

ON RATIONAL FUNCTIONS ORTHOGONAL TO ALL POWERS OF A GIVEN RATIONAL FUNCTION ON A CURVE

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ABSTRACT. In this paper we study the generating function $f(t)$ for the sequence of the moments $\int_{\gamma} P^i(z)q(z)dz$, $i \geq 0$, where $P(z)$, $q(z)$ are rational functions of one complex variable and γ is a curve in \mathbb{C} . We calculate an analytical expression for $f(t)$ and provide conditions implying that $f(t)$ is rational or vanishes identically. In particular, for $P(z)$ in generic position we give an explicit criterion for a function $q(z)$ to be orthogonal to all powers of $P(z)$ on γ . As an application, we prove a stronger form of the Wermer theorem, describing analytic functions satisfying the system of equations $\int_{S^1} h^i(z)g^j(z)g'(z)dz = 0$, $i \geq 0$, $j \geq 0$, in the case where the functions $h(z)$, $g(z)$ are rational. We also generalize the theorem of Duistermaat and van der Kallen about Laurent polynomials $L(z)$ whose integer positive powers have no constant term, and prove other results about Laurent polynomials $L(z)$, $m(z)$ satisfying $\int_{S^1} L^i(z)m(z)dz = 0$, $i \geq i_0$.

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1. INTRODUCTION

In this paper we study the generating function $f(t) = \sum_{i=0}^{\infty} m_i t^i$ for the sequence of the moments

$$m_i = \int_{\gamma} P^i(z)q(z)dz, \quad i \geq 0, \quad (1)$$

where $P(z)$, $q(z)$ are rational functions of one complex variable and γ is a curve in \mathbb{C} . In particular, we study conditions under which $f(t)$ is a rational function, a polynomial, or an identical zero.

Our main motivation for such a study is related to differential equations and may be described as follows. Let $F(x, y)$, $G(x, y)$ be real-valued analytical functions vanishing at the origin together with their first derivatives. Then the classical

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Poincaré problem is to find conditions under which all solutions of the system

$$\begin{cases} \dot{x} = -y + F(x, y), \\ \dot{y} = x + G(x, y), \end{cases} \quad (2)$$

around zero are closed (see e.g. the recent survey [11] and the bibliography therein). This problem remains wide open even in the case where $F(x, y)$ and $G(x, y)$ are polynomials of degree 3, and any advances in its understanding are of a great interest.

It is known [9] that if $F(x, y)$, $G(x, y)$ are homogeneous polynomials of the same degree, then one can construct trigonometric polynomials $f(\cos \varphi, \sin \varphi)$, $g(\cos \varphi, \sin \varphi)$ such that (2) has a center if and only if all solutions of the trigonometric Abel equation

$$\frac{dr}{d\varphi} = f(\cos \varphi, \sin \varphi) r^2 + g(\cos \varphi, \sin \varphi) r^3 \quad (3)$$

with $r(0)$ small enough are periodic on $[0, 2\pi]$. In its turn, the trigonometric Abel equation can be transformed by an exponential substitution into the equation

$$\frac{dy}{dz} = l(z)y^2 + m(z)y^3, \quad (4)$$

where $l(z)$ and $m(z)$ are Laurent polynomials. Furthermore, all solutions of (3) with $r(0)$ small enough are periodic on $[0, 2\pi]$ if and only if all solutions of (4) with $y(1)$ small enough are non-ramified along S^1 .

In the series of papers [3]–[6] the following modification of the center problem for equation (4) was proposed: find conditions under which for any solution $y(z)$ of the Abel differential equation

$$\frac{dy}{dz} = p(z)y^2 + q(z)y^3 \quad (5)$$

with *polynomial* coefficients $p(z)$, $q(z)$ the equality $y(1) = y(0)$ holds whenever $y(0)$ is small enough. This modification seems to be easier than the initial problem at the same time keeping its main features, and in this context can be considered as a simplified form of the classical center-focus problem of Poincaré.

The center problem for Abel equation (5) naturally leads to the following “polynomial moment problem”, which is a prototype of problems considered in this paper: find conditions under which polynomials $P = \int p(z)dz$ and $Q = \int q(z)dz$ satisfy the system of equations

$$\int_0^1 P^i(z)dQ(z) = 0, \quad i \geq 0, \quad (6)$$

or, in other words, find conditions implying that the corresponding generating function of moments vanishes identically (for a detailed description of relations between the center problem for Abel equation (5) and the polynomial moment problem see [8] and the bibliography therein).

The polynomial moment problem has been studied in the papers [3]–[7], [10], [23]–[27], [35] and was completely solved in the recent papers [31], [29]. The solution

involves the following “composition condition” imposed on $P(z)$ and $Q(z)$: there exist polynomials \tilde{P}, \tilde{Q}, W such that

$$P = \tilde{P}(W(z)), \quad Q = \tilde{Q}(W(z)), \quad W(0) = W(1). \tag{7}$$

It is easy to see using the change $z \rightarrow W(z)$ that (7) implies (6) and it was shown in [31] that if polynomials $P(z), Q(z)$ satisfy (6), then there exist polynomials $Q_j(z)$ such that $Q(z) = \sum_j Q_j(z)$ and

$$P(z) = \tilde{P}_j(W_j(z)), \quad Q_j(z) = \tilde{Q}_j(W_j(z)), \quad W_j(0) = W_j(1) \tag{8}$$

for some polynomials $\tilde{P}_j(z), \tilde{Q}_j(z), W_j(z)$. Moreover, in [29] polynomial solutions of (8) were described in a very explicit form suitable for applications.

In the same way as the center problem for Abel equation (5) leads to the polynomial moment problem, the original center problem for equations (3), (4) leads to the moment problem for trigonometric polynomials or, more generally, to the following “Laurent polynomial moment problem”, which is the main motivation for investigations of this paper: describe Laurent polynomials $L(z), m(z)$ such that

$$\int_{S^1} L^i(z)m(z)dz = 0, \quad i \geq 0. \tag{9}$$

An analogue of condition (7) in this setting is that there exist a Laurent polynomial $W(z)$ and polynomials $\tilde{L}(z), \tilde{m}(z)$ such that the equalities

$$L(z) = \tilde{L}(W(z)), \quad m(z) = \tilde{m}(W(z))W'(z) \tag{10}$$

hold. Clearly, (10) implies (9). Furthermore, if for given $L(z)$ there exist several such $m(z)$, then (9) is satisfied for their sum. However, in distinction with the polynomial moment problem other mechanisms for (9) to be satisfied also exist. For example, if $L(z) = \tilde{L}(z^d)$ for some $d > 1$, then the residue calculation shows that condition (9) is satisfied for any Laurent polynomial $m(z)$ containing no terms cz^n with $n \equiv -1 \pmod{d}$.

Notice that questions concerning the function $f(t)$ appear also in domains not related to differential equations, providing additional motivations for investigations of this paper. Let us mention for example the following particular case of the Mathieu conjecture concerning compact Lie groups [21] proved by Duistermaat and van der Kallen [14]: if all integer positive powers of a Laurent polynomial $L(z)$ have no constant term, then $L(z)$ is either a polynomial in z , or a polynomial in $1/z$. Clearly, the assumption of this theorem is equivalent to the condition that the function $f(t)$ for $P(z) = L(z), m(z) = 1/z$, and $\gamma = S^1$ is a constant. Another related question is the problem of description of polynomials $P(z), q(z)$ for which the function $f(t)$ is equal to a constant, and not just to zero as in the polynomial moment problem, which was recently raised by Zhao [41] in connection with his conjecture about images of commuting differential operators. Finally, notice that the classical Wermer theorem [36], [37] describing analytic functions on S^1 satisfying

$$\int_{S^1} h^i(z)g^j(z)g'(z)dz = 0, \quad i, j \geq 0, \tag{11}$$

in the case where the functions $h(z)$, $g(z)$ are rational, obviously is also related to the subject of this paper.

Since the paper is rather long and involves many different problems, in the next section we give a detailed description of the results obtained in each section.

2. MAIN RESULTS

2.1. In Section 3 we fix the notation and introduce basic sufficient conditions implying the rationality and the vanishing of $f(t)$. We also calculate an explicit analytical expression for $f(t)$.

We start from introducing an auxiliary function $I_\infty(t)$ which is defined near infinity by the Cauchy type integral

$$I_\infty(t) = I_\infty(q, P, \gamma, t) = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{q(z)dz}{P(z) - t}. \quad (12)$$

The calculation of the Taylor series of $I_\infty(t)$ at infinity shows that

$$I_\infty(t) = -\frac{1}{2\pi\sqrt{-1}} \frac{1}{t} f\left(\frac{1}{t}\right)$$

implying that we may study $I_\infty(t)$ instead of $f(t)$.

Then we introduce the so-called “*composition condition*” relating the functions $P(z)$ and $q(z)$. In its most general form the composition condition is defined as follows: there exist rational functions $\tilde{q}(z)$, $\tilde{P}(z)$, $W(z)$ such that $\deg W(z) > 1$ and

$$P(z) = \tilde{P}(W(z)), \quad q(z) = \tilde{q}(W(z))W'(z). \quad (13)$$

In case if the composition condition is satisfied, the change of variable $z \rightarrow W(z)$ reduces questions about the function $I_\infty(q, P, \gamma, t)$ to similar questions for the function $I_\infty(\tilde{q}, \tilde{P}, W(\gamma), t)$, which is defined by rational functions of lesser degrees. Furthermore, in certain cases the new integration path $W(\gamma)$ turns out more convenient for investigation than the initial one. Notice that if the indefinite integral $\int q(z)dz$ is a rational function, then the composition condition is equivalent to the condition that

$$P(z) = \tilde{P}(W(z)), \quad Q(z) = \tilde{Q}(W(z)) \quad (14)$$

for some rational functions $\tilde{Q}(z)$, $\tilde{P}(z)$, $W(z)$ with $\deg W(z) > 1$.

Finally, we introduce basic sufficient topological conditions for rationality and vanishing of $f(t)$. Let P be a rational function and γ be a curve. Say that poles of $P(z)$ lie “*on one side*” (resp. “*outside*”) of γ , if γ is closed and homologous to zero in \mathbb{CP}^1 (resp. in \mathbb{C}) with poles of $P(z)$ removed. It is not hard to prove that if poles of $P(z)$ lie on one side of γ , then the function $f(t)$ is rational for any $q(z)$ (Corollary 3.2). More generally, $I_\infty(t)$ is rational whenever $P(z)$ and $q(z)$ satisfy composition condition (13) and poles of the function $\tilde{P}(z)$ lie on one side of $W(\gamma)$. On the other hand, if $P(z)$ and $q(z)$ satisfy composition condition (13) and poles of both $\tilde{P}(z)$ and $\tilde{q}(z)$ lie outside of $W(\gamma)$, then $I_\infty(t)$ vanishes identically by the Cauchy theorem.

The main results of Section 3 (Theorem 2.1 and Theorem 2.4) are explicit analytical expressions for $f(t)$. In particular, these expressions imply that if γ is closed, then $f(t)$ is an algebraic function from the field K_P generated over $\mathbb{C}(t)$ by the branches $P_i^{-1}(t)$, $1 \leq i \leq n$, of the algebraic function $P^{-1}(t)$ inverse to $P(z)$. On the other hand, if γ is non-closed, then the function $f(t)$ is a linear combination of branches of the logarithm with coefficients from K_P . As an application we obtain an important necessary condition for rationality of $f(t)$ for non-closed γ (Proposition 3.5) which generalizes the conditions for vanishing of $f(t)$ obtained earlier in [24], [33], [27].

2.2. In Section 4 we give a criterion (Theorem 4.1) for rationality of $f(t)$ which is more convenient for the use than explicit formulas obtained in Section 3. Namely, we show that $f(t)$ is rational if and only if superpositions of the rational function $(q/P')(z)$ with branches of $P^{-1}(t)$ satisfy a system of equations

$$\sum_{i=1}^n f_{s,i} \left(\frac{q}{P'} \right) (P_i^{-1}(t)) = 0, \quad f_{s,i} \in \mathbb{Z}, \quad 1 \leq s \leq k, \tag{15}$$

where $f_{s,i}$ and k are calculated in an effective way and depend on $P(z)$ and γ only. This result generalizes the corresponding criterion for polynomials given in [27] and relies on similar ideas. In particular, we use combinatorial objects called “constellations” (similar to what is called “Dessins d’enfants”) which represent the monodromy group G_P of the algebraic function $P^{-1}(z)$ in a combinatorial way. Notice that expressions (15) can be interpreted as “Abelian integrals along zero-dimensional cycles” [16].

We also show (Theorem 4.1) that if $P(z)$ and $q(z)$ satisfy the conditions

$$q^{-1}\{\infty\} \subseteq P^{-1}\{\infty\}, \quad P(\infty) = \infty, \tag{16}$$

then the rationality of $f(t)$ yields that $f(t) \equiv 0$. Notice that this result implies immediately the theorem of Duistermaat and van der Kallen cited above. Indeed, if $L(z)$ is not a polynomial in z or in $1/z$, then for $P(z) = L(z)$, $q(z) = 1/z$ conditions (16) are satisfied and therefore the equality $f(t) = c$ would imply that $c = 0$ in contradiction with the fact that for $i = 0$ the integral in (9) is equal to $2\pi i$. Another corollary is the following statement, which gives the answer to the question of Zhao: if the polynomial moments in (6) vanish for all $i \geq i_0$, then they vanish for all $i \geq 0$.

2.3. In Section 5 we solve the problems of rationality and vanishing of $f(t)$ in the case where the function $P(z)$ is in “generic position”. Clearly, we can assume that such $P(z)$ is indecomposable, that is can not be represented as a composition $P(z) = \tilde{P}(W(z))$ of rational functions of lesser degrees. In particular, for such $P(z)$ the composition condition (13) is equivalent to the condition that

$$q(z) = \tilde{q}(P(z))P'(z). \tag{17}$$

Further, we may assume that $P(z)$ has only simple branch points. Under these conditions we show (Corollary 5.4) that, unless poles of $P(z)$ lie on one side of γ , the function $f(t)$ is rational if and only if the curve $P(\gamma)$ is closed and the function

$q(z)$ has the form (17). We also show (Corollary 5.7) that $f(t)$ vanishes identically if and only if poles of the function $\tilde{q}(z)$ lie outside of the curve $P(\gamma)$.

2.4. In Section 6 we prove two results which can be considered as more precise versions of the Wermer theorem [36], [37], describing analytic functions on S^1 satisfying (11), in the case where the functions $h(z), g(z)$ are rational.

Let $P(z), Q(z)$ be rational functions and γ be an arbitrary curve in \mathbb{C} . For fixed $j \geq 0$ denote by $I_j(t)$ the generating function for the sequence of the moments

$$m_i = \int_{\gamma} P^i(z)Q^j(z)Q'(z) dz, \quad i \geq 0. \tag{18}$$

Our first result (Theorem 6.1) states that all the functions $I_j(t), j \geq 0$, are rational if and only if composition condition (14) holds and poles of $\tilde{P}(z)$ lie on one side of the curve $W(\gamma)$. In fact, we show that it is enough to assume that $I_j(t)$ are rational for all j in the finite interval $j_0 \leq j \leq j_0 + n - 1$, where j_0 is any non-negative integer and $n = \deg P(z)$.

