (5), the corresponding solutions of (4) have the form (2). However, other solutions of (5) also exist, providing solutions of (4) which do not have the form (2).

Roughly speaking, the main result of this note states that in fact all solutions of (4) can be obtained from solutions of (5). More generally, the following statement holds.

**Theorem 1.1.** Let \( f_1, f_2, \ldots, f_n \) be non-special rational functions of degree at least two on \( \mathbb{CP}^1 \). Then they share the measure of maximal entropy if and only if some their iterates \( F_1, F_2, \ldots, F_n \) can be represented in the form
\[
F_1 = X_1 \circ A, \quad F_2 = X_2 \circ A, \quad \ldots, \quad F_n = X_n \circ A, \tag{7}
\]
where \( A \) and \( X_1, X_2, \ldots, X_n \) are rational functions such that
\[
A \circ X_1 = A \circ X_2 = \cdots = A \circ X_n \tag{8}
\]
and \( \mathbb{C}(X_1, X_2, \ldots, X_n) = \mathbb{C}(z) \).

Theorem 1.1 shows that “up to iterates” describing pairs of rational functions \( f \) and \( g \) with \( \mu_f = \mu_g \) reduces to describing solutions of (5). In particular, since by the well-known result which goes back to Ritt polynomial solutions of (5) satisfy (6), we immediately recover the result that polynomials \( f, g \) with \( \mu_f = \mu_g \) satisfy (2). Nevertheless, the problem of describing solutions of (5) for arbitrary rational \( A, X, Y \) is still widely open. In fact, a complete description of solutions of (5) is obtained only in the case where \( A \) is a polynomial (while \( X \) and \( Y \) can be arbitrary rational functions) in the paper [5] by Avanzi and Zannier. The approach of [5] is based on listing \( A \) for which the genus of the irreducible algebraic curve
\[
\mathcal{C}_A : \frac{A(x) - A(y)}{x - y} = 0 \tag{9}
\]
is zero, and analyzing situations where \( \mathcal{C}_A \) is reducible but has a component of genus zero. Although the same strategy can be applied for arbitrary rational \( A \), both its stages become much more complicated and no general results are known.
Note that the problem of describing solutions of equation (5) for rational \( A \) and meromorphic on the complex plane \( X, Y \) was posed in the paper of Lyubich and Minsky (see [16], p. 83) in the context of studying the action of rational functions on the “universal space” of non-constant functions meromorphic on \( \mathbb{C} \). In algebraic terms, this problem reduces to describing rational functions \( A \) such that (9) has a component of genus zero or one.

Theorem 1.1 implies an interesting corollary concerning dynamical characteristics of rational functions sharing the measure of maximal entropy. Recall that the multiplier spectrum of a rational function \( f \) of degree \( d \) is a function which assigns to each \( s \geq 1 \) the unordered list of multipliers at all \( d^s + 1 \) fixed points of \( f^s \) taken with appropriate multiplicity. Two rational functions are called isospectral if they have the same multiplier spectrum.

**Corollary 1.1.** If non-special rational functions \( f_1, f_2, \ldots, f_n \) of degree at least two share the measure of maximal entropy, then some their iterates \( F_1, F_2, \ldots, F_n \) are isospectral.

The rest of this note is organized as follows. In the second section, we prove Theorem 1.1 and Corollary 1.1. Then, in the third section, we prove two results concerning equation (5) and system (8). The first result states that if the curve \( C_A \) is irreducible and rational functions \( X, Y \) provide a generically one-to-one parametrization of \( C_A \), then \( X = Y \circ \mu \) for some involution \( \eta \in Aut(\mathbb{C}P^1) \). The second result states that if \( A \) and \( X_1, X_2, \ldots, X_n \) are rational functions such that (8) holds and \( X_1, X_2, \ldots, X_n \) are distinct, then \( n \leq \deg A \), and \( n = \deg A \) only if the Galois closure of the field extension \( \mathbb{C}(z)/\mathbb{C}(A) \) has genus zero or one. In fact, we prove these results in the more general setting, allowing the functions \( X, Y \) and \( X_1, X_2, \ldots, X_n \) to be meromorphic on \( \mathbb{C} \).

2. Functions sharing the measure of maximal entropy

In this section we deduce Theorem 1.1 and Corollary 1.1 from the criterion (4) and the following four lemmas.

**Lemma 2.1.** Let \( A_1, A_2, \ldots, A_n \) and \( Y_1, Y_2, \ldots, Y_n \) be rational functions such that
\[
A_i \circ Y_1 = A_i \circ Y_2 = \cdots = A_i \circ Y_n, \quad i = 1, \ldots, n,
\]
and
\[
C(A_1, A_2, \ldots, A_n) = \mathbb{C}(z).
\]
Then
\[
Y_1 = Y_2 = \cdots = Y_n.
\]

