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RESEARCH CONTRIBUTION

On Rational Functions Sharing the Measure of Maximal Entropy

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Abstract
We show that describing rational functions $f_1, f_2, \ldots, f_n$ sharing the measure of maximal entropy reduces to describing solutions of the functional equation $A \circ X_1 = A \circ X_2 = \cdots = A \circ X_n$ in rational functions. We also provide some results about solutions of this equation.

Keywords Measure of maximal entropy · Rational functions · Functional equations

1 Introduction

Let f be a rational function of degree $d \geq 2$ on $\mathbb{CP}^1$. It was proved by Freire et al. (1983), and independently by Ljubich (1983) that there exists a unique probability measure μ_f on $\mathbb{CP}^1$, which is invariant under f, has support equal to the Julia set $J(f)$ of f, and achieves maximal entropy $\log d$ among all f-invariant probability measures. In this note, we study rational functions sharing the measure of maximal entropy, that is rational functions f and g such that $\mu_f = \mu_g$, and more generally rational functions $f_1, f_2, \ldots, f_n$ such that $\mu_{f_1} = \mu_{f_2} = \cdots = \mu_{f_n}$. We assume that considered functions are *non-special* in the following sense: they are neither Lattès maps nor conjugate to $z^{\pm n}$ or $\pm T_n$.

In case if f and g are polynomials, the condition $\mu_f = \mu_g$ is equivalent to the condition $J(f) = J(g)$. In turn, for non-special polynomials f and g the equality $J(f) = J(g) = J$ holds if and only if there exists a polynomial h such that $J(h) = J$

To Misha Lyubich, on the occasion of his 60th birthday.

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and

$$f = \eta_1 \circ h^{\circ s}, \quad g = \eta_2 \circ h^{\circ t} \tag{1}$$

for some integers $s, t \geq 1$ and rotational symmetries η_1, η_2 of J (see Atela and Hu 1996; Schmidt and Steinmetz 1995, and also Atela 1998; Baker and Eremenko 1987; Beardon 1990 for other related results). Note that a similar conclusion remains true if instead of the condition $J(f) = J(g)$ one were to assume only that f and g share a completely invariant compact set in $\mathbb{C}$ (see Pakovich 2008). Note also that in the polynomial case any of the conditions $J(f) = J(g)$ and (1) is equivalent to the condition that

$$f^{\circ k} = \eta \circ g^{\circ l}, \tag{2}$$

for some integers $k, l \geq 1$ and Möbius transformation η such that $\eta(J(g)) = J(g)$.

Since $\mu_f = \mu_{f^{\circ k}}$, the equality $\mu_f = \mu_g$ holds whenever f and g share an iterate, that is satisfy

$$f^{\circ k} = g^{\circ l} \tag{3}$$

for some integers $k, l \geq 1$. Moreover, $\mu_f = \mu_g$ whenever f and g commute. However, the latter condition in fact is a particular case of the former one, since non-special commuting f and g always satisfy (3) by the result of Ritt (1923). Note that in distinction with the polynomial case rational solutions of (3) not necessarily have the form (1) (see Ritt 1923; Pakovich 2019b).

The problem of describing rational functions f and g with $\mu_f = \mu_g$ can be expressed in algebraic terms. Specifically, the results of Levin (1990) and Levin and Przytycki (1997) imply that for non-special f and g the equality $\mu_f = \mu_g$ holds if and only if some of their iterates $F = f^{\circ k}$ and $G = g^{\circ l}$ satisfy the system of functional equations

$$F \circ F = F \circ G, \quad G \circ G = G \circ F \tag{4}$$

(see Ye 2015 for more detail).

Examples of rational functions f, g with $\mu_f = \mu_g$, which do not have the form (2), were constructed by Ye (2015). These examples are based on the following remarkable observation: if X, Y, and A are rational functions such that

$$A \circ X = A \circ Y, \tag{5}$$

then the functions

$$F = X \circ A, \quad G = Y \circ A$$

satisfy (4). The simplest examples of solutions of (5) can be obtained from rational functions satisfying $A \circ \eta = A$ for some Möbius transformation η, by setting

$$X = \eta \circ Y. \tag{6}$$

In this case, the corresponding solutions of (4) have the form (2). However, other solutions of (5) also exist, allowing to construct solutions of (4) which do not have the form (2).

Roughly speaking, the main result of this note states that in fact *all* solutions of (4) can be obtained from solutions of (5). More generally, the following statement holds.