Our second result (Theorem 6.2) states that all the moments (18) vanish if and only if composition condition (14) holds and poles of $\tilde{P}(z)$ and $\tilde{Q}(z)$ lie on one side of the curve $W(\gamma)$. Moreover, we show that it is enough to assume that moments (18) vanish for all $i \geq i_0, j \geq j_0$, where i_0, j_0 are some non-negative integers.

2.5. In Section 7 we study the Laurent polynomial moment problem (9). In particular, we prove the following generalization of the theorem of Duistermaat and van der Kallen (Theorem 7.1): if $L(z)$ and $m(z)$ are Laurent polynomials such that the coefficient of the term $1/z$ in $m(z)$ is distinct from zero and (9) holds for all $i \geq i_0$, where $i_0 \geq 0$, then $L(z)$ is either a polynomial in z , or a polynomial in $1/z$. Notice that this result implies that if $L(z)$ is a proper Laurent polynomial (that is, not a polynomial in z or in $1/z$) and $m(z)$ is a Laurent polynomial such that (9) holds, then $M(z) = \int m(z)dz$ is a Laurent polynomial. In particular, equalities (9) may be written in the form

$$\int_{S^1} L^i(z)dM(z) = 0, \quad i \geq 0. \tag{19}$$

Another result proved in Section 4 is a convenient necessary and sufficient condition for equalities (19) to be satisfied. Namely, denoting by $M_0(z)$ the principal part of $M(z)$ at zero and by $M_{\infty}(z)$ the difference $M(z) - M_0(z)$ we prove (Theorem 7.5) that $M(z)$ satisfies (19) if and only if

$$\sum_{i \in J_0} M_{\infty}(L_i^{-1}(t)) \equiv \sum_{i \in J_{\infty}} M_0(L_i^{-1}(t)),$$

where J_0 (resp. J_{∞}) is a subset of $\{1, 2, \dots, r\}, r = \deg L(z)$, consisting of all $i \in \{1, 2, \dots, r\}$ such that for t close to infinity, $L_i^{-1}(t)$ is close to 0 (resp. to ∞). We also show that for $L(z)$ in generic position any solution $M(z)$ of (19) has the form $M(z) = \tilde{M}(L(z))$, where $\tilde{M}(z)$ is a polynomial, and prove some partial results about solutions of (9) in the case where $M(z)$ is a polynomial (Theorem 7.6 and Theorem 7.7).

2.6. In Section 8 we study the following problem: how many first integrals in (19) should vanish in order to conclude that all of them vanish. Using the fact that the corresponding function $f(t)$ is contained in the field K_L , we give a bound (Theorem 8.1) which depends on degrees of Laurent polynomial $L(z)$ and $M(z)$ only.

2.7. By the results of Section 5, for $P(z)$ in generic position and γ such that poles of $P(z)$ do not lie on one side of γ , the rationality of $f(t)$ always implies condition (13). In Section 9, using a general algebraic result of Girstmair [18], we establish a relationship between rationality of $f(t)$ and condition (13) in the general case (Theorem 9.3), and discuss an explanatory example.

3. ANALYTIC EXPRESSION FOR $I_\infty(t)$

3.1. Notation. The main object studied in this paper is the generating function

$$f(t) = \sum_{i=0}^{\infty} m_i t^i$$

for the sequence of the moments

$$m_i = \int_{\gamma} P^i(z)q(z)dz, \tag{20}$$

where $P(z), q(z)$ are rational functions considered as mappings from \mathbb{CP}^1 to \mathbb{CP}^1 and γ is an oriented piecewise-smooth curve in \mathbb{C} containing no poles of $P(z)$ or $q(z)$. We set $n = \deg P(z)$ and always will assume that $n > 0, q(z) \not\equiv 0$ and that γ has only transversal self-intersections. We study conditions under which $f(t)$ is an identical zero, a polynomial, or a rational function. Clearly, the first two conditions are equivalent to the conditions that $m_i = 0$ for $i \geq 0$ or $m_i = 0$ for all $i \geq i_0$, while the third one is equivalent to the condition that m_i satisfy a linear recurrence relation with constant coefficients.

Actually, instead of studying the function $f(t)$ directly, we will study an auxiliary function $I_\infty(t)$ defined near infinity by the integral

$$I_\infty(t) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} \frac{q(z)dz}{P(z) - t}. \tag{21}$$

More precisely, integral (21) defines a holomorphic function in each domain of the complement of $P(\gamma)$ in \mathbb{CP}^1 and, by definition, $I_\infty(t)$ is a function defined in the domain U_∞ containing infinity. Calculating the Taylor series of $I_\infty(t)$ at infinity we see that

$$I_\infty(t) = -\frac{1}{2\pi\sqrt{-1}} \frac{1}{t} f\left(\frac{1}{t}\right). \tag{22}$$

Therefore, the study of $f(t)$ near zero is equivalent to the study of $I_\infty(t)$ near infinity and vice versa.

Under certain conditions the function $I_\infty(t) = I_\infty(q, P, \gamma, t)$ coincides with a similar function $I_\infty(\tilde{q}, \tilde{P}, \tilde{\gamma}, t)$, where $\tilde{P}(z), \tilde{q}(z)$ are rational functions of lesser

degrees. Namely, suppose that there exist rational functions $\tilde{q}(z), \tilde{P}(z), W(z)$ such that $\deg W(z) > 1$ and

$$P(z) = \tilde{P}(W(z)), \quad q(z) = \tilde{q}(W(z))W'(z). \tag{23}$$

Then changing the variable z to $W(z)$ we see that

$$I_\infty(q, P, \gamma, t) = I_\infty(\tilde{q}, \tilde{P}, W(\gamma), t), \tag{24}$$

or equivalently

$$\int_\gamma P^i(z)q(z)dz = \int_{W(\gamma)} \tilde{P}^i(z)\tilde{q}(z)dz, \quad i \geq 0. \tag{25}$$

Notice that in the case where a rational function $Q(z)$ such that $q(z) = Q'(z)$ exists, the condition (23) is equivalent to the condition that

$$P(z) = \tilde{P}(W(z)), \quad Q(z) = \tilde{Q}(W(z)) \tag{26}$$

for some rational functions $\tilde{Q}(z), \tilde{P}(z), W(z), \deg W(z) > 1$. If condition (26) is satisfied, we will say that rational functions $P(z), Q(z)$ have a *non-trivial common compositional right factor*.

Let $C_P \subset \mathbb{CP}^1$ be the set of branch points of the algebraic function $P^{-1}(t)$ inverse to $P(z)$. Throughout this paper, U will always denote a fixed simply connected subdomain of \mathbb{C} whose boundary contains the points C_P and ∞ . We also will assume that U does not contain the point $P(\infty)$. Notice that the condition $\infty \in \partial U$ implies that $U \cap U_\infty$ is not empty. Furthermore, since U is simply connected, it follows from $C_P \cap U = \emptyset$ that in U there exist n single valued analytical branches of $P^{-1}(t)$. We will denote these branches by $P_i^{-1}(t), 1 \leq i \leq n$.

Under the analytic continuation along a closed curve the set $P_i^{-1}(t), 1 \leq i \leq n$, transforms to itself and this induces a homomorphism

$$\pi_1(\mathbb{CP}^1 \setminus C_P, c) \rightarrow S_n, \quad c \in U. \tag{27}$$

The image of the group $\pi_1(\mathbb{CP}^1 \setminus C_P, c)$ under homomorphism (27) is called the monodromy group of $P(z)$. We will denote this group by G_P and use the notation $\sigma(i) = j$ for $\sigma \in G_P$ if $P_i^{-1}(t)$ transforms to $P_j^{-1}(t)$ by a preimage of σ under (27). Recall that the group G_P is permutation equivalent to the Galois group of the algebraic equation $P(t) - z = 0$ over the ground field $\mathbb{C}(z)$.

3.2. Calculation of $I_\infty(t)$ for closed γ . In this subsection we will assume that γ is a closed curve. In this case for any point $z \in \mathbb{CP}^1 \setminus \gamma$ the winding number of γ around z is well defined. We will denote this number by $\mu(\gamma, z)$. Let z_1^q, \dots, z_l^q be finite poles of $q(z)$. For $s, 1 \leq s \leq l$, denote by $q_s(z)$ the principal part of the Laurent series of $q(z)$ at z_s^q , and set

$$\psi_s(t) = \sum_{i=1}^n \left(\frac{q_s}{P^i} \right) (P_i^{-1}(t)), \quad t \in U.$$

Clearly, $\psi_s(t)$ is invariant with respect to the action of the group G_P . Therefore, by the main theorem of the Galois theory this is a rational function.

Furthermore, denote by z_1^P, \dots, z_r^P finite poles of $P(z)$ and define $J_e, 1 \leq e \leq r$, as a subset of $\{1, 2, \dots, n\}$ consisting of all $i \in \{1, 2, \dots, n\}$ such that for t close

to infinity $P_i^{-1}(t)$ is close to z_e^P . Clearly, the cardinality of J_e , $1 \leq e \leq r$, coincides with the multiplicity of $P(z)$ at z_e^P .

Theorem 3.1. *Let γ be a closed curve. Then for $t \in U \cap U_\infty$ the equality*

$$I_\infty(t) = \sum_{e=1}^r \mu(\gamma, z_e^P) \sum_{i \in J_e} \left(\frac{q}{P'} \right) (P_i^{-1}(t)) - \sum_{s=1}^l \mu(\gamma, z_s^q) \psi_s(t) + \frac{1}{2\pi\sqrt{-1}} \frac{1}{(P(\infty) - t)} \int_\gamma q(z) dz \quad (28)$$

holds.

Proof. It is enough to prove equality (28) for $t \in U \cap U_\infty$ close enough to infinity. For such t , poles of the function

$$\frac{q(z)}{P(z) - t}$$

split into two *disjointed* groups one of which contains poles of $q(z)$ while the other one contains the points $P_i^{-1}(t)$, $1 \leq i \leq n$. Moreover, restrictions imposed on U imply that the poles from the second group are simple and finite. In particular, this implies that

$$\frac{1}{P(z) - t} = \frac{1}{P(\infty) - t} + \sum_{i=1}^n \frac{1}{P'(P_i^{-1}(t))} \frac{1}{(z - P_i^{-1}(t))}. \quad (29)$$

Setting

$$W(z, t) = \sum_{i=1}^n \frac{1}{P'(P_i^{-1}(t))} \frac{1}{(z - P_i^{-1}(t))},$$

we have:

$$\begin{aligned} I_\infty(t) &= \frac{1}{2\pi\sqrt{-1}} \frac{1}{(P(\infty) - t)} \int_\gamma q(z) dz + \frac{1}{2\pi\sqrt{-1}} \int_\gamma q(z) W(z, t) dz = \\ &= \frac{1}{2\pi\sqrt{-1}} \frac{1}{(P(\infty) - t)} \int_\gamma q(z) dz + \sum_{i=1}^n \mu(\gamma, P_i^{-1}(t)) \operatorname{Res}_{P_i^{-1}(t)} \{q(z) W(z, t)\} + \\ &\quad + \sum_{s=1}^l \mu(\gamma, z_s^q) \operatorname{Res}_{z_s^q} \{q(z) W(z, t)\}. \quad (30) \end{aligned}$$

Clearly,

$$\operatorname{Res}_{P_i^{-1}(t)} \{q(z) W(z, t)\} = \left(\frac{q}{P'} \right) (P_i^{-1}(t)), \quad 1 \leq i \leq n. \quad (31)$$

On the other hand,

$$\begin{aligned} \operatorname{Res}_{z_s^q} \{q(z) W(z, t)\} &= \operatorname{Res}_{z_s^q} \{q_s(z) W(z, t)\} = \\ &= - \sum_{i=1}^n \operatorname{Res}_{P_i^{-1}(t)} \{q_s(z) W(z, t)\} - \operatorname{Res}_\infty \{q_s(z) W(z, t)\} = \\ &= -\psi_s(t) - \operatorname{Res}_\infty \{q_s(z) W(z, t)\}, \quad 1 \leq s \leq l. \quad (32) \end{aligned}$$

Furthermore, since for any $s, 1 \leq s \leq l$, both functions $q_s(z)$ and $W(z, t)$ have zeros at infinity the residue of $q_s(z)W(z, t)$ at infinity equals zero. Therefore, since for t close enough to infinity and $i \in J_e$ the equality $\mu(\gamma, P_i^{-1}(t)) = \mu(\gamma, z_e^P)$ holds, formula (28) follows from formulas (30), (31), and (32). \square

Following [33], we will say that points $x_1, x_2, \dots, x_k \in \mathbb{CP}^1$ lie on one side of a curve γ , if γ is closed and homologous to zero in $\mathbb{CP}^1 \setminus \{x_1, x_2, \dots, x_k\}$. An equivalent condition is that γ is closed and

$$\mu(\gamma, x_1) = \mu(\gamma, x_2) = \dots = \mu(\gamma, x_k). \tag{33}$$

Further, if $x_1, x_2, \dots, x_k \in \mathbb{CP}^1$ lie on one side of γ and all numbers in (33) equal zero, then we will say that x_1, x_2, \dots, x_k lie outside γ .

Corollary 3.2. *If poles of $P(z)$ lie on one side of γ , then for any rational function $q(z)$ the function $I_\infty(t)$ is rational. If poles of $P(z)$ and $q(z)$ lie outside γ , then $I_\infty(t) \equiv 0$.*

Proof. If poles of $P(z)$ and $q(z)$ lie outside γ , then it follows directly from the Cauchy theorem applied to coefficients (20) of $I_\infty(t)$ that $I_\infty(t) \equiv 0$. On the other hand, if poles of $P(z)$ lie on one side of γ , then (33) implies that the expression

$$\sum_{e=1}^r \mu(\gamma, z_e^P) \sum_{i \in J_e} \left(\frac{q}{P^i} \right) (P_i^{-1}(t))$$

in formula (28) is invariant with respect to the action of the group G_P and therefore is a rational function. Since other terms of (28) also are rational, this implies the rationality of $I_\infty(t)$. \square

3.3. Calculation of $I_\infty(t)$ for non-closed γ . In this subsection we will assume that γ is a non-closed curve with the starting point a and the ending point b .

Lemma 3.3. *Let $q(z)$ be a rational function and γ be a curve. Then the function*

$$\hat{q}(t) = \int_\gamma \frac{q(z) - q(t)}{z - t} dz \tag{34}$$

is rational.

Proof. Since $q(z)$ may be represented as a linear combination of rational functions $q(z) = (z - \beta)^{-l}$, $l \geq 0$, and monomials, it is enough to prove the lemma for such functions. If $q(z)$ is a monomial, then

$$\frac{q(z) - q(t)}{z - t}$$

is a polynomial in z, t implying that function (34) is a polynomial in t . On the other hand,

$$\frac{(z - \beta)^{-l} - (t - \beta)^{-l}}{z - t} = \frac{(t - \beta)^l - (z - \beta)^l}{(z - t)(t - \beta)^l(z - \beta)^l} = \frac{R(z, t)}{(t - \beta)^l(z - \beta)^l},$$

where $R(z, t)$ is a polynomial. Therefore, if $q(z) = (z - \beta)^{-l}$, $l \geq 0$, then function (34) is rational. \square

For fixed $t \in U$ denote by $\text{Log}_{1,i}(z - P_i^{-1}(t))$, $1 \leq i \leq n$, a branch of the logarithm defined in a neighborhood of the point $z = a$ and by $\text{Log}_{2,i}(z - P_i^{-1}(t))$, $1 \leq i \leq n$, its analytical continuation along γ to a neighborhood of b .

