ON THE POLYNOMIAL MOMENT PROBLEM

F. Pakovich

1. Introduction

In this paper we treat the following “polynomial moment problem”: for complex polynomials $P(z), Q(z) = \int q(z)dz$ and distinct $a, b \in \mathbb{C}$ such that $P(a) = P(b), Q(a) = Q(b)$ to find conditions under which

$$\int_{a}^{b} P^i(z)q(z)dz = 0$$

for all integer non-negative $i$.

The polynomial moment problem was proposed in the series of papers of M. Briskin, J.-P. Francoise and Y. Yomdin [1]-[5] as an infinitesimal version of the center problem for the polynomial Abel equation in the complex domain in the frame of a programme concerning the classical Poincaré center-focus problem for the polynomial vector field on the plane. It was suggested that the following “composition condition” imposed on $P(z)$ and $Q(z) = \int q(z)dz$ is necessary and sufficient for the pair $P(z), q(z)$ to satisfy (*): there exist polynomials $\tilde{P}(z), \tilde{Q}(z), W(z)$ such that

$$P(z) = \tilde{P}(W(z)), \quad Q(z) = \tilde{Q}(W(z)), \quad \text{and} \quad W(a) = W(b).$$

It is easy to see that the composition condition is sufficient: since after the change of variable $z \to W(z)$ the way of integration becomes closed, the sufficiency follows from the Cauchy theorem. The necessity of the composition condition in the case when $a, b$ are not critical points of $P(z)$ was proved by C. Christopher in [6] (see also the paper of N. Roytvarf [12] for a similar result) and in some other special cases by M. Briskin, J.-P. Francoise and Y. Yomdin in the papers cited above.

Nevertheless, in general the composition conjecture fails to be true. Namely, in the paper [9] a class of counterexamples to the composition conjecture was constructed. These counterexamples exploit polynomials $P(z)$ which admit double decompositions: $P(z) = A(B(z)) = C(D(z))$, where $A(z), B(z), C(z), D(z)$,
$D(z)$ are non-linear polynomials. If $P(z)$ is such a polynomial and, in addition, $B(a) = B(b)$, $D(a) = D(b)$ then for any polynomial $Q(z)$ which can be represented