Theorem 1.1 *Let $f_1, f_2, \ldots, f_n$ be non-special rational functions of degree at least two on $\mathbb{CP}^1$. Then they share the measure of maximal entropy if and only if some of their iterates $F_1, F_2, \ldots, F_n$ can be represented in the form*

$$F_1 = X_1 \circ A, \quad F_2 = X_2 \circ A, \ldots, F_n = X_n \circ A, \tag{7}$$

where A and $X_1, X_2, \ldots, X_n$ are rational functions such that

$$A \circ X_1 = A \circ X_2 = \cdots = A \circ X_n \tag{8}$$

and $\mathbb{C}(X_1, X_2, \ldots, X_n) = \mathbb{C}(z)$.

Theorem 1.1 shows that "up to iterates" describing pairs of rational functions f and g with $\mu_f = \mu_g$ reduces to describing solutions of (5). In particular, since polynomial solutions of (5) satisfy (6), we immediately recover the result that polynomials f, g with $\mu_f = \mu_g$ satisfy (2). Nevertheless, the problem of describing solutions of (5) for arbitrary rational A, X, Y is still widely open. In fact, a complete description of solutions of (5) is obtained only in the case where A is a polynomial (while X and Y can be arbitrary rational functions) in the paper by Avanzi and Zannier (2003). The approach of Avanzi and Zannier (2003) is based on describing polynomials A for which the genus of an irreducible algebraic curve

$$C_A : \frac{A(x) - A(y)}{x - y} = 0 \tag{9}$$

is zero, and analyzing situations where C_A is reducible but has a component of genus zero. Although the same strategy can be applied to an arbitrary rational function A, both its stages become much more complicated and no general results are known to date.

Note that the problem of describing solutions of equation (5) for rational A and *meromorphic* on the complex plane X, Y was posed in the paper of Lyubich and Minsky (see Lyubich and Minsky 1997, p. 83) in the context of studying the action of rational functions on the "universal space" of non-constant functions meromorphic on $\mathbb{C}$. In algebraic terms, the last problem is equivalent to describing rational functions A such that (9) has a component of genus zero *or one*.

Theorem 1.1 implies an interesting corollary, concerning dynamical characteristics of rational functions sharing the measure of maximal entropy. Recall that the *multiplier spectrum* of a rational function f of degree d is a function which assigns to each $s \geq 1$ the unordered list of multipliers at all $d^s + 1$ fixed points of $f^{\circ s}$ taken with appropriate multiplicity. Two rational functions are called *isospectral* if they have the same multiplier spectrum.

Corollary 1.1 *If non-special rational functions $f_1, f_2, \dots, f_n$ of degree at least two share the measure of maximal entropy, then some of their iterates $F_1, F_2, \dots, F_n$ are isospectral.*

The rest of this note is organized as follows. In the second section, we prove Theorem 1.1 and Corollary 1.1. Then, in the third section, we prove two results concerning equation (5) and system (8). The first result states that if the curve C_A is irreducible and rational functions X, Y provide a generically one-to-one parametrization of C_A, then $X = Y \circ \eta$ for some involution $\eta \in Aut(\mathbb{CP}^1)$. The second result states that if A and $X_1, X_2, \dots, X_n$ are rational functions such that (8) holds and $X_1, X_2, \dots, X_n$ are distinct, then $n \leq \deg A$, and $n = \deg A$ only if the Galois closure of the field extension $\mathbb{C}(z)/\mathbb{C}(A)$ has genus zero or one. In fact, we prove these results in the more general setting, allowing the functions X, Y and $X_1, X_2, \dots, X_n$ to be meromorphic on $\mathbb{C}$.

2 Functions sharing the measure of maximal entropy

In this section, we deduce Theorem 1.1 and Corollary 1.1 from the criterion (4) and the following four lemmas.

Lemma 2.1 *Let $A_1, A_2, \dots, A_n$ and $Y_1, Y_2, \dots, Y_n$ be rational functions such that*

$$A_i \circ Y_1 = A_i \circ Y_2 = \cdots = A_i \circ Y_n, \quad i = 1, \dots n, \tag{10}$$

and

$$\mathbb{C}(A_1, A_2, \dots, A_n) = \mathbb{C}(z). \tag{11}$$

Then

$$Y_1 = Y_2 = \cdots = Y_n. \tag{12}$$

***Proof*.** By (11), there exists a rational function $P \in \mathbb{C}(z_1, z_2, \dots, z_n)$ such that

$$z = P(A_1, A_2, \dots, A_n),$$

implying that

$$Y_j = P(A_1 \circ Y_j, A_2 \circ Y_j, \dots, A_n \circ Y_j), \quad 1 \leq j \leq n. \tag{13}$$