Theorem 3.4. *Let γ be a non-closed curve with the starting point a and the ending point b . Then in a neighborhood of $t \in U \cap U_\infty$ the equality*

$$I_\infty(t) = \frac{1}{2\pi\sqrt{-1}} \frac{1}{(P(\infty) - t)} \int_\gamma q(z) dz + \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^n \left(\frac{\widehat{q}}{P'}\right)(P_i^{-1}(t)) + \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^n \left(\frac{q}{P'}\right)(P_i^{-1}(t)) [\text{Log}_{2,i}((b - P_i^{-1}(t))) - \text{Log}_{1,i}((a - P_i^{-1}(t)))] \quad (35)$$

holds.

Proof. Since in U equality (29) holds, we have:

$$I_\infty(t) = \frac{1}{2\pi\sqrt{-1}} \frac{1}{(P(\infty) - t)} \int_\gamma q(z) dz + \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^n \frac{1}{P'(P_i^{-1}(t))} \int_\gamma \frac{q(z) dz}{z - P_i^{-1}(t)}.$$

Writing now $q(z)$ as

$$q(z) = q(z) - q(P_i^{-1}(t)) + q(P_i^{-1}(t)), \quad 1 \leq i \leq n,$$

we obtain (35). □

Define J_a (resp. J_b) as a subset of $\{1, 2, \dots, n\}$ consisting of all i , $1 \leq i \leq n$, such that for z close to $P(a)$ (resp. to $P(b)$), $P_i^{-1}(z)$ is close to a (resp. b). The following statement is a more general version of previous results proved in [24], [33], [27].

Proposition 3.5. *If γ is non-closed and $P(a) \neq P(b)$, then the rationality of $I_\infty(t)$ implies the equalities*

$$\sum_{i \in J_a} \left(\frac{q}{P'}\right)(P_i^{-1}(t)) = 0, \quad \sum_{i \in J_b} \left(\frac{q}{P'}\right)(P_i^{-1}(t)) = 0. \quad (36)$$

On the other hand, if γ is non-closed and $P(a) = P(b)$, then the rationality of $I_\infty(t)$ implies the equality

$$\frac{1}{d_b} \sum_{i \in J_b} \left(\frac{q}{P'}\right)(P_i^{-1}(t)) = \frac{1}{d_a} \sum_{i \in J_a} \left(\frac{q}{P'}\right)(P_i^{-1}(t)). \quad (37)$$

Proof. Denote a multivalued analytical function obtained by the complete analytical continuation of $I_\infty(t)$ by $I(t)$. Clearly, $I_\infty(t)$ is rational if and only if $I(t)$ is rational. Suppose first that $P(a) \neq P(b)$. Then Theorem 3.4 implies that near $P(a)$ any branch of $I(t)$ has the form

$$I(t) = -\frac{1}{2\pi\sqrt{-1}} \sum_{i \in J_a} \left(\frac{q}{P'}\right)(P_i^{-1}(t)) \text{Log}(a - P_i^{-1}(t)) + \chi(t),$$

where $\text{Log}(z)$ is a branch of the logarithm and $\chi(t)$ is a branch of a function which has only a finite ramification at $P(a)$. Furthermore, if $i_0 \in J_a$ is a fixed index, then it is easy to see using Puiseux series that for any $i \in J_a$ the limit

$$\lim_{t \rightarrow P(a)} \frac{a - P_i^{-1}(t)}{a - P_{i_0}^{-1}(t)}$$

is a finite number distinct from zero and hence

$$\text{Log}(a - P_i^{-1}(t)) = \text{Log}(a - P_{i_0}^{-1}(t)) + \psi_i(t),$$

where

$$\psi_i(t) = \text{Log}\left(\frac{a - P_i^{-1}(t)}{a - P_{i_0}^{-1}(t)}\right)$$

is a function analytical at $P(a)$. This implies that

$$I(t) = -\frac{1}{2\pi\sqrt{-1}} \text{Log}(a - P_{i_0}^{-1}(t)) \sum_{i \in J_a} \left(\frac{q}{P'}\right)(P_i^{-1}(t)) + \chi_1(t), \tag{38}$$

where $\chi_1(t)$ is a branch of a function which has only a finite ramification at $P(a)$.

Since $\text{Log}(z)$ has an infinite ramification at 0 it follows from (38) that if $I(t)$ is rational (or just has a finite ramification at $P(a)$), then the first equality in (36) holds.

Similarly, near $P(b)$ any branch of $I(t)$ has the form

$$I(t) = \frac{1}{2\pi\sqrt{-1}} \text{Log}(b - P_{j_0}^{-1}(t)) \sum_{j \in J_b} \left(\frac{q}{P'}\right)(P_j^{-1}(t)) + \chi_2(t),$$

where $j_0 \in J_b$ and $\chi_2(t)$ is a branch of a function which has only a finite ramification at $P(b)$. Therefore, if $I(t)$ is rational, then the second equality in (36) holds.

Finally, if $P(a) = P(b)$, then setting $x = P(a) = P(b)$, we obtain that near x any branch of $I(t)$ has the form

$$I(t) = \frac{1}{2\pi\sqrt{-1}} \text{Log}(b - P_{j_0}^{-1}(t)) \sum_{j \in J_b} \left(\frac{q}{P'}\right)(P_j^{-1}(t)) - \frac{1}{2\pi\sqrt{-1}} \text{Log}(a - P_{i_0}^{-1}(t)) \sum_{i \in J_a} \left(\frac{q}{P'}\right)(P_i^{-1}(t)) + \chi_3(t),$$

where $\chi_3(t)$ is a branch of a function which has only a finite ramification at x . Furthermore, if d_a (resp. d_b) is the multiplicity of $P(z)$ at a (resp. at b), then for the functions

$$f(t) = (a - P_{i_0}^{-1}(t))^{d_a}, \quad g(t) = (b - P_{j_0}^{-1}(t))^{d_b}$$

the inequalities

$$\lim_{t \rightarrow x} \frac{f(t)}{g(t)} \neq 0, \quad \lim_{t \rightarrow x} \frac{f(t)}{g(t)} \neq \infty$$

hold.

Therefore, near x

$$\begin{aligned} \operatorname{Log}(b - P_{j_0}^{-1}(t)) &= \frac{1}{d_b} \operatorname{Log}(g(t)) = \frac{1}{d_b} \operatorname{Log}\left(f(t) \frac{g(t)}{f(t)}\right) = \\ &= \frac{d_a}{d_b} \operatorname{Log}(a - P_{i_0}^{-1}(t)) + \psi(t), \end{aligned}$$

where

$$\psi(t) = \frac{1}{d_b} \operatorname{Log}\left(\frac{g(t)}{f(t)}\right)$$

is a function analytical at x , and hence

$$I(t) = \frac{1}{2\pi\sqrt{-1}} \operatorname{Log}(a - P_{i_0}^{-1}(t)) \left[\frac{d_a}{d_b} \sum_{i \in J_b} \left(\frac{q}{P'}\right)(P_i^{-1}(t)) - \sum_{i \in J_a} \left(\frac{q}{P'}\right)(P_i^{-1}(t)) \right] + \chi_4(t),$$

where $\chi_4(t)$ is a branch of a function which has only a finite ramification at the point x . This implies that if $I(t)$ has a finite ramification at x , then (37) holds. \square

Corollary 3.6. *If γ is non-closed, then the rationality of $I_\infty(t)$ implies that either $P(a) = P(b)$ or both points a, b are ramification points of P .*

Proof. Indeed, if $P(a) \neq P(b)$ and say a is not a ramification point of P , then the first equality (36) has the form $(q/P')(P_i^{-1}(t)) \equiv 0$ for some $i \in \{1, 2, \dots, n\}$, which is impossible unless $q \equiv 0$. \square

4. CONDITIONS FOR $I_\infty(t)$ TO BE RATIONAL AND TO VANISH

Let $P(z)$ be a rational function and γ be a curve in \mathbb{C} . In this section we construct a finite system of equations

$$\sum_{i=1}^n f_{s,i} \left(\frac{q}{P'}\right)(P_i^{-1}(z)) = 0, \quad f_{s,i} \in \mathbb{Z}, \quad 1 \leq s \leq k, \quad (39)$$

where $f_{s,i}$ and k depend on $P(z)$ and γ only, such that for a given rational function $q(z)$ the function $I_\infty(t) = I_\infty(q, P, \gamma, t)$ is rational if and only if (39) holds. We also show that if functions $P(z), q(z)$ satisfy the conditions $q^{-1}\{\infty\} \subseteq P^{-1}\{\infty\}$ and $P(\infty) = \infty$, then (39) holds if and only if $I_\infty(t)$ vanishes identically.

Let $P(z)$ be a rational function of degree n . Define an embedded into the Riemann sphere graph λ_P , associated with $P(z)$, setting $\lambda_P = P^{-1}\{S\}$, where S is a “star” joining a non-branch point c of $P^{-1}(z)$ with all its *finite* branch points c_1, c_2, \dots, c_k by non-intersecting oriented arcs $\gamma_1, \gamma_2, \dots, \gamma_k$. More precisely, define vertices of λ_P as preimages of the points c and $c_s, 1 \leq s \leq k$, under the mapping $P(z): \mathbb{C}P^1 \rightarrow \mathbb{C}P^1$, and edges of λ_P as preimages of the arcs $\gamma_s, 1 \leq s \leq k$. Furthermore, for each $s, 1 \leq s \leq k$, mark vertices of λ_P which are preimages of the point c_s by the number s (see Fig. 1).

By construction, the restriction of $P(z)$ on $\mathbb{C}P^1 \setminus \lambda_P$ is a covering of the topological punctured disk $\mathbb{C}P^1 \setminus \{S \cup \infty\}$ and therefore $\mathbb{C}P^1 \setminus \lambda_P$ is a disjointed union of punctured disks (see e.g. [15]). This implies that the graph λ_P is connected and the faces $f_e, 1 \leq e \leq r$, of λ_P are in a one-to-one correspondence with poles $z_e, 1 \leq e \leq r$, of $P(z)$. Define a *star* of λ_P as a subset of edges of λ_P consisting of

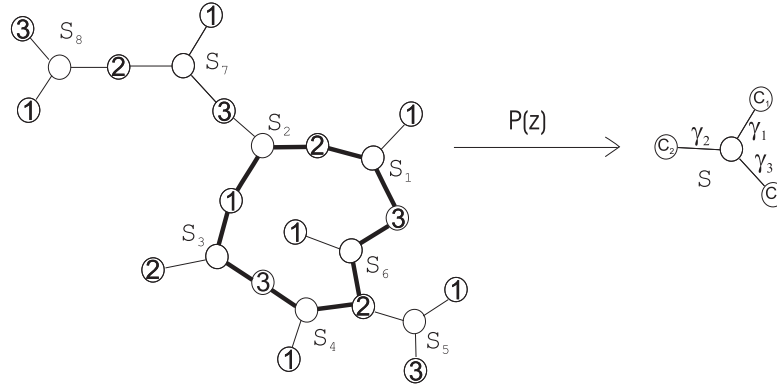


FIGURE 1.

edges adjacent to some non-marked vertex. We will assume that the interior of S is contained in the domain U defined above. Then the set of stars of λ_P may be naturally identified with the set of single-valued branches of $P^{-1}(z)$ in U as follows: to the branch $P_i^{-1}(z)$, $1 \leq i \leq n$, corresponds the star S_i such that $P_i^{-1}(z)$ maps bijectively the interior of S to the interior of S_i . Notice that the Riemann existence theorem implies that a rational function $P(z)$ is defined by c_1, c_2, \dots, c_k and λ_P up to a composition $P(z) \rightarrow P(\mu(z))$, where $\mu(z)$ is a Möbius transformation. The graph λ_P constructed above is known under the name “constellation” and is closely related to what is called “Dessins d’enaftns” (see [20] for further details and other versions of the construction)

Below we will use the graph λ_P for the study of the function $I_\infty(q, P, \gamma, t)$ which depends on the curve γ . By this reason it is more convenient to define the graph λ_P in such a way that the end points a, b of non-closed γ would be marked vertices of λ_P even in the case if $P(a)$ or $P(b)$ is not a branch point of $P^{-1}(z)$. So, in the case where γ is non-closed redefine the collection c_1, c_2, \dots, c_k in the definition of λ_P as the set of finite branch points of $P(z)$ supplemented if necessary by $P(a)$ or $P(b)$ (or by both of them) if the corresponding point is not a critical value of $P(z)$. Clearly, λ_P is still connected and the points a, b are vertices of λ_P .

Denote by $\tilde{I}_\infty(t)$ an auxiliary function obtained from $I_\infty(t)$ by the change of γ to a new integration path $\tilde{\gamma}$ such that $\tilde{\gamma}$ is completely contained in the graph λ_P and poles of $P(z)$ lie on one side of $\tilde{\gamma} - \gamma$ (see Fig. 2). More precisely, if γ is closed, set

$$\tilde{\gamma} = \sum_{e=1}^r \mu(\gamma, z_e) \delta_e, \tag{40}$$

where the curve $\delta_e \subset \lambda_P$ is the boundary of the face f_e of λ_P which contains the pole z_e , while if γ is non-closed, set

$$\tilde{\gamma} = \delta + \sum_{e=1}^r \mu(\gamma - \delta, z_e) \delta_e,$$

where $\delta \subset \lambda_P$ is a path such that $\gamma - \delta$ is closed.

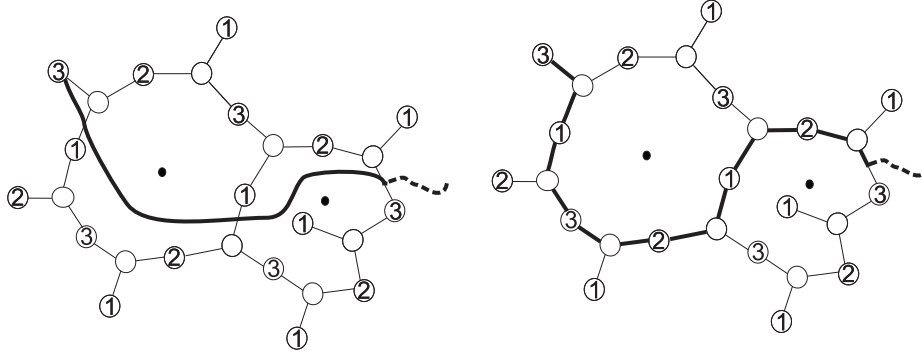


FIGURE 2.