**Proof.** By (11), there exist a rational function \( P \in \mathbb{C}(z_1, z_2, \ldots, z_n) \) such that
\[
z = P(A_1, A_2, \ldots, A_n),
\]
implying that
\[
Y_j = P(A_1 \circ Y_j, A_2 \circ Y_j, \ldots, A_n \circ Y_j), \quad 1 \leq j \leq n.
\]
Now (12) follows from (13) and (10). \( \square \)
Lemma 2.2. Let $F_1, F_2, \ldots, F_n$ be rational functions such that
\[(14)\quad F_i \circ F_1 = F_i \circ F_2 = \cdots = F_i \circ F_n, \quad i = 1, \ldots, n.\]
Then there exist rational functions $A$ and $X_1, X_2, \ldots, X_n$ such that
\[(15)\quad F_i = X_i \circ A, \quad i = 1, \ldots, n,\]
\[(16)\quad C(X_1, X_2, \ldots, X_n) = C(z),\]
and
\[(17)\quad A \circ X_1 = A \circ X_2 = \cdots = A \circ X_n.\]

Proof. By the Lüroth theorem,
\[C(F_1, F_2, \ldots, F_n) = C(A)\]
for some rational function $A$, implying that equalities (15) hold for some rational functions $X_1, X_2, \ldots, X_n$ satisfying (16). Substituting now (15) in (14) we see that
\[X_i \circ (A \circ X_1) = X_i \circ (A \circ X_2) = \cdots = X_i \circ (A \circ X_n), \quad i = 1, \ldots, n.\]
Applying now Lemma 2.1 to the last system we obtain (17). □

Lemma 2.3. Let $A$ and $B$ be rational functions such that the equality
\[A \circ A = A \circ B\]
holds. Then
\[A^{\circ l} \circ A^{\circ l} = A^{\circ l} \circ B^{\circ l}\]
for any $l \geq 1$.

Proof. The proof is by induction on $l$. Assuming that the statement is true for $l \leq n$, we have:
\[A^{\circ(n+1)} \circ B^{\circ(n+1)} = A^{\circ n} \circ (A \circ B) \circ B^{\circ n} = A^{\circ n} \circ A^{\circ 2} \circ B^{\circ n} = A^{\circ n} \circ B^{\circ n} = A^{\circ 2} \circ A^{\circ n} \circ B^{\circ n} = A^{\circ n} \circ A^{\circ 2n} = A^{\circ 2n+2}.\] □

Lemma 2.4. Let $d_i \geq 2, 1 \leq i \leq n$, and $n_{i,j}, 1 \leq i, j \leq n, i \neq j$, be integers such that
\[d_i^{n_{i,j}} = d_j^{n_{j,i}}, \quad 1 \leq i, j \leq n, \quad i \neq j.\]
Then there exist integers $l_i \geq 1, 1 \leq i \leq n$, such that
\[d_1^{l_1} = d_2^{l_2} = \cdots = d_n^{l_n}.\]

Proof. Using induction, we may assume that there exist $a_i, 1 \leq i \leq n - 1$, and $b_i, 2 \leq i \leq n$, such that
\[d_1^{a_1} = d_2^{a_2} = \cdots = d_n^{a_{n-1}}\]
and
\[d_2^{b_2} = d_3^{b_3} = \cdots = d_n^{b_n},\]
implying that
\[d_1^{a_1b_2} = d_2^{a_2b_2} = \cdots = d_n^{a_{n-1}b_2}\]
and
\[d_2^{b_2a_2} = d_3^{b_3a_2} = \cdots = d_n^{b_na_2}.\]
Therefore,
\[d_1^{a_1b_2} = d_2^{b_2a_2} = d_3^{b_3a_2} = \cdots = d_n^{b_na_2}.\] □
Proof of Theorem 1.1 and Corollary 1.1. For any rational functions \( A \) and \( X_1, X_2, \ldots, X_n \) satisfying (8) the corresponding functions (7) satisfy system (14). In particular, for any pair \( i, j \) \( 1 \leq i, j \leq n, i \neq j \), the equalities

\[
F_i \circ F_i = F_i \circ F_j, \quad F_j \circ F_j = F_j \circ F_i, \quad 1 \leq i, j \leq n,
\]

hold, implying that the functions \( f_i, f_j \) share the measure of maximal entropy. Therefore, all \( f_1, f_2, \ldots, f_n \) share the measure of maximal entropy.

In the other direction, if \( \mu_{f_1} = \mu_{f_2} = \cdots = \mu_{f_n} \), then using the criterion (4) we can find integers \( n_{i,j}, 1 \leq i, j \leq n, i \neq j \), such that

\[
\begin{align*}
&f_{i}^{m_{i,j}} \circ f_{j}^{m_{i,j}} = f_{i}^{m_{i,j}} \circ f_{j}^{m_{i,j}}, \\
&f_{j}^{m_{i,j}} \circ f_{j}^{m_{i,j}} = f_{j}^{m_{i,j}} \circ f_{j}^{m_{i,j}},
\end{align*}
\]

which, under different assumptions on \( A, B \) and \( X, Y \), has been studied in many papers (see e. g. [1], [4], [10], [12], [17], [19], [20], [22], [26]). Nevertheless, to our best knowledge precisely equation (5) was the subject of only two papers. One of them is the paper of Avanzi and Zannier cited in the introduction. The other one is the paper [28] by Ritt, written eighty years earlier, where some partial results were obtained. In particular, Ritt observed that solutions of (5), not satisfying (6), can be obtained starting from finite subgroup of \( \text{Aut}(\mathbb{C}P^1) \). Consider, for example,
the dihedral group $D_{2n}$ generated by $z \to 1/z$ and $z \to \varepsilon z$, where $\varepsilon = e^{\frac{2\pi i}{n}}$, and the corresponding invariant rational function