Now (12) follows from (13) and (10). □

Lemma 2.2 *Let $F_1, F_2, \ldots, F_n$ be rational functions such that*

$$F_i \circ F_1 = F_i \circ F_2 = \cdots = F_i \circ F_n, \quad i = 1, \ldots n. \tag{14}$$

Then there exist rational functions A and $X_1, X_2, \ldots, X_n$ such that

$$F_i = X_i \circ A, \quad i = 1, \ldots n, \tag{15}$$

$$\mathbb{C}(X_1, X_2, \ldots, X_n) = \mathbb{C}(z), \tag{16}$$

and

$$A \circ X_1 = A \circ X_2 = \cdots = A \circ X_n. \tag{17}$$

Proof. By the Lüroth theorem,

$$\mathbb{C}(F_1, F_2, \ldots, F_n) = \mathbb{C}(A)$$

for some rational function A, implying that equalities (15) hold for some rational functions $X_1, X_2, \ldots, X_n$ satisfying (16). Substituting now (15) in (14) we see that

$$X_i \circ (A \circ X_1) = X_i \circ (A \circ X_2) = \cdots = X_i \circ (A \circ X_n), \quad i = 1, \ldots n.$$

Applying now Lemma 2.1 to the last system we obtain (17). □

Lemma 2.3 *Let A and B be rational functions such that the equality*

$$A \circ A = A \circ B$$

holds. Then

$$A^{\circ l} \circ A^{\circ l} = A^{\circ l} \circ B^{\circ l}$$

for any $l \geq 1$.

Proof. The proof is by induction on l. Assuming that the lemma is true for $l = k$, we have:

$$A^{\circ(k+1)} \circ B^{\circ(k+1)} = A^{\circ k} \circ (A \circ B) \circ B^{\circ k} = A^{\circ k} \circ A^{\circ 2} \circ B^{\circ k} =$$
$$= A^{\circ 2} \circ A^{\circ k} \circ B^{\circ k} = A^{\circ 2} \circ A^{\circ 2k} = A^{2k+2}.$$
□

Lemma 2.4 *Let $d_i \geq 2$, $1 \leq i \leq n$, and $n_{i,j} \geq 1$, $1 \leq i, j \leq n$, $i \neq j$, be integers such that*

$$d_i^{n_{i,j}} = d_j^{n_{j,i}}, \quad 1 \leq i, j \leq n, \quad i \neq j.$$

Then there exist integers $l_i \geq 1$, $1 \leq i \leq n$, such that

$$d_1^{l_1} = d_2^{l_2} = \cdots = d_n^{l_n}.$$

***Proof*.** The proof is by induction on n. For $n = 2$, we obviously can set

$$l_1 = n_{1,2}, \quad l_2 = n_{2,1}.$$

Assuming that the lemma is true for $n = k$, we can find integers a_i, $1 \leq i \leq k$, and b_i, $2 \leq i \leq k+1$, such that

$$d_1^{a_1} = d_2^{a_2} = \cdots = d_k^{a_k}$$

and

$$d_2^{b_2} = d_3^{b_3} = \cdots = d_{k+1}^{b_{k+1}},$$

implying that

$$d_1^{a_1 b_2} = d_2^{a_2 b_2} = \cdots = d_k^{a_k b_2}$$

and

$$d_2^{b_2 a_2} = d_3^{b_3 a_2} = \cdots = d_{k+1}^{b_{k+1} a_2}.$$

Therefore,

$$d_1^{a_1 b_2} = d_2^{b_2 a_2} = d_3^{b_3 a_2} = \cdots = d_{k+1}^{b_{k+1} a_2},$$

and hence the lemma is true for $n = k+1$. $\square$

Proof of Theorem 1.1 For any rational functions A and $X_1, X_2, \ldots, X_n$ satisfying (8) the corresponding functions (7) satisfy system (14). In particular, for any pair i, j $1 \leq i, j \leq n$, $i \neq j$, the equalities

$$F_i \circ F_i = F_i \circ F_j, \quad F_j \circ F_j = F_j \circ F_i, \quad 1 \leq i, j \leq n,$$

hold, implying that the functions f_i, f_j share the measure of maximal entropy. Therefore, all $f_1, f_2, \ldots, f_n$ share the measure of maximal entropy.