Since

$$\tilde{I}_\infty(t) = \int_{\tilde{\gamma}} \frac{q(z)dz}{P(z)-t} = \int_{\tilde{\gamma}} \frac{q(z)dP(z)}{P'(z)(P(z)-t)}, \tag{41}$$

making the change of variable $z \rightarrow P(z)$ we can represent $\tilde{I}_\infty(t)$ in the form

$$\tilde{I}_\infty(t) = \sum_{s=1}^k \int_{\gamma_s} \frac{\varphi_s(z)}{z-t} dz, \tag{42}$$

where $\varphi_s(z)$, $1 \leq s \leq k$, are linear combinations of the functions $(q/P')(P_i^{-1}(z))$, $1 \leq i \leq n$, in U . In more details, let $c_{s,i}$, $1 \leq i \leq n$, $1 \leq s \leq k$, be the unique vertex of the star S_i marked by the number s . Then

$$\varphi_s(z) = \sum_{i=1}^n f_{s,i} \left(\frac{q}{P'} \right) (P_i^{-1}(z)), \quad 1 \leq s \leq k, \tag{43}$$

where $f_{s,i}$ is a sum of “signed” appearances of $c_{s,i}$ on the path $\tilde{\gamma}$. By definition, this means that an appearance is taken with the sign plus if the center of S_i is followed by $c_{s,i}$ and minus if $c_{s,i}$ is followed by the center of S_i . For example, for the graph λ_P shown on Fig. 1 and the path $\tilde{\gamma} \subset \lambda_P$ pictured by the fat line we have:

$$\begin{aligned} \varphi_1(z) &= - \left(\frac{q}{P'} \right) (P_2^{-1}(z)) + \left(\frac{q}{P'} \right) (P_3^{-1}(z)), \\ \varphi_2(z) &= \left(\frac{q}{P'} \right) (P_2^{-1}(z)) - \left(\frac{q}{P'} \right) (P_1^{-1}(z)) + \left(\frac{q}{P'} \right) (P_6^{-1}(z)) - \left(\frac{q}{P'} \right) (P_4^{-1}(z)), \\ \varphi_3(z) &= \left(\frac{q}{P'} \right) (P_1^{-1}(z)) - \left(\frac{q}{P'} \right) (P_6^{-1}(z)) + \left(\frac{q}{P'} \right) (P_4^{-1}(z)) - \left(\frac{q}{P'} \right) (P_3^{-1}(z)). \end{aligned}$$

Theorem 4.1. *Let $P(z)$, $q(z)$ be rational functions and γ be a curve. Then the function $I_\infty(t)$ is rational if and only if*

$$\varphi_s(z) \equiv 0, \quad 1 \leq s \leq k. \tag{44}$$

Furthermore, if $q^{-1}\{\infty\} \subseteq P^{-1}\{\infty\}$ and $P(\infty) = \infty$, then $I_\infty(t)$ is rational if and only if it vanishes identically.

Proof. First of all observe that without loss of generality we may assume that $q^{-1}\{\infty\} \subseteq P^{-1}\{\infty\}$. Indeed, we always may find a polynomial $R(z)$ such that for the function $r(z) = q(z)R(P(z))$ the inclusion $r^{-1}\{\infty\} \subseteq P^{-1}\{\infty\}$ becomes true. On the other hand, since

$$I_\infty(qP, P, \gamma, t) = \frac{I_\infty(q, P, \gamma, t) - \int_\gamma q(z)dz}{t},$$

it is easy to see inductively that the function $I_\infty(r, P, \gamma, t)$ is rational if and only if the function $I_\infty(q, P, \gamma, t)$ is rational.

Assume now that $q^{-1}\{\infty\} \subseteq P^{-1}\{\infty\}$. Deforming if necessary the star S in the definition of λ_P , without loss of generality we may assume that either infinity is an interior point of some face f_{e_0} of λ_P or $P(\infty) = c_{s_0}$ for some s_0 . In the first case, expressions (41), (42) for the auxiliary function $\tilde{I}_\infty(t)$ are well-defined. Further, by construction

$$\mu(\tilde{\gamma}, z_e) = \mu(\gamma, z_e) - \mu(\gamma, z_{e_0}), \quad 1 \leq e \leq r. \tag{45}$$

Therefore, poles of $P(z)$ lie on one side of $\tilde{\gamma} - \gamma$ implying by Corollary 3.2 that $\tilde{I}_\infty(t) - I_\infty(t)$ is a rational function. Moreover, if $P(\infty) = \infty$, then $\mu(\gamma, z_{e_0}) = 0$ implying that $\tilde{I}_\infty(t) \equiv I_\infty(t)$. Finally, it is easy to see that $\tilde{I}_\infty(t)$ is rational if and only if equalities (44) hold. Indeed, by the well-known boundary property of Cauchy type integrals (see e.g. [22]), for any $s, 1 \leq s \leq k$, and any interior point z_0 of γ_s we have:

$$\lim_{t \rightarrow z_0}^+ \tilde{I}_\infty(t) - \lim_{t \rightarrow z_0}^- \tilde{I}_\infty(t) = \varphi_s(z_0), \tag{46}$$

where the limits are taken when t approaches z_0 from the “right” and “left” side of γ_s correspondingly. Therefore, if the function $\tilde{I}_\infty(t)$ is rational, then the limits above are equal and hence (44) holds. On the other hand, if (44) holds, then it follows directly from formula (42) that $\tilde{I}_\infty(t) \equiv 0$. In particular, we see that $\tilde{I}_\infty(t)$ is rational if and only if it vanishes identically.

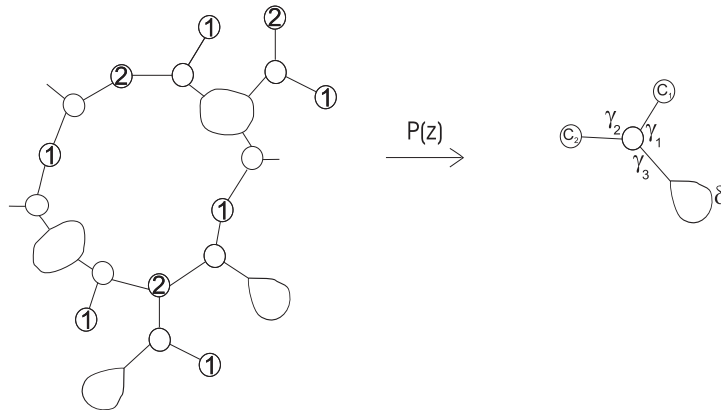


FIGURE 3.

In the case where $P(\infty) = c_{s_0}$ for some s_0 , in order to avoid the situation where $\tilde{\gamma}$ passes through infinity, consider a small loop δ around c_{s_0} and deform the star S and the graph λ_P as it is shown on Fig. 3. Making obvious changes in the definition of $\tilde{\gamma}$ we obtain that, up to addition of a rational function, the equality

$$I_\infty(t) = \sum_{s=1}^k \int_{\gamma_s} \frac{\varphi_s(z)}{z-t} dz + \int_\delta \frac{\psi(z)}{z-t} dz$$

holds, where $\psi(z)$ is a function obtained by the analytical continuation along δ of some linear combinations of the functions $(q/P')(P_i^{-1}(z))$, $1 \leq i \leq n$.

Clearly, the rationality of $I_\infty(t)$ still implies (44) by (46). On the other hand, it follows from Theorem 3.1 and Theorem 3.4 that $I_\infty(t)$ may branch only at the points c_s , $1 \leq s \leq k$, and that if $I_\infty(t)$ does not branch at these points, then $I_\infty(t)$ is rational. Since (44) implies that $I_\infty(t)$ does not branch at c_s , $1 \leq s \leq k$, by (46), we conclude that (44) implies the rationality of $I_\infty(t)$. \square

Corollary 4.2. *Suppose that poles of $P(z)$ do not lie on one side of γ . Then there exist integer numbers f_i , $1 \leq i \leq n$, not all equal between themselves such that the equality*

$$\sum_{i=1}^n f_i \left(\frac{q}{P'} \right) (P_i^{-1}(z)) \equiv 0 \tag{47}$$

holds whenever the function $I_\infty(t) = I_\infty(q, P, \gamma, t)$ is rational.

Proof. Indeed, by construction, for any s , $1 \leq s \leq k$, the numbers $f_{s,i}$, $1 \leq i \leq n$, in (43) satisfy either the equality $\sum_{i=1}^n f_{s,i} = 0$ or the equality $\sum_{i=1}^n f_{s,i} = \pm 1$, where the last case has the place if and only if γ is non-closed and exactly one of the end points a, b of γ is an s -vertex of λ_P . This implies that for any s coefficients of $\varphi_s(z)$ are not all equal, unless they all equal to zero. Further, if γ is non-closed and x is the starting point or the ending point of γ , then it follows from the construction that for s , $1 \leq s \leq k$, such that $P(x) = c_s$ and i , $1 \leq i \leq n$, such that $x \in S_i$, the coefficient $f_{s,i}$ of the equation $\varphi_s(z)$ is distinct from zero. Therefore, it is enough to show that if γ is a closed curve such that all the equations $\varphi_s(z) = 0$, $1 \leq s \leq k$, have zero coefficients, then poles of $P(z)$ lie on one side of γ .

Assume that γ is a such a curve. It follows from (40) that for any s , $1 \leq s \leq k$, the equation $\varphi_s(z)$ is obtained as a sum

$$\varphi_s(z) = \sum_{j=1}^r \mu(\gamma, z_e) \varphi_{s,e}(z),$$

where $\varphi_{s,e}(z)$ is an equation similar to $\varphi_s(z)$ but written for δ_e , $1 \leq e \leq r$. Since by condition all $f_{s,i}$ equal zero, this implies that if $\mu(\gamma, z_{j_0})$ is distinct from zero, then for any face f_{i_0} of λ_P adjacent to f_{j_0} the equality $\mu(\gamma, z_{j_0}) = \mu(\gamma, z_{i_0})$ holds. Since we can join any two faces of λ_P by a connected chain of faces, we conclude that all $\mu(\gamma, z_e)$, $1 \leq e \leq r$, are equal. \square

Corollary 4.3. *Let $P(z), q(z)$ be rational functions and γ be a curve such that $q^{-1}\{\infty\} \subseteq P^{-1}\{\infty\}$, $P(\infty) = \infty$, and*

$$\int_{\gamma} P^i(z)q(z)dz = 0 \tag{48}$$

for all $i \geq i_0$, where $i_0 \geq 0$. Then (48) holds for all $i \geq 0$.

Proof. Since (48) implies that $I_{\infty}(t)$ is a rational function, the statement follows from Theorem 4.1. \square

Notice that conditions of Corollary 4.3 are satisfied in particular if $P(z), q(z)$ are polynomials or if $P(z), q(z)$ are Laurent polynomials such that $P(z)$ is not a polynomial in z or in $1/z$.

In conclusion of this section observe that if there exists a rational function $Q(z)$ such that $Q'(z) = q(z)$, then conditions (44) implying the rationality of $I_{\infty}(t)$ can be written in a slightly different form, which is used in [27], [31].

Proposition 4.4. *Suppose that γ is closed and $q(z) = Q'(z)$ for some rational function $Q(z)$. Then $I_{\infty}(t)$ is rational if and only if the equalities*

$$\sum_{i=1}^n f_{s,i}Q(P_i^{-1}(z)) = 0, \quad 1 \leq s \leq k, \tag{49}$$

hold for any choice of $Q(z) = \int q(z)dz$.

Proof. Since equations (44) are obtained from (49) by differentiation, condition (49) is clearly sufficient. Furthermore, Theorem 4.1 implies that if $I_{\infty}(t)$ is rational, then for any $s, 1 \leq s \leq k$, we have:

$$\sum_{i=1}^n f_{s,i}Q(P_i^{-1}(z)) = d_s, \quad d_s \in \mathbb{C}. \tag{50}$$

On the other hand, since γ is closed the construction of system (44) yields that the limit of the left side of (50) as z tends to c_s is zero. Therefore, $d_s = 0$. \square

Proposition 4.5. *The conclusion of Proposition 4.4 remains true for non-closed γ if $q^{-1}\{\infty\} \subseteq P^{-1}\{\infty\}$, $P(\infty) = \infty$, and $Q(z) = \int q(z)dz$ is chosen in such a way that $Q(a) = 0$.*

Proof. Indeed, if γ is non-closed, then calculating the limit of the left side of (50) as z tends to c_s , we obtain one of the following equalities: $d_s = 0$ if $c_s \neq P(a), P(b)$, $d_s = -Q(a)$ if $c_s = P(a)$ and $c_s \neq P(b)$, $d_s = Q(b)$ if $c_s = P(b)$ and $c_s \neq P(a)$, or $d_s = Q(b) - Q(a)$ if $c_s = P(b) = P(a)$. Furthermore, if $I_{\infty}(t)$ is rational, then Theorem 4.1 implies that $Q(a) = Q(b)$ by (48) taken for $i = 0$. Therefore, the equality $d_s = 0$ holds in all cases whenever $Q(a) = 0$. \square

5. CASE OF GENERIC POSITION

In this section we give criteria for $I_{\infty}(t)$ to be rational or to vanish identically under certain conditions which are satisfied in particular if the function $P(z)$ is indecomposable and has only simple branch points. Since the last two conditions

are obviously satisfied for $P(z)$ in generic position, this solves the problem in this case.

We start from recalling the following simple fact (see e.g. [31, Lemma 2.3]) explaining the theoretic-functional meaning of the equality

$$Q(P_{i_1}^{-1}(z)) = Q(P_{i_2}^{-1}(z)), \quad 1 \leq i_1, i_2 \leq n, \tag{51}$$

where $P(z), Q(z)$ are rational functions, $\deg P(z) = n$.

Lemma 5.1. *Let $P(z), Q(z)$ be non-constant rational functions. Then $P(z)$ and $Q(z)$ have a non-trivial compositional right factor if and only if equality (51) holds for some $i_1 \neq i_2$. In particular, $Q(z) = \tilde{Q}(P(z))$ for some rational function $\tilde{Q}(z)$ if and only if all the functions $Q(P_i^{-1}(z)), 1 \leq i \leq n$, are equal. \square*

Let $G \subset S_n$ be a transitive permutation group and K be a field which is supposed to be either \mathbb{Q} or \mathbb{C} . Recall that the permutation matrix representation of G over K is the homomorphism $\rho_G: G \rightarrow \text{GL}(K^n)$, where $\rho_G(g), g \in G$, is defined as a linear map which sends a vector $\vec{a} = (a_1, a_2, \dots, a_n)$ to the vector $\vec{a}_g = (a_{g(1)}, a_{g(2)}, \dots, a_{g(n)})$. Notice that K^n always has at least two ρ_G -invariant subspaces: the subspace $E_K \subset K^n$ generated by the vector $(1, 1, \dots, 1)$, and its orthogonal complement E_K^\perp with respect to the inner product

$$(\vec{u}, \vec{v}) = u_1 v_1 + u_2 v_2 + \dots + u_n v_n, \quad \vec{v} = (v_1, v_2, \dots, v_n), \quad \vec{u} = (u_1, u_2, \dots, u_n).$$

Theorem 5.2. *Let $P(z), q(z)$ be rational functions and $\gamma \subset \mathbb{C}$ be a curve. Assume that $E_{\mathbb{Q}}$ and $E_{\mathbb{Q}}^\perp$ are the only invariant subspaces with respect to the permutation matrix representation of the monodromy group G_P of $P(z)$ over \mathbb{Q} . Then the function $I_\infty(t)$ is rational if and only if either γ is closed and poles of $P(z)$ lie on one side of γ , or $P(\gamma)$ is closed and $q(z) = \tilde{q}(P(z))P'(z)$ for some rational function $\tilde{q}(z)$.*

Proof. The sufficiency follows from Corollary 3.2 taking into account that if $q(z) = \tilde{q}(P(z))P'(z)$, then

$$I_\infty(q, P, \gamma, t) = I_\infty(\tilde{q}, z, P(\gamma), t). \tag{52}$$

Assume now that $I_\infty(t)$ is rational and poles of $P(z)$ do not lie on one side of γ . In this case by Corollary 4.2 there exist integers $f_i, 1 \leq i \leq s$, not all equal between themselves, such that equality (47) holds. Furthermore, acting on equality (47) by an element $\sigma^{-1}, \sigma \in G_P$, we see that the equality

$$\sum_{i=1}^n f_{\sigma(i)} \left(\frac{q}{P'} \right) (P_i^{-1}(z)) = 0$$

holds for any $\sigma \in G_P$.