$$Z_n = \frac{1}{2} \left( z^n + \frac{1}{z^n} \right).$$

Since the function $Z_n$ can be decomposed as $Z_n = T_n \circ Z_1$, the equality $Z_n = Z_n \circ \varepsilon z$ gives rise to the solution

$$T_n \circ Z_1 = T_n \circ (Z_1 \circ \varepsilon z),$$

of (5). Ritt also constructed solutions of (5) using rational functions arising from the formulas for the period transformations of the Weierstrass functions $\wp(z)$ for lattices with symmetries of order greater than two.

In this note we do not make an attempt to obtain an explicit classification of solutions of (5) in spirit of [5]. Instead, we prove two general results which emphasize the role of symmetries in the problem.

Theorem 3.1. Let $A$ be a rational function and $\varphi, \psi$ functions meromorphic on $\mathbb{C}$ such that

$$A \circ \varphi = A \circ \psi.$$ 

Assume in addition that the algebraic curve $\mathcal{C}_A$ is irreducible. Then the desingularization $R$ of $\mathcal{C}_A$ has genus zero or one and there exist holomorphic functions $\varphi_1 : R \to \mathbb{C}P^1$, $\psi_1 : R \to \mathbb{C}P^1$ and $h : \mathbb{C} \to R$ such that

$$\varphi = \varphi_1 \circ h, \quad \psi = \psi_1 \circ h,$$

and the map from $R$ to $\mathcal{C}_A$ given by $z \to (\varphi_1(z), \psi_1(z))$ is generically one-to-one. Moreover,

$$\varphi_1 = \psi_1 \circ \mu$$

for some involution $\mu : R \to R$.

Proof. The first conclusion of the theorem holds for any parametrization of an algebraic curve by functions meromorphic on $\mathbb{C}$ (see e. g. [9], Theorem 1 and Theorem 2), so we only must show the existence of an involution $\mu$ satisfying (20).

Since the equation of $\mathcal{C}_A$ is invariant under the exchange of variable, along with the meromorphic parametrization $z \to (\varphi_1, \psi_1)$ the curve $\mathcal{C}_A$ admits the meromorphic parametrization $z \to (\psi_1, \varphi_1)$. Since the desingularization $R$ is defined up to an automorphism, it follows now from the first part of the theorem that

$$\varphi_1 = \psi_1 \circ \mu, \quad \psi_1 = \varphi_1 \circ \mu$$

for some $\mu \in \text{Aut}(R)$, implying that

$$\varphi_1 = \varphi_1 \circ (\mu \circ \mu), \quad \psi_1 = \psi_1 \circ (\mu \circ \mu).$$

Finally, $\mu \circ \mu = z$ since otherwise (21) contradicts to the condition that the map $z \to (\varphi_1(z), \psi_1(z))$ is generically one-to-one.

Theorem 3.2. Let $A$ be a rational function of degree $d$ and $\varphi_1, \varphi_2, \ldots, \varphi_n$ distinct meromorphic functions on $\mathbb{C}$ such that

$$A \circ \varphi_1 = A \circ \varphi_2 = \cdots = A \circ \varphi_n.$$

Then $n \leq d$. Moreover, if $n = d$, then the Galois closure of the field extension $\mathbb{C}(z)/\mathbb{C}(A)$ has genus zero or one.
Proof. Since for any $z_0 \in \mathbb{C}P^1$ the preimage $A^{-1}(z_0)$ contains at most $d$ distinct points, if (22) holds for $n > d$, then for every $z \in \mathbb{C}P^1$ at most $d$ of the values $\varphi_1(z), \varphi_2(z), \ldots, \varphi_n(z)$ are distinct, implying that at most $d$ of the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are distinct.

The second part of the theorem is the “if” part of the following criterion (see [23], Theorem 2.3). For a rational function $A$ of degree $d$, the Galois closure of the field extension $\mathbb{C}(z)/\mathbb{C}(A)$ has genus zero or one if and only if there exist $d$ distinct functions $\psi_1, \psi_2, \ldots, \psi_d$ meromorphic on $\mathbb{C}$ such that

$$A \circ \psi_1 = A \circ \psi_2 = \cdots = A \circ \psi_d.$$  

Note that rational functions $A$ for which the genus $g_A$ of the Galois closure of the field extension $\mathbb{C}(z)/\mathbb{C}(A)$ is zero can be listed explicitly, while functions with $g_A = 1$ admit a simple geometric description (see [21]). The simplest examples of rational functions with $g_A \leq 1$ are $z^n, T_n, Z_n$, and Lattès maps.

References

20. F. Pakovich, On the equation $P(f)=Q(g)$, where $P,Q$ are polynomials and $f,g$ are entire functions, Amer. Journal of Math., 132 (2010), no. 6, 1591-1607.