In the other direction, if $\mu_{f_1} = \mu_{f_2} = \cdots = \mu_{f_n}$, then using the criterion (4) we can find integers $n_{i,j}$, $1 \leq i, j \leq n$, $i \neq j$, such that

$$f_i^{\circ n_{i,j}} \circ f_i^{\circ n_{i,j}} = f_i^{\circ n_{i,j}} \circ f_j^{\circ n_{j,i}}, \quad f_j^{\circ n_{j,i}} \circ f_j^{\circ n_{j,i}} = f_j^{\circ n_{j,i}} \circ f_i^{\circ n_{i,j}}. \tag{18}$$

Suppose first that

$$\deg f_1 = \deg f_2 = \cdots = \deg f_n. \tag{19}$$

Then (18) and (19) imply that $n_{i,j} = n_{j,i}$, $1 \le i, j \le n$. Applying now Lemma 2.3 to (18), we see that for any integer number M divisible by all the numbers $n_{i,j}$, $1 \le i, j \le n$, the equalities

$$f_i^{\circ M} \circ f_i^{\circ M} = f_i^{\circ M} \circ f_j^{\circ M}, \quad 1 \le i, j \le n,$$

hold. Thus, the functions $F_i = f_i^{\circ M}$, $1 \le i \le n$, satisfy system (14), implying by Lemma 2.2 that equalities (15), (16), and (17) hold.

For arbitrary rational functions $f_1, f_2, \dots, f_n$ sharing the measure of maximal entropy, we still can write system (18), implying that

$$(\deg f_i)^{n_{i,j}} = (\deg f_j)^{n_{j,i}}, \quad 1 \le i, j \le n, \quad i \ne j.$$

Applying Lemma 2.4, we can find l_i, $1 \le i \le n$, such that the rational functions $f_i^{\circ l_i}$, $1 \le i \le n$, have the same degree. Since these functions along with the functions $f_1, f_2, \dots, f_n$ share the measure of maximal entropy, we can write system (18) for these functions. Using now the already proved part of the theorem, we conclude that there exist m_i, $1 \le i \le n$, such that the rational functions $F_i = f_i^{\circ m_i}$, $1 \le i \le n$, satisfy (14), implying (15), (16), and (17). □

Proof of Corollary 1.1 The corollary follows from the statement of the theorem and the fact that for any rational functions U and V the rational functions $U \circ V$ and $V \circ U$ are isospectral (see Pakovich 2019a, Lemma 2.1). □

3 Functional equation $A(\phi) = A(\psi)$

Equation (5) is a particular case of the functional equation

$$A \circ X = B \circ Y,$$

which, under different assumptions on A, B and X, Y, has been studied in many papers (see e.g. An and Diep 2013; Avanzi and Zannier 2001; Bilu and Tichy 2000; Fried 1973; Ng and Wang 2013; Pakovich 2009, 2010, 2018b; Ritt 1922). Nevertheless, to our best knowledge precisely equation (5) was the subject of only two papers. One of them is the paper of Avanzi and Zannier cited in the introduction. The other one is the paper by Ritt (1924), written 80 years earlier, where some partial results were obtained. In particular, Ritt observed that solutions of (5) with $X \ne Y$ can be obtained using finite subgroups of $Aut(\mathbb{CP}^1)$ as follows. Let Γ be a finite subgroup of $Aut(\mathbb{CP}^1)$ and θ_Γ its invariant function, that is a rational function such that $\theta_\Gamma(x) = \theta_\Gamma(y)$ if and only if $y = \sigma(x)$ for some $\sigma \in \Gamma$. Then for any subgroup $\Gamma' \subset \Gamma$ the equality

$$\theta_\Gamma = \psi \circ \theta_{\Gamma'} \tag{20}$$

holds for some $\psi \in \mathbb{C}(z)$, implying that

$$\psi \circ \theta_{\Gamma'} = \psi \circ (\theta_{\Gamma'} \circ \sigma)$$

for every $\sigma \in \Gamma$. Nevertheless, $\theta_{\Gamma'} \neq \theta_{\Gamma'} \circ \sigma$ unless $\sigma \in \Gamma'$. For example, for the dihedral group D_{2n}, generated by $z \to 1/z$ and $z \to \varepsilon z$, where $\varepsilon = e^{\frac{2\pi i}{n}}$, and its subgroup D_2 equality (20) takes the form

$$\frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = T_n \circ \frac{1}{2}\left(z + \frac{1}{z}\right)$$

giving rise to the solution

$$T_n \circ \frac{1}{2}\left(z + \frac{1}{z}\right) = T_n \circ \frac{1}{2}\left(\varepsilon z + \frac{1}{\varepsilon z}\right)$$

of (5) not satisfying to (6). Ritt also constructed solutions of (5) using rational functions arising from the formulas for the period transformations of the Weierstrass functions $\wp(z)$ for lattices with symmetries of order greater than two.