Let V be a subspace of \mathbb{Q}^n generated by the vectors $\vec{v}_\sigma, \sigma \in G_P$, where

$$\vec{v}_\sigma = (f_{\sigma(1)}, f_{\sigma(2)}, \dots, f_{\sigma(n)}).$$

Clearly, V is ρ_{G_P} -invariant and for any $\vec{v} = (v_1, v_2, \dots, v_n)$ from V the equality

$$\sum_{i=1}^n v_i \left(\frac{q}{P'} \right) (P_i^{-1}(z)) = 0 \tag{53}$$

holds. Since $f_i, 1 \leq i \leq s$, are not all equal between themselves, $V \neq E_{\mathbb{Q}}$. Furthermore, we may assume that $V \neq \mathbb{Q}^n$ since otherwise the elements $\vec{e}_i, 1 \leq i \leq n$, of the Euclidean basis of \mathbb{Q}^n are contained in V , and (53) yields that

$$\left(\frac{q}{P'}\right)(P_1^{-1}(z)) \equiv \left(\frac{q}{P'}\right)(P_2^{-1}(z)) \equiv \dots \equiv \left(\frac{q}{P'}\right)(P_n^{-1}(z)) \equiv 0$$

implying that $q(z) \equiv 0$. Thus, $V = E_{\mathbb{Q}}^{\perp}$. Since this implies that V contains the vectors $\vec{e}_i - \vec{e}_j, 1 \leq i, j \leq n$, it follows from (53) that

$$\left(\frac{q}{P'}\right)(P_1^{-1}(z)) \equiv \left(\frac{q}{P'}\right)(P_2^{-1}(z)) \equiv \dots \equiv \left(\frac{q}{P'}\right)(P_n^{-1}(z)). \tag{54}$$

Therefore, $(q/P')(z) = \tilde{q}(P(z))$ for some rational function $\tilde{q}(z)$ by Lemma 5.1. Finally, it follows from equality (52) and Corollary 3.6 that $P(\gamma)$ is closed. \square

Recall that a permutation group G acting on a set C is called doubly transitive if it acts transitively on the set of pairs of elements of C . Notice that the full symmetric group is obviously doubly transitive.

Corollary 5.3. *Let $P(z), q(z)$ be rational functions and $\gamma \subset \mathbb{C}$ be a curve. Assume that the monodromy group G_P of $P(z)$ is doubly transitive. Then the function $I_{\infty}(t)$ is rational if and only if either γ is closed and poles of $P(z)$ lie on one side of γ , or $P(\gamma)$ is closed and $q(z) = \tilde{q}(P(z))P'(z)$ for some rational function $\tilde{q}(z)$.*

Proof. Indeed, it is known (see e.g. [38, Th. 29.9]) that a permutation group G is doubly transitive if and only if the subspaces $E_{\mathbb{C}}$ and $E_{\mathbb{C}}^{\perp}$ are the only ρ_G -invariant subspaces with respect to the permutation matrix representation of G_P over \mathbb{C} . Since $\mathbb{Q} \subset \mathbb{C}$, this implies that if G_P is doubly transitive, then $E_{\mathbb{Q}}$ and $E_{\mathbb{Q}}^{\perp}$ are the only invariant subspaces with respect to the permutation matrix representation of G_P over \mathbb{Q} and hence the corollary follows from Theorem 5.2. \square

Recall that a branch point x of a rational function $f(z)$ of degree n is called simple if $f^{-1}\{x\}$ contains $n - 1$ points.

Corollary 5.4. *Assume that $P(z)$ is indecomposable and has only simple branch points. Then the function $I_{\infty}(t)$ is rational if and only if either γ is closed and poles of $P(z)$ lie on one side of γ , or $P(\gamma)$ is closed and $q(z) = \tilde{q}(P(z))P'(z)$ for some rational function $\tilde{q}(z)$.*

Proof. Restrictions imposed on the function $P(z)$ imply that its monodromy group is primitive and contains a transposition. Since a primitive permutation group containing a transposition is the full symmetric group (see e.g. Theorem 13.3 of [38]) the corollary follows now from Corollary 5.3. \square

Notice that the conditions for $P(z)$ to be indecomposable and to have only simple branch points are satisfied if $P(z)$ is in generic position in the sense of algebraic geometry.

Theorem 5.5. *Let $P(z), q(z)$ be rational functions and $\gamma \subset \mathbb{C}$ be a curve such that poles of $P(z)$ do not lie on one side of γ , and subspaces $E_{\mathbb{Q}}$ and $E_{\mathbb{Q}}^{\perp}$ are the*

only invariant subspaces with respect to the permutation matrix representation of the monodromy group G_P of $P(z)$ over \mathbb{Q} . Then equalities

$$\int_{\gamma} P^i(z)q(z)dz = 0, \quad i \geq 0, \tag{55}$$

hold if and only if $P(\gamma)$ is closed and $q(z) = \tilde{q}(P(z))P'(z)$ for some rational function $\tilde{q}(z)$ whose poles lie outside the curve $P(\gamma)$.

Proof. The “if” part follows from Corollary 3.2. In other direction, if (55) holds, then by Theorem 5.2 the curve $P(\gamma)$ is closed and $q(z) = \tilde{q}(P(z))P'(z)$ for some rational function $\tilde{q}(z)$. Furthermore, it follows from (52) and Theorem 3.1 that

$$I_{\infty}(t) = - \sum_{s=1}^{\tilde{l}} \mu(P(\gamma), z_s^{\tilde{q}}) \tilde{q}_s,$$

where $z_1^{\tilde{q}}, z_2^{\tilde{q}}, \dots, z_{\tilde{l}}^{\tilde{q}}$ are finite poles of $\tilde{q}(z)$ and $\tilde{q}_s(z)$ is the principal part of $\tilde{q}(z)$ at \tilde{z}_s . Therefore, the equality $I_{\infty}(t) = 0$ implies that

$$\mu(P(\gamma), z_s^{\tilde{q}}) = 0, \quad 1 \leq s \leq \tilde{l}. \quad \square$$

The Corollaries 5.6 and 5.7 below are obtained from Theorem 5.5 in the same way as Corollaries 5.3 and 5.4 are obtained from Theorem 5.2.

Corollary 5.6. *Let $P(z), q(z)$ be rational functions and $\gamma \subset \mathbb{C}$ be a curve such that poles of $P(z)$ do not lie on one side of γ and the monodromy group G_P of $P(z)$ is doubly transitive. Then equalities (55) hold if and only if $P(\gamma)$ is closed and $q(z) = \tilde{q}(P(z))P'(z)$ for some rational function $\tilde{q}(z)$ whose poles lie outside the curve $P(\gamma)$. \square*

Corollary 5.7. *Let $P(z)$ be a rational function whose poles do not lie on one side of γ . Assume that $P(z)$ is indecomposable and has only simple branch points. Then the equalities (55) hold if and only if $P(\gamma)$ is closed and $q(z) = \tilde{q}(P(z))P'(z)$ for some rational function $\tilde{q}(z)$ whose poles lie outside of the curve $P(\gamma)$. \square*

Remark. Notice that if $P(z)$ is a polynomial, then the requirement of Theorem 5.2 imposed on ρ_{G_P} -invariant subspaces of \mathbb{Q}^n may be weakened to the requirement of indecomposability of $P(z)$ via the Schur theorem (see [24]). However, there exist indecomposable rational functions $P(z)$ for which the conclusion of Theorem 5.2 fails to be true (see Section 9).

6. DOUBLE MOMENTS OF RATIONAL FUNCTIONS

In this section we prove two results which can be considered as versions of the Wermer theorem [36], [37], describing analytic functions on S^1 satisfying

$$\int_{S^1} h^i(z)g^j(z)g'(z)dz = 0, \quad i, j \geq 0,$$

in the case where the functions $h(z), g(z)$ are rational while the integration path is allowed to be an arbitrary curve in \mathbb{C} . Notice that for rational functions the Wermer

theorem is equivalent to the following statement [2]: rational functions $P(z)$ and $Q(z)$ satisfy equalities

$$\int_{S^1} P^i(z)Q^j(z)Q'(z)dz = 0, \quad i \geq 0, j \geq 0,$$

if and only there exist rational functions $\tilde{P}(z), \tilde{Q}(z), W(z)$ such that (7) holds and poles of $\tilde{P}(z)$ and $\tilde{Q}(z)$ lie on one side of $W(S^1)$.

For given $P(z), Q(z)$ and $j \geq 0$ denote by $I_j(t)$ the generating functions for the sequence of the moments

$$m_i = \int_{\gamma} P^i(z)Q^j(z)Q'(z) dz, \quad i \geq 0. \tag{56}$$

Theorem 6.1. *Let $P(z), Q(z)$ be rational functions and γ be a curve such that the functions $I_j(t)$ are rational for any j in the interval $j_0 \leq j \leq j_0 + n - 1$, where $j_0 \geq 0$ and $n = \deg P(z)$. Then there exist rational functions $\tilde{P}(z), \tilde{Q}(z), W(z)$ such that:*

$$P(z) = \tilde{P}(W(z)), \quad Q(z) = \tilde{Q}(W(z)), \tag{57}$$

the curve $W(\gamma)$ is closed, and poles of $\tilde{P}(z)$ lie on one side of $W(\gamma)$.

Proof. First, observe that if $\mathbb{C}(P(z), Q(z)) = \mathbb{C}(W(z))$, where $\deg W(z) > 1$, then the corresponding functions $\tilde{P}(z), \tilde{Q}(z)$ in (57) have no common compositional right factor and

$$\int_{\gamma} P^i(z)Q^j(z)Q'(z) dz = \int_{W(\gamma)} \tilde{P}^i(z)\tilde{Q}^j(z)\tilde{Q}'(z) dz, \quad i \geq 0, j \geq 0.$$

Therefore, it is enough to show that if $P(z)$ and $Q(z)$ have no common compositional right factor, then γ is closed and poles of $P(z)$ lie on one side of γ . Assume the inverse. Then it follows from Corollary 4.2 applied to the functions $P(z)$ and $Q^j(z)Q'(z), j_0 \leq j \leq j_0 + n - 1$, that the system

$$\sum_{s=1}^n f_s Q^j(P_s^{-1}(z)) \left(\frac{Q'}{P'}\right)(P_s^{-1}(z)) = 0, \quad j_0 \leq j \leq j_0 + n - 1, \tag{58}$$

considered as a system of linear equations over the field K_P , generated over $\mathbb{C}(z)$ by $P_i^{-1}(z), 1 \leq i \leq n$, has a non-trivial solution f_1, f_2, \dots, f_n . Since the determinant of system (58) is a product of the Vandermonde determinant $D = \|Q^j(P_i^{-1}(z))\|$ and a non-zero function

$$\prod_{i=1}^n Q^{j_0}(P_i^{-1}(z)) \left(\frac{Q'}{P'}\right)(P_i^{-1}(z)),$$

this implies that

$$Q(P_{i_1}^{-1}(z)) \equiv Q(P_{i_2}^{-1}(z)) \tag{59}$$

for some $i_1 \neq i_2, 1 \leq i_1, i_2 \leq n$, and hence $P(z), Q(z)$ have a common compositional right factor by Lemma 5.1. The obtained contradiction proves the theorem. \square

Theorem 6.2. *Let $P(z), Q(z)$ be rational functions and γ be a curve. Then equalities*

$$\int_{\gamma} P^i(z)Q^j(z)Q'(z)dz = 0 \tag{60}$$

hold for all $i \geq i_0, j \geq j_0$, where $i_0 \geq 0, j_0 \geq 0$, if and only if there exist rational functions $\tilde{P}(z), \tilde{Q}(z), W(z)$ such that:

$$P(z) = \tilde{P}(W(z)), \quad Q(z) = \tilde{Q}(W(z)), \tag{61}$$

the curve $W(\gamma)$ is closed, and poles of $\tilde{P}(z)$ and $\tilde{Q}(z)$ lie on one side of the curve $W(\gamma)$. In particular, if equalities (60) hold for all $i \geq i_0, j \geq j_0$, then they hold for all $i \geq 0, j \geq 0$.

Proof. Using the same reduction as in the proof of Theorem 6.1, it is enough to show that if $P(z)$ and $Q(z)$ have no common compositional right factor, then (60) holds if and only if the curve γ is closed and poles of $P(z)$ and $Q(z)$ lie on one side of γ .

Assume first that poles of $P(z)$ and $Q(z)$ lie on one side of γ and show that this implies that $I_j(t) = 0$ for any $j \geq 0$. Applying Theorem 3.1 to $q(z) = Q^j(z)Q'(z)$ and taking into account that the equality

$$q(z) = \left(\frac{Q^{j+1}(z)}{j+1} \right)' \tag{62}$$

implies the equality $\int_{\gamma} q(z)dz = 0$, we see that

$$I_j(t) = \mu \sum_{i=1}^n \left(\frac{q_{\infty}}{P^i} \right) (P_i^{-1}(t)), \tag{63}$$

where

$$q_{\infty}(z) = q(z) - \sum_s^l q_s(z)$$

and μ equals to the common winding number of poles of $P(z)$ and $Q(z)$. If ∞ is a pole of $Q(z)$, then $\mu = 0$ implying that $I_j(t) = 0$. On the other hand, if ∞ is not a pole of $Q(z)$, then $q_{\infty}(z)$ is a constant which is actually zero in view of (62).

In another direction, assume that (60) holds and show that then points from the set $P^{-1}\{\infty\} \cup Q^{-1}\{\infty\}$ lie on one side of γ . Clearly, for any $s \geq i_0$ the function

$$R(z) = P^s(z) + Q(z)$$

satisfies the equalities

$$\int_{\gamma} R^i(z)Q^j(z)Q'(z)dz = 0, \quad i \geq 0, j \geq j_0.$$

Furthermore,

$$R^{-1}\{\infty\} \subseteq P^{-1}\{\infty\} \cup Q^{-1}\{\infty\}$$

and, if s is big enough, then

$$R^{-1}\{\infty\} = P^{-1}\{\infty\} \cup Q^{-1}\{\infty\}.$$

Therefore, since we may apply Theorem 6.1 to the functions $R(z)$, $Q(z)$, it is enough to prove that for any $k \geq 1$ there exists $s \geq k$ such that $P^s(z)$ and $Q(z)$, or equivalently $R(z)$ and $Q(z)$, have no common compositional right factor.

In order to prove the last statement observe that since the monodromy group of a rational function has only finite number of imprimitivity systems, there exist a finite number of right factors $Q_j(z)$, $\deg Q_j(z) > 1$, of $Q(z)$ such that any other right factor of $Q(z)$ of degree greater than one has the form $\mu(Q_j(z))$ for some $Q_j(z)$ and a Möbius transformation μ . Further, it is easy to see that if $P^s(z)$ and $Q(z)$ have a common compositional right factor $\mu(Q_j(z))$, then s is a multiple of a *minimal* number s_j such that $Q_j(z)$ is a common compositional right factor of $P^{s_j}(z)$ and $Q(z)$. Since by assumption $P(z)$ and $Q(z)$ have no common compositional right factor, the inequalities $s_j > 1$ hold and hence $R(z)$ and $P(z)$ have no common compositional right factor whenever s is not a multiple of any s_j . \square

Notice that the above results also generalize Theorem 6.1 and Corollary 6.2 of the paper [33].

7. LAURENT POLYNOMIAL MOMENT PROBLEM

In this subsection we study the following problem: *for a given Laurent polynomial $L(z)$ describe Laurent polynomials $m(z)$ such that*

$$\int_{S^1} L^i(z)m(z)dz = 0, \quad (64)$$

for all $i \geq i_0$, where $i_0 \geq 0$. It is easy to see that if $L(z)$ is a polynomial in z or in $1/z$, then for any $m(z)$ there exists $i_0 \geq 0$ such that (64) holds. So, the interesting case is the one where $L(z)$ is not a polynomial in z or in $1/z$. We will call such Laurent polynomials *proper*.

We start from a generalizations of the following result proved by Duistermaat and van der Kallen [14]: if all integer positive powers of a Laurent polynomial $L(z)$ have no constant term, then $L(z)$ is either a polynomial in z , or a polynomial in $1/z$. Clearly, the condition that all powers of $L(z)$ have no constant term is equivalent to the condition that integrals in (64) vanish for $m(z) = 1/z$ and $i \geq 1$.

Theorem 7.1. *Let $L(z)$ and $m(z)$ be Laurent polynomials such that the coefficient of the term $1/z$ in $m(z)$ is distinct from zero and (64) holds for all $i \geq i_0$, where $i_0 \geq 0$. Then $L(z)$ is either a polynomial in z , or a polynomial in $1/z$.*

Proof. Assume the inverse. Then by Corollary 4.3 equalities (64) hold for all $i \geq 0$. On the other hand, for $i = 0$ the integral in (64) does not vanish since it coincides with the coefficient of the term $1/z$ in $m(z)$ multiplied by $2\pi\sqrt{-1}$. \square

Corollary 7.2. *Let $L(z)$ be a proper Laurent polynomial and $m(z)$ be a Laurent polynomial such that (64) holds for all $i \geq i_0$, where $i_0 \geq 0$. Then there exists a Laurent polynomial $M(z)$ such that $m(z) = M'(z)$. \square*

Let

$$L(z) = a_{n_1}z^{n_1} + a_{n_1+1}z^{n_1+1} + \dots + a_{n_2}z^{n_2}, \quad a_{n_1} \neq 0, \quad a_{n_2} \neq 0,$$

be a Laurent polynomial. Define the *bi-degree* of $L(z)$ as the ordered pair (n_1, n_2) of integers n_1, n_2 . Notice that if $M(z)$ is another Laurent polynomial whose bi-degree is (m_1, m_2) , then the bi-degree of the product $L(z)M(z)$ is $(n_1 + m_1, n_2 + m_2)$.

The next result provides yet another generalization of the theorem of Duistermaat and van der Kallen.

Theorem 7.3. *Let $L(z)$ be a Laurent polynomial of bi-degree (n_1, n_2) and $m(z)$ be a Laurent polynomial of bi-degree (m_1, m_2) such that (64) holds for all $i \geq i_0$, where $i_0 \geq 0$. Assume additionally that $m(z)$ is either a polynomial in z such that $m_1 \equiv -1 \pmod{n_1}$, or a polynomial in $1/z$ such that $m_2 \equiv -1 \pmod{n_2}$. Then $L(z)$ is either a polynomial in z , or a polynomial in $1/z$.*

Proof. Assume the converse. Observe that then in particular $n_1 < 0$ and $n_2 > 0$. Furthermore, Corollary 4.3 implies that (64) hold for all $i \geq 0$. Therefore, in order to prove the theorem it is enough to show that if $m(z)$ has the form as above, then there exists $k \geq 0$ such that for $i = k$ integral in (64) is distinct from zero.

If $m(z)$ is a polynomial in z and $l_1 \geq 0$ is a number such that $m_1 + n_1 l_1 = -1$, then the integral in (64) is distinct from zero for $i = l_1$ since the bi-degree of $L^{l_1}(z)m(z)$ is $(-1, m_2 + n_2 l_1)$ implying that the residue of $L^{l_1}(z)m(z)$ at zero does not vanish. Similarly, if $m(z)$ is a polynomial in $1/z$ and $l_2 \geq 0$ is a number such $m_2 + n_2 l_2 = -1$, then the integral in (64) is distinct from zero for $i = l_2$, since the bi-degree of $L^{l_2}(z)m(z)$ is $(m_1 + n_1 l_2, -1)$. \square

Corollary 7.4. *Let $L(z)$ be a Laurent polynomial of bi-degree (n_1, n_2) and d be either a non-negative integer such that $d \equiv 0 \pmod{n_2}$, or a non-positive integer such that $d \equiv 0 \pmod{n_1}$. Suppose that for all $i \geq i_0$, where $i_0 \geq 0$, the coefficient of z^d in $L^i(z)$ vanishes. Then $L(z)$ is either a polynomial in z , or a polynomial in $1/z$. \square*

For a Laurent polynomial $M(z)$ denote by $M_0(z)$ the principal part of $M(z)$ at zero and by $M_\infty(z)$ the difference $M(z) - M_0(z)$. Taking into account Corollary 7.2 in the following we usually will write system (64) in the form

$$\int_{S^1} L^i(z) dM(z) = 0, \tag{65}$$

$i \geq i_0$, where it is always assumed that $M_\infty(0) = 0$.

Define J_0 (resp. J_∞) as a subset of $\{1, 2, \dots, r\}$, $r = \deg L(z)$, consisting of all $i \in \{1, 2, \dots, r\}$ such that for t close to infinity, $L_i^{-1}(t)$ is close to 0 (resp. to ∞). Notice that $\{1, 2, \dots, r\} = J_0 \cup J_\infty$. The theorem below summarizes general results about $I_\infty(t)$ obtained above in the particular case where $I_\infty(t)$ corresponds to moments in (65).

Theorem 7.5. *Let $L(z)$ be a proper Laurent polynomial and $M(z)$ be a Laurent polynomial such that (65) holds for all $i \geq i_0$, where $i_0 \geq 0$. Then (65) holds for all $i \geq 0$. Furthermore, condition (65) is equivalent to the condition*

$$\sum_{i \in J_0} M_\infty(L_i^{-1}(t)) \equiv \sum_{i \in J_\infty} M_0(L_i^{-1}(t)). \tag{66}$$

Finally, if the monodromy group of $L(z)$ is doubly transitive, or more generally, if $E_{\mathbb{Q}}$ and $E_{\mathbb{Q}}^{\perp}$ are the only invariant subspaces with respect to the permutation matrix representation of the monodromy group G_L of $L(z)$ over \mathbb{Q} , then (65) holds if and only if $M(z) = \tilde{M}(L(z))$, where $\tilde{M}(z)$ is a polynomial.

Proof. The first statement follows from Corollary 4.3. Furthermore, it follows from Theorem 3.1, taking into account the equality $\int_{S_1} dM(z) = 0$, that

$$\begin{aligned} I_{\infty}(t) &= \sum_{i \in J_0} \left(\frac{M'}{L'} \right) (L_i^{-1}(t)) - \sum_{i=1}^{\deg L} \left(\frac{M'_0}{L'} \right) (L_i^{-1}(t)) = \\ &= \sum_{i \in J_0} \left(\frac{M'_{\infty}}{L'} \right) (L_i^{-1}(t)) - \sum_{i \in J_{\infty}} \left(\frac{M'_0}{L'} \right) (L_i^{-1}(t)). \end{aligned} \tag{67}$$

Taking the primitive of both sides of this equality we see that condition (65) is equivalent to the condition

$$\sum_{i \in J_0} M_{\infty}(L_i^{-1}(t)) - \sum_{i \in J_{\infty}} M_0(L_i^{-1}(t)) = c, \tag{68}$$

where $c \in \mathbb{C}$. Furthermore, since the limit of the left part of (68) as t tends to infinity is $M_{\infty}(0)|J_0| = 0$, we conclude that (65) is equivalent to (66).

Finally, Theorem 5.5 implies that if $E_{\mathbb{Q}}$ and $E_{\mathbb{Q}}^{\perp}$ are the only invariant subspaces with respect to the permutation matrix representation of G_L of $L(z)$ over \mathbb{Q} , then there exists a rational function $N(z)$ such that

$$M'(z) = N(L(z))L'(z). \tag{69}$$

Since $M'(z)$ is a Laurent polynomial it follows from (69) that $N(L(z))$ also is a Laurent polynomial, implying that $N(z)$ is a polynomial for otherwise $N(L(z))$ would have a pole distinct from $0, \infty$. Therefore, $M(z) = \tilde{M}(L(z))$, where $\tilde{M}(z) = \int N(z)dz$ is a polynomial. \square

Notice that if $L(z)$ is decomposable, then Laurent polynomials $M(z)$ which satisfy (65) but are not polynomials in $L(z)$ always exist. Indeed, it is easy to see that if $L(z) = A(B(z))$ is a decomposition of a Laurent polynomial $L(z)$ into a composition of rational functions $A(z)$ and $B(z)$, with $\deg A(z) > 1$, $\deg B(z) > 1$, then the condition $B^{-1}\{A^{-1}\{\infty\}\} = \{0, \infty\}$ implies that there exists a Möbius transformation $\mu(z)$ such that either $A(\mu(z))$ is a polynomial and $\mu^{-1}(B(z))$ is a Laurent polynomial, or $A(\mu(z))$ is a Laurent polynomial and $\mu^{-1}(B(z)) = z^d$, for some $d > 1$ ¹. Therefore, if $L(z)$ is decomposable, then either there exist a polynomial $\tilde{L}(z)$ and a Laurent polynomial $L_1(z)$ such that $L(z) = \tilde{L}(L_1(z))$, or there exists a Laurent polynomial $L_1(z)$ such that $L(z) = L_1(z^d)$ for some $d > 1$. In the first case it is easy to see that any Laurent polynomial $M(z) = \tilde{M}(L_1(z))$, where $\tilde{M}(z)$ is a polynomial, satisfies (65) for all $i \geq 0$. On the other hand, in the second case the residue calculation shows that any Laurent polynomial $M(z)$ containing no terms of degrees which are multiples of d satisfies (65). Furthermore, if $L(z)$

¹For a comprehensive decomposition theory of Laurent polynomials generalizing the decompositions theory of polynomials developed by Ritt [34] we refer the reader to [28]

admits several decompositions, then the sum of corresponding $M(z)$ also satisfies (65).

It seems natural to start the investigation of the Laurent polynomial moment problem from a description of its *polynomial* solutions, and two theorems below are initial results in this direction. Another interesting subproblem is to describe solutions of the Laurent polynomial moment problem in the case where $L(z)$ is indecomposable. In the last case “expected” solutions have the form $M(z) = \tilde{M}(L(z))$, where $\tilde{M}(z)$ is a polynomial. However, one can show (see Section 9 and the paper [32]) that other solutions also may exist.

Theorem 7.6. *Let $L(z)$ be a proper Laurent polynomial of bi-degree (n_1, n_2) such that either $n_1 = -1$ or $n_2 = 1$. Then a Laurent polynomial $M(z)$ which is a polynomial in z may not satisfy (65) for $i \geq i_0$, where $i_0 \geq 0$, unless $M(z) \equiv 0$.*

Proof. Indeed, if $M(z)$ is a polynomial, then (66) is equivalent to

$$\sum_{i \in J_0} M(L_i^{-1}(t)) \equiv 0. \tag{70}$$

If $n_1 = -1$, then (70) immediately implies that $M(z) \equiv 0$ since in this case J_0 contains a single element. Suppose now that $n_2 = 1$ and denote by $L_\infty^{-1}(z)$ a unique branch of $L^{-1}(z)$ for which $\lim_{z \rightarrow \infty} L_\infty^{-1}(z) = \infty$. It follows from the transitivity of the monodromy group G_L of $L(z)$ that there exists $\sigma \in G_L$ such that acting on equality (70) by σ we obtain the equality

$$M(L_\infty^{-1}(t)) + \sum_{i \in J_0 \setminus j} M(L_i^{-1}(t)) = 0, \tag{71}$$

where $j \in J_0$. Since for any $M(z) \not\equiv 0$ we have:

$$\lim_{t \rightarrow \infty} M(L_\infty^{-1}(t)) = \infty$$

while

$$\lim_{t \rightarrow \infty} M(L_i^{-1}(t)) = 0, \quad i \in J_0,$$

equality (71) implies that $M(z) \equiv 0$. □

Theorem 7.7. *Let $L(z)$ be a proper Laurent polynomial of bi-degree (n, p) , where p is a prime, and $M(z) \not\equiv 0$ be a polynomial in z such that (65) holds for $i \geq i_0$, where $i_0 \geq 0$. Then $L(z) = L_1(z^p)$ for some Laurent polynomial $L_1(z)$ while $M(z)$ is a linear combination of the monomials z^j , where $j \not\equiv 0 \pmod{p}$.*

Proof. Show first that J_∞ is a block of an imprimitivity system for the monodromy group G_L of $L(z)$. Indeed, if J_∞ is not a block then there exists $\sigma \in G_L$ such that $\sigma\{J_\infty\} \cap J_\infty \neq \emptyset$ and $\sigma\{J_\infty\} \cap J_0 \neq \emptyset$. This implies that acting on equality (70) by σ we obtain the equality

$$\sum_{i \in A} M(L_i^{-1}(t)) + \sum_{i \in B} M(L_i^{-1}(t)) = 0, \tag{72}$$

where A is a subset of J_0 and B is a proper subset of J_∞ .

Without loss of generality we may assume that $L_i^{-1}(t)$, $1 \leq i \leq p + n$, are numbered in such a way that $J_\infty = \{1, 2, \dots, p\}$ and that to the loop around infinity corresponds the element

$$(12 \dots p)(p + 1 \ p + 2 \dots p + n) \tag{73}$$

of G_L . Then Puiseux series of $L_i^{-1}(t)$, $1 \leq i \leq p$, at infinity have the form

$$L_i^{-1}(z) = \sum_{k=-1}^{\infty} u_k \varepsilon_p^{(i-1)k} \left(\frac{1}{z}\right)^{\frac{k}{p}},$$

where $\varepsilon_p = \exp(\frac{2\pi\sqrt{-1}}{p})$ and $u_{-1} \neq 0$. Therefore,

$$M(L_i^{-1}(z)) = \beta \varepsilon_p^{(i-1)m} z^{\frac{m}{p}} + o(z^{\frac{m}{p}}), \quad 1 \leq i \leq p,$$

where $m = \deg M(z)$, $\beta = u_{-1}^m \neq 0$. On the other hand, for any i , $p + 1 \leq i \leq n + p$, near infinity we have:

$$M(L_i^{-1}(z)) = o(1).$$

Therefore, for the coefficient γ of $z^{\frac{m}{p}}$ in the Puiseux series of the function in the left side of (72) the equality

$$\gamma = \beta \sum_{j \in B} \varepsilon_p^{(j-1)m} \tag{74}$$

holds. Thus, if we will show that $\gamma \neq 0$, the contradiction obtained will imply that J_∞ is a block.