In this note, we do not make an attempt to obtain an explicit classification of solutions of (5) in spirit of Avanzi and Zannier (2003). Instead, we prove two general results which emphasize the role of symmetries in the problem.

Theorem 3.1 *Let A be a rational function and φ, ψ distinct functions meromorphic on $\mathbb{C}$ such that*

$$A \circ \varphi = A \circ \psi.$$

Assume in addition that the algebraic curve C_A is irreducible. Then the desingularization R of C_A has genus zero or one and there exist holomorphic functions $\varphi_1 : R \to \mathbb{CP}^1$, $\psi_1 : R \to \mathbb{CP}^1$ and $h : \mathbb{C} \to R$ such that

$$\varphi = \varphi_1 \circ h, \quad \psi = \psi_1 \circ h,$$

and the map from R to C_A given by $z \to (\varphi_1(z), \psi_1(z))$ is generically one-to-one. Moreover,

$$\varphi_1 = \psi_1 \circ \eta \tag{21}$$

for some involution $\eta : R \to R$.

Proof. The first conclusion of the theorem holds for any parametrization of an algebraic curve by functions meromorphic on $\mathbb{C}$ (see e.g. Beardon and Ng 2006, Theorem 1 and Theorem 2), so we only must show the existence of an involution μ satisfying (21).

Since the equation of C_A is invariant under the exchange of variable, along with the meromorphic parametrization $z \to (\varphi_1, \psi_1)$ the curve C_A admits the meromorphic parametrization $z \to (\psi_1, \varphi_1)$. Since the desingularization R is defined up to an automorphism, it follows now from the first part of the theorem that

$$\varphi_1 = \psi_1 \circ \eta, \quad \psi_1 = \varphi_1 \circ \eta$$

for some $\eta \in Aut(R)$, implying that

$$\varphi_1 = \varphi_1 \circ (\eta \circ \eta), \quad \psi_1 = \psi_1 \circ (\eta \circ \eta). \tag{22}$$

Finally, $\eta \circ \eta = z$ since otherwise (22) contradicts to the condition that the map $z \to (\varphi_1(z), \psi_1(z))$ is generically one-to-one. □

Theorem 3.2 *Let A be a rational function of degree d and $\varphi_1, \varphi_2, \ldots, \varphi_n$ distinct meromorphic functions on $\mathbb{C}$ such that*

$$A \circ \varphi_1 = A \circ \varphi_2 = \cdots = A \circ \varphi_n. \tag{23}$$

Then $n \leq d$. Moreover, if $n = d$, then the Galois closure of the field extension $\mathbb{C}(z)/\mathbb{C}(A)$ has genus zero or one.

Proof. Since for any $z_0 \in \mathbb{CP}^1$ the preimage $A^{-1}(z_0)$ contains at most d distinct points, if (23) holds for $n > d$, then for every $z \in \mathbb{CP}^1$ at most d of the values $\varphi_1(z), \varphi_2(z), \ldots, \varphi_n(z)$ are distinct, implying that at most d of the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are distinct.

The second part of the theorem is the “if” part of the following criterion (see Pakovich 2018c, Theorem 2.3). For a rational function A of degree d, the Galois closure of the field extension $\mathbb{C}(z)/\mathbb{C}(A)$ has genus zero or one if and only if there exist d distinct functions $\psi_1, \psi_2, \ldots, \psi_d$ meromorphic on $\mathbb{C}$ such that

$$A \circ \psi_1 = A \circ \psi_2 = \cdots = A \circ \psi_d.$$ □

Note that rational functions A for which the genus g_A of the Galois closure of the field extension $\mathbb{C}(z)/\mathbb{C}(A)$ is zero are exactly all possible “compositional left factors” of Galois coverings of $\mathbb{CP}^1$ by $\mathbb{CP}^1$ and can be listed explicitly. On the other hand, functions with $g_A = 1$ admit a simple geometric description in terms of projections of maps between elliptic curves (see Pakovich 2018a). The simplest examples of rational functions with $g_A \leq 1$ are z^n, T_n, $\frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$, and Lattès maps.

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