Set

$$r(z) = \sum_{j \in B} z^{j-1}.$$

Clearly, $\gamma = \beta r(\varepsilon_p^m)$. Since p is a prime, the number ε_p^m is either 1 or a primitive p -th root of unity. In the first case obviously $\gamma \neq 0$. On the other hand, in the second case the equality $r(\varepsilon_p^m) = 0$ implies that the p -th cyclotomic polynomial $\Phi_p(z)$ divides $r(z)$ in the ring $\mathbb{Z}[z]$. However, since

$$\Phi_p(z) = 1 + z + z^2 + \dots + z^{p-1},$$

the bi-degree (t_1, t_2) of the product $\Phi_p(z)f(z)$ for any non-constant polynomial $f(z)$ satisfies the inequality $t_2 - t_1 \geq p$. Since by assumption B is a proper subset of $J_\infty = \{1, 2, \dots, p\}$ this implies that the equality $r(\varepsilon_p^m) = 0$ is impossible. Therefore, $\gamma \neq 0$, and hence J_∞ is a block.

Since the group G_L contains a block, the Laurent polynomial $L(z)$ may be decomposed into a composition $L(z) = A(B(z))$ of rational functions $A(z)$ and $B(z)$ of degree greater than one. Furthermore, since element (73) transforms the block J_∞ of the corresponding imprimitivity \mathcal{B} system to itself, the function $A(z)$ is not a polynomial for otherwise (73) would permute blocks of \mathcal{B} cyclically. Therefore, taking into account that p is a prime, we conclude that there exists a Laurent polynomial $L_1(z)$ such that $L(z) = L_1(z^p)$, where the bi-degree of $L_1(z)$ is $(n/p, 1)$. Clearly, $M(z)$ can be written as

$$M(z) = M_1(z^p) + M_2(z),$$

where $M_1(z)$ is a polynomial in z and $M_2(z)$ is a combination of the monomials z^j , where $j \not\equiv 0 \pmod{p}$. Furthermore, clearly $M_2(z)$ satisfies (65). Therefore, $M_1(z^p)$ also should satisfy (65). Since after the change of variable this implies that $M_1(z)$ satisfies (65) for $L(z) = L_1(z)$ it follows from Theorem 7.6 that $M_1(z) \equiv 0$. \square

8. BAUTIN INDEX FOR THE LAURENT POLYNOMIAL MOMENT PROBLEM

In this section we study the following problem: for Laurent polynomials $L(z), M(z)$ to find a number i_0 such that the vanishing of the integrals

$$\int_{S^1} L^i(z) dM(z) = 0, \tag{75}$$

for i satisfying $0 \leq i \leq i_0$ implies that they vanish for all $i \geq 0$.

A progress in the study of a similar problem for *polynomials* was achieved in the recent paper [19]. The approach of [19] is based on the fact that the function $I_\infty(t) = I_\infty(q, P, \gamma, t)$ satisfies a Fuchsian linear differential equation (see e.g. [33, p. 250]). Since Theorems 3.1, 3.4 give an explicit expression for the function $I_\infty(t)$, they provide an approach to the problem in the general case. We demonstrate this approach below in the case where $L(z), M(z)$ are Laurent polynomials.

For a Laurent polynomial $L(z)$ of degree n define numbers $\tau_i, 1 \leq i \leq n$, as follows: $\tau_i, 1 \leq i \leq n$, equals 1 if $i \in J_0$ and 0 otherwise. Further, define a number $N(L)$ as the number of different vectors in the collection

$$(\tau_{\sigma(1)}, \tau_{\sigma(2)}, \dots, \tau_{\sigma(n)}), \quad \sigma \in G_L. \tag{76}$$

Notice that obviously $N(L) \leq |G_L| \leq n!$. Finally, for a function $\psi(t)$ whose Puiseux series at infinity is

$$\psi(t) = \sum_{k=j}^{\infty} w_k \left(\frac{1}{t}\right)^{\frac{k}{l}}, \tag{77}$$

where $l \geq 1$ and $w_j \neq 0$, set $\text{ord}_\infty \psi(t) = j/l$.

Theorem 8.1. *Let $L(z), M(z), \deg L(z) = n, \deg M(z) = m$, be Laurent polynomials such that the equality*

$$\int_{S^1} L^i(z) dM(z) = 0, \tag{78}$$

holds for all i satisfying $0 \leq i \leq m(N(L) - 1) + 1$. Then (78) holds for all $i \geq 0$. In particular, equalities (78) hold for all $i \geq 0$ whenever they hold for all i satisfying $0 \leq i \leq m(n! - 1) + 1$.

Proof. Set

$$\psi(t) = \int I_\infty(t) dt = \frac{1}{2\pi i} \sum_{k=1}^{\infty} \frac{m_k}{k} \left(\frac{1}{t}\right)^k.$$

Clearly, we only must show that if

$$\text{ord}_\infty \psi(t) > m(N(L) - 1), \tag{79}$$

then $\psi(t) \equiv 0$.

Equality (67) implies that

$$\psi(t) = \sum_{i \in J_0} M(L_i^{-1}(t)) - \sum_{i=1}^n M_0(L_i^{-1}(t)) \tag{80}$$

is an algebraic function. Therefore, $\psi(t)$ satisfies an irreducible algebraic equation

$$y^N(t) + a_1(t)y^{N-1}(t) + \dots + a_N(t) = 0, \quad a_j(t) \in \mathbb{C}(t), \tag{81}$$

whose roots $\psi_j(t)$, $1 \leq j \leq N$, are all possible analytic continuations of $\psi(t)$ and whose coefficients are elementary symmetric functions of $\psi_j(t)$, $1 \leq j \leq N$. Furthermore, since $M(t)$, $L(t)$ are Laurent polynomials, it follows from (80) that the functions $\psi_j(t)$, $1 \leq j \leq N$, have no poles in \mathbb{C} implying that the functions $a_j(t)$, $1 \leq j \leq N$, are polynomials. Finally, since the second sum in (80) is a rational function, the inequality

$$N \leq N(L) \tag{82}$$

holds.

Let (n_1, n_2) be the bi-degree of $L(t)$. Then Puiseux series at infinity of branches $L_i^{-1}(t)$, $i \in J_0$, have the form

$$\sum_{k=1}^{\infty} v_{k,i} \left(\frac{1}{t}\right)^{\frac{k}{n_1}}, \tag{83}$$

where $v_{1,i}$ is distinct from zero, while Puiseux series of branches $L_i^{-1}(t)$, $i \in J_{\infty}$, have the form

$$\sum_{k=-1}^{\infty} \tilde{v}_{k,i} \left(\frac{1}{t}\right)^{\frac{k}{n_2}}, \tag{84}$$

where $\tilde{v}_{-1,i}$ is distinct from zero. Since Puiseux series at infinity of the functions $M(L_i^{-1}(t))$, $1 \leq j \leq N$, and $M_0(L_i^{-1}(t))$, $1 \leq j \leq N$, are obtained by the substitution of series (83), (84) into $M(t)$ and $M_0(t)$, this implies that for any j , $1 \leq j \leq N$, the inequality

$$\text{ord}_{\infty} \psi_j(t) \geq -m \tag{85}$$

holds. Therefore, since $a_j(t)$, $1 \leq j \leq N$, are elementary symmetric functions of $\psi_j(t)$, $1 \leq j \leq N$, the inequalities

$$\text{ord}_{\infty} a_j(t) \geq -mj, \quad 1 \leq j \leq N,$$

hold and hence

$$\deg a_j(t) = -\text{ord}_{\infty} a_j(t) \leq mj, \quad 1 \leq j \leq N. \tag{86}$$

Now we are ready to show that if (79) holds, then $\psi(t) \equiv 0$. Indeed, assume the inverse. Then the coefficient $a_N(t)$ in (81) does not vanish and the inequality $\text{ord}_{\infty} a_N(t) \leq 0$ holds. On the other hand, (79) and (82) imply the inequality $\text{ord}_{\infty} \psi(t) > m(N - 1)$ and hence for any i , $1 \leq i \leq N$, taking into account inequalities (86), we have:

$$\begin{aligned} \text{ord}_{\infty} \{a_{N-i}(t)\psi^i(t)\} &\geq \text{ord}_{\infty} \{a_{N-i}(t)\psi(t)\} = \\ &= \text{ord}_{\infty} \psi(t) - \deg a_{N-i}(t) \geq \text{ord}_{\infty} \psi(t) - m(N - 1) > 0 \end{aligned}$$

(here we set $a_0(t) \equiv 1$). Therefore,

$$\text{ord}_\infty\{\psi^N(t) + a_1(t)\psi^{N-1}(t) + \dots + a_{N-1}(t)\psi(t)\} > 0$$

in contradiction with

$$\psi^N(t) + a_1(t)\psi^{N-1}(t) + \dots + a_N(t) = 0. \quad \square$$

Remark. The proof of Theorem 8.1 uses the same approach as Section 2.4 of [27]. Notice that corresponding formulas in [27] on the page 758 contain misprints. Namely, all printed powers of the expression m/n are actually its factors.

9. INTERRELATIONS BETWEEN THE RATIONALITY OF $I_\infty(q, P, \gamma, t)$ AND THE COMPOSITION CONDITION

9.1. Definition of the subspace $M_{P,\gamma}$. Let γ be a curve and $P(z)$ be a rational function such that poles of $P(z)$ do not lie on one side of γ . By Corollary 5.4, if $P(z)$ is in generic position, then the rationality of $I_\infty(q, P, \gamma, t)$ for a rational function $q(z)$ implies that $q(z) = \tilde{q}(P(z))$ for some rational function $\tilde{q}(z)$. For arbitrary $P(z)$ such a statement fails to be true. However, in many cases the rationality of $I_\infty(q, P, \gamma, t)$ still implies that composition condition (23) holds. For brevity, if (23) holds, we will say that the function $I_\infty(q, P, \gamma, t)$ is *reducible*.

For example, if γ is a non-closed curve such that its end points a, b are not ramification points of $P(z)$, then the rationality of $I_\infty(q, P, \gamma, t)$ implies its reducibility. Indeed, it follows from Corollary 3.6 that $P(a) = P(b)$. Furthermore, equality (37) from Proposition 3.5 reduces to the equality

$$\left(\frac{q}{P'}\right)(P_{i_1}^{-1}(z)) = \left(\frac{q}{P'}\right)(P_{i_2}^{-1}(z)) \tag{87}$$

for some $i_1 \neq i_2, 1 \leq i_1, i_2 \leq n$. Therefore, by Lemma 5.1 there exist rational functions $R(z), \tilde{P}(z)$, and $W(z)$ with $\deg W(z) > 1$ such that

$$\left(\frac{q}{P'}\right)(z) = R(W(z)), \quad P(z) = \tilde{P}(W(z)) \tag{88}$$

and hence (23) holds for $\tilde{q}(z) = R(z)\tilde{P}'(z)$ since (88) yields that

$$q(z) = R(W(z))P'(z) = R(W(z))\tilde{P}'(W(z))W'(z) = \tilde{q}(W(z))W'(z).$$

In this section we in a sense describe the class of pairs $P(z), \gamma$ for which the rationality of $I_\infty(q, P, \gamma, t)$ implies its reducibility. For given $P(z)$ and γ such that poles of $P(z)$ do not lie on one side of γ , a natural necessary condition for the existence of $q(z)$ such that $I_\infty(q, P, \gamma, t)$ is rational but is not reducible may be formulated as follows. Let

$$\sum_{i=1}^n f_{s,i} \left(\frac{q}{P'}\right)(P_i^{-1}(z)) = 0, \quad 1 \leq s \leq k,$$

be the system of equations from Theorem 4.1 and let $M_{P,\gamma}$ be a linear subspace of \mathbb{Q}^n generated by the vectors

$$(f_{s,\sigma(1)}, f_{s,\sigma(2)}, \dots, f_{s,\sigma(n)}), \quad \sigma \in G_P, \quad 1 \leq s \leq k,$$

where G_P is the monodromy group of $P(z)$ and $n = \deg P(z)$. By Corollary 4.2 the subspace $M_{P,\gamma}$ is not zero-dimensional. Furthermore, by construction $M_{P,\gamma}$ is invariant with respect to the permutation representation of G_P over \mathbb{Q} and it follows from Theorem 4.1 by the analytic continuation that for any vector $\vec{v} \in M_{P,\gamma}$, $\vec{v} = (v_1, v_2, \dots, v_n)$, the equality

$$\sum_{i=1}^n v_i \left(\frac{q}{P^i} \right) (P_i^{-1}(z)) = 0 \tag{89}$$

holds.

Observe now that if $M_{P,\gamma}$ contains a vector of the form $\vec{e}_i - \vec{e}_j$, $i \neq j$, where \vec{e}_i , $1 \leq i \leq n$, denote vectors of the Euclidean basis of \mathbb{Q}^n , then the rationality of $I_\infty(q, P, \gamma, t)$ implies its reducibility since for such a vector equality (89) implies (87) and (88). Therefore, a necessary condition for the existence of $q(z)$ such that $I_\infty(q, P, \gamma, t)$ is rational but is not reducible is that $M_{P,\gamma}$ contains no vectors of the form $\vec{e}_i - \vec{e}_j$, $i \neq j$, and in this section, using a general result of [18], we prove (Theorem 9.3 below) that this condition is also sufficient. As an application we show that the requirement of Theorem 5.2 can not be weakened to the requirement of indecomposability of $P(z)$ already for Laurent polynomials (see the remark before Section 6).

9.2. Girstmair’s theorem. Let $f(t) \in K[t]$ be an irreducible polynomial over a field of characteristic zero K . Denote by x_1, x_2, \dots, x_n roots of $f(t)$, by L the field $K(x_1, x_2, \dots, x_n)$, and by G the Galois group $\text{Gal}(L/K)$. We will identify G with a permutation group acting on the set $\{1, 2, \dots, n\}$ setting $\sigma(i) = j$ if $x_j = \sigma(x_i)$, $1 \leq i, j \leq n$. In this subsection, following [18], we sketch a solution of the following problem: *under what conditions on a collection W of vectors from K^n there exists a rational function $R(t) \in K(t)$ such that for all $\vec{w} \in W$, $\vec{w} = (w_1, w_2, \dots, w_n)$, the equality*

$$w_1 R(x_1) + w_2 R(x_2) + \dots + w_n R(x_n) = 0 \tag{90}$$

holds, and $R(x_i) \neq R(x_j)$ for any $i \neq j$, $1 \leq i, j \leq n$. If such a function $R(t)$ exists we will say that W is *admissible*.

Notice that if (90) holds for vectors \vec{w}_1, \vec{w}_2 , then it holds for any vector of the form $\alpha \vec{w}_1 + \beta \vec{w}_2$, $\alpha, \beta \in K$. Furthermore, for any element $\sigma \in G$, acting on equality (90) by σ^{-1} , we obtain the equality

$$w_{\sigma(1)} R(x_1) + w_{\sigma(2)} R(x_2) + \dots + w_{\sigma(n)} R(x_n) = 0.$$

Therefore, equality (90) holds for all $\vec{w} \in W$ if and only if it holds for all vectors from the linear subspace of K^n generated by the vectors

$$\vec{w}^\sigma = (w_{\sigma(1)}, w_{\sigma(2)}, \dots, w_{\sigma(n)}), \quad w \in W, \sigma \in G.$$

Thus, without loss of generality we may assume that the collection W is a linear subspace of K^n invariant with respect to the permutation representation of G .

We start from reformulating the problem above in the form it was considered in [18]. Fix a root x of $f(t)$. Denote by H the stabilizer G_x of x in G and by $G/H = \{\bar{s} : s \in G\}$ the set of left cosets $\bar{s} = sH$ of the subgroup H in G . Further,

denote by $K[G]$ the group ring of G over K and by $K[G/H]$ a K -module with the basis $(\bar{s} : \bar{s} \in G/H)$. Thus, elements of $K[G]$ have the form

$$\lambda = \sum_{s \in G} l_s s, \quad l_s \in K, \tag{91}$$

while elements of $K[G/H]$ have the form

$$\alpha = \sum_{\bar{s} \in G/H} a_{\bar{s}} \bar{s}, \quad a_{\bar{s}} \in K. \tag{92}$$

Notice that $K[G/H]$ is a $K[G]$ -module with respect to the scalar multiplication defined by the formula

$$g\bar{s} = \overline{gs}, \quad g \in G, \bar{s} \in G/H.$$

If $y \in L$ satisfies $G_y = H$, then for any $\alpha \in K[G/H]$ defined by (92) the expression

$$\alpha y = \sum_{\bar{s} \in G/H} a_{\bar{s}} s(y)$$

is a well defined element of L . We say that a subset M of $K[G/H]$ is *admissible* if there exists $y \in L$ such that $G_y = H$ and for any $\alpha \in M$ the equality $\alpha y = 0$ holds. Clearly, M is admissible if and only if the $K[G]$ -submodule of $K[G/H]$ generated by M is admissible so without loss of generality we may assume that M is a $K[G]$ -submodule of $K[G/H]$.

Recall that linear subspaces of K^n invariant with respect to the permutation representation of G on K^n are in one-to-one correspondence with $K[G]$ -submodules of $K[G/H]$. Namely, to a subspace W corresponds a submodule \widehat{W} consisting of elements

$$\alpha = \sum_{i=1}^n a_i \bar{s}_i,$$

where $s_i, 1 \leq i \leq n$, is an element of G which transforms 1 to i and $\vec{a} = (a_1, a_2, \dots, a_n)$ runs elements of W .

Proposition 9.1. *A linear subspace W of K^n , invariant with respect to the permutation representation of G on K^n , is admissible if and only if the corresponding $K[G]$ -submodule \widehat{W} of $K[G/H]$ is admissible.*

Proof. Show first that if W is admissible, then we can set $y = R(x_1)$. Indeed, since $y \in K(x_1)$ we have $H \subseteq G_y$. Furthermore, H may not be a proper subgroup of G_y since otherwise the length of the orbit of y under the action of G would be strictly less than n in contradiction with the conditions that all $R(x_i), 1 \leq i \leq n$, are different between themselves. Finally, $\alpha y = 0$ for any $\alpha \in \widehat{W}$.

In other direction, if \widehat{W} is admissible and y is an element of L such that $\alpha y = 0$ for all $\alpha \in \widehat{W}$, then $G_y = H$ implies that $y \in K(x_1)$. Therefore, there exists $R(t) \in K(t)$ such that $y = R(x_1)$ and for such $R(z)$ equality (90) holds for all $\vec{w} \in W$. Furthermore, since the length of the orbit of y under the action of G equals n , all $R(x_i), 1 \leq i \leq n$, are different between themselves. \square

Theorem 9.2 [18]. *A $K[G]$ -submodule \widehat{W} of $K[G/H]$ is admissible if and only if \widehat{W} contains no elements $\bar{s}_1 - \bar{s}_2$, $s_1, s_2 \in G$, unless $\bar{s}_1 = \bar{s}_2$. Equivalently, a linear subspace W of K^n , invariant with respect to the permutation representation of G on K^n , is admissible if and only if W contains no vectors $\vec{e}_i - \vec{e}_j$, $1 \leq i, j \leq n$, unless $i = j$.*

Proof. If W contains a vector $\vec{w} = \vec{e}_i - \vec{e}_j$, $i \neq j$, $1 \leq i, j \leq n$, then equality (90) implies that $R(x_i) = R(x_j)$.

In another direction, assume that \widehat{W} contains no elements $\bar{s}_1 - \bar{s}_2$, $s_1, s_2 \in G$, such that $\bar{s}_1 \neq \bar{s}_2$. Consider the canonical $K[G]$ -linear map

$$\rho: K[G] \rightarrow K[G/H]$$

which maps s to \bar{s} , and let $\gamma = \rho^{-1}(\widehat{W})$ be the inverse image of \widehat{W} . Since $K[G]$ is semisimple, the ideal γ is generated by an idempotent element ε . Notice that for any $\lambda \in \gamma$ the equalities $\lambda = a\varepsilon$, $a \in K[G]$, and $\varepsilon^2 = \varepsilon$ imply that $\lambda\varepsilon = \lambda$. Set $\mu = 1 - \varepsilon$.

For any $s \in H$ the element $s - 1$ is in the kernel of ρ and therefore in γ . Hence $s - 1 = (s - 1)\varepsilon$ and $(s - 1)\mu = 0$ implying that $H \subseteq G_\mu$. On the other hand, for any $s \in G_\mu$ we have $(s - 1)\mu = 0$. Therefore, $s - 1 = (s - 1)\varepsilon$ and $s - 1 \in \gamma$. This implies that $\bar{s} - \bar{1}$ is in \widehat{W} and therefore $s \in H$ by the assumption. This proves that $G_\mu = H$.

Since $G_\mu = H$, for any $\alpha \in \widehat{W}$ the expression

$$\alpha\mu = \sum_{\bar{s} \in G/H} a_{\bar{s}} s\mu$$

is a well defined element of $K[G]$. Furthermore, in fact for any $\alpha \in \widehat{W}$ the equality $\alpha\mu = 0$ holds. Indeed, if λ is an element of γ such that $\rho(\lambda) = \alpha$, then we have:

$$\alpha\mu = \lambda\mu = \lambda - \lambda\varepsilon = 0.$$

Finally, let us show that from the existence of $\mu \in K[G]$ such that $H = G_\mu$ and $\alpha\mu = 0$ for any $\alpha \in \widehat{W}$, it follows that \widehat{W} is admissible. For this purpose observe that by the normal basis theorem there exists an element $x \in L$ such that gx , $g \in G$, is a basis of L over K . Set now $y = \mu x$. Then obviously for any $\alpha \in \widehat{W}$ the equality $\alpha y = 0$ holds and $H \subseteq G_y$. Furthermore, $H = G_y$. Indeed, if there exists $g_0 \in G$ such that $g_0 y = y$ but $g_0 \mu \neq \mu$, then the equalities $g_0 \mu x = y$, $\mu x = y$ imply that gx , $g \in G$, are linearly dependent over K . \square

9.3. Existence of $q(z)$ with rational but not reducible $I_\infty(q, P, \gamma, t)$. Theorem 9.2 permits to solve the problem posed in Subsection 9.1 as follows.

Theorem 9.3. *Let γ be a curve and $P(z)$ be a rational function of degree n such that poles of $P(z)$ do not lie on one side of γ . Then a rational function $q(z)$ such that $I_\infty(q, P, \gamma, t)$ is rational but is not reducible exists if and only if the subspace $M_{P,\gamma}$ contains no vectors $\vec{e}_i - \vec{e}_j$, $1 \leq i, j \leq n$, unless $i = j$.*

Proof. As it was already observed in Subsection 9.1 the requirement of the theorem is necessary. On the other hand, since vectors with rational coefficients which are

linear independent over \mathbb{Q} remain linearly independent over $\mathbb{C}(z)$, if this requirement is satisfied, then the subspace $\tilde{M}_{P,\gamma}$ of $(\mathbb{C}(z))^n$, generated over $\mathbb{C}(z)$ by the same vectors

$$(f_{s,\sigma(1)}, f_{s,\sigma(2)}, \dots, f_{s,\sigma(n)}), \quad \sigma \in G_P, \quad 1 \leq s \leq k,$$

which generate $M_{P,\gamma}$ over \mathbb{Q} , still contains no vectors $\vec{e}_i - \vec{e}_j$, $1 \leq i, j \leq n$, unless $i = j$.

Therefore, applying Theorem 9.2 to the roots

$$x_1 = P_1^{-1}(z), \quad x_2 = P_2^{-1}(z), \quad \dots, \quad x_n = P_n^{-1}(z)$$

of the polynomial $P(x) - z = 0$ over the field $\mathbb{C}(z)$, we conclude that there exists a rational function $\tilde{R}(t)$ over the field $\mathbb{C}(z)$ such that for any vector $\vec{v} \in \tilde{M}_{P,\gamma}$, $\vec{v} = (v_1, v_2, \dots, v_n)$, the equality

$$\sum_{i=1}^n v_i \tilde{R}(P_i^{-1}(z)) = 0 \tag{93}$$

holds and all $\tilde{R}(P_i^{-1}(z))$, $1 \leq i \leq n$, are different between themselves. Furthermore, since $z = P(P_i^{-1}(z))$, $1 \leq i \leq n$, we can write any rational function in z as a rational function in $P_i^{-1}(z)$, $1 \leq i \leq n$, and hence there exists a rational function $R(t)$ over \mathbb{C} such that

$$R(P_i^{-1}(z)) = \tilde{R}(P_i^{-1}(z)), \quad 1 \leq i \leq n.$$

Setting now $q(z) = R(z)P'(z)$ we see that for any vector $\vec{v} \in M_{P,\gamma}$ the equality (89) holds, implying that $I_\infty(q, P, \gamma, t)$ is a rational function. Furthermore, equality (23) is impossible since otherwise the functions $P(z)$ and $(q/P')(z)$ have a non-trivial common right factor and Lemma 5.1 implies that

$$R(P_i^{-1}(z)) = R(P_j^{-1}(z))$$

for some $i \neq j$, $1 \leq i, j \leq n$. □

Using Theorem 9.3 one can prove the existence of an indecomposable Laurent polynomial $L(z)$ for which there exists a rational function $q(z)$ such that $I_\infty(q, P, \gamma, t)$ is rational but is not reducible, without an actual calculation $L(z)$ and $q(z)$. Indeed, let $L(z)$ be a Laurent polynomial whose constellation is shown on Fig. 4 (since $L(z)$ has only two finite critical values, in correspondence with the no-

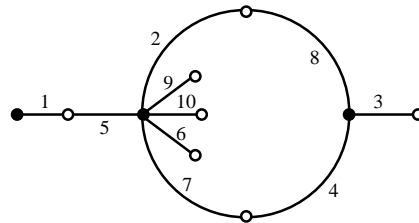


FIGURE 4.

tation of “dessins d’enfants” theory, we picture here 1-vertices as “black”, 2-vertices as “white”, and do not mark non-numerated vertices at all). Such a choice of $L(z)$

is motivated by the fact that the action of the monodromy group G of $L(z)$ on branches of $L^{-1}(z)$, generated by the permutations $\alpha = (2, 5, 7, 6, 10, 9)(3, 8, 4)$ and $\beta = (1, 5)(2, 8)(4, 7)$, is permutation equivalent to the action of the group S_5 on two element subsets of $\{1, 2, 3, 4, 5\}$. Since the last action is primitive while the corresponding matrix representation of dimension 10 over \mathbb{Q} is not a sum of $V_{\mathbb{Q}}$ and $V_{\mathbb{Q}}^{\perp}$, one can expect that $L(z)$ provides a desired example.

By Theorem 4.1 the function $I_{\infty}(q, P, \gamma, t)$ is rational if and only if the equality

$$Q(L_2^{-1}(z)) - Q(L_7^{-1}(z)) + Q(L_4^{-1}(z)) - Q(L_8^{-1}(z)) \equiv 0, \tag{94}$$

holds. Therefore, the subspace M_{L,S^1} is generated by the single vector

$$\vec{v} = (0, 1, 0, 1, 0, 0, -1, -1, 0, 0).$$

Show now that M_{L,S^1} may not contain a vector w of the form

$$w = \vec{e}_i - \vec{e}_j, \quad i \neq j, \quad 1 \leq i, j \leq n. \tag{95}$$

Consider the vector subspace V of \mathbb{Q}^{10} generated by the vectors

$$\begin{aligned} \vec{v}_1 &= (1, 0, 0, 0, 1, 1, 0, 0, 1, 0), \\ \vec{v}_2 &= (1, 1, 0, 0, 0, 0, 1, 0, 0, 1), \\ \vec{v}_3 &= (0, 1, 1, 0, 0, 1, 0, 1, 0, 0), \\ \vec{v}_4 &= (0, 0, 1, 1, 0, 0, 1, 0, 1, 0), \\ \vec{v}_5 &= (0, 0, 0, 1, 1, 0, 0, 1, 0, 1). \end{aligned}$$

Since α and β permute the vectors \vec{v}_i , $1 \leq i \leq 5$, between themselves, V is ρ_G -invariant. Furthermore, since \vec{v} is orthogonal to \vec{v}_i , $1 \leq i \leq 5$, the inclusion

$$M_{L,S^1} \subseteq V^{\perp}$$

holds. On the other hand, it is easy to see that for any vector (95) there exists \vec{v}_i , $1 \leq i \leq 5$, such that $(w, v_i) \neq 0$. Indeed, since G is transitive and permute \vec{v}_i , $1 \leq i \leq 5$, it is enough to verify this property only for w whose first coordinate equals 1 and for such w we may take one of the vectors v_1, v_2 . Therefore, M_{L,S^1} may not contain vectors (95) and hence by Theorem 9.3 there exists a rational function $q(z)$ such that $I_{\infty}(q, P, \gamma, t)$ is rational but is not reducible.

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ADDENDUM

This paper essentially coincides with the preprint [30] published in 2009. Below we briefly mention some related publications that appeared afterwards.

In the paper [17] — among other things — the moment problem for meromorphic functions on compact Riemann surfaces was investigated. In particular, some of results of Sections 4 and 5 were generalized to this case.

In the papers [32], [1] were constructed explicit examples of indecomposable Laurent polynomials $L(z)$, the existence of which was shown in the Section 9, such

that $I_\infty(q, P, \gamma, t)$ is rational for some rational function $q(z)$ but is not reducible. The paper [1] also contains some further results concerning Laurent polynomial moment problem. Notice that both these papers make use of the bound on the Bautin index for Laurent polynomial moment problem obtained in Section 8.

In the paper [40] the results of Section 4 were applied to the study of Mathieu subspaces of associative algebras.

Finally, in the recent preprints [13], [12] the problem of vanishing of double moments for real trigonometric polynomials on a segment was investigated in the context of the center problem for the Abel equation. In particular, it was shown in [13] that if f, g are real trigonometric polynomials such that all the moments $\int_0^{2\pi} f^i g^j dg$ ($i, j \geq 0$) vanish, then there exist polynomials \tilde{f}, \tilde{g} and a trigonometric polynomial h such that $f = \tilde{f}(h)$, $g = \tilde{g}(h)$. Notice that this result follows from results of Section 6 by an exponential substitution. However, the proof obtained in [13] for the real case is much easier.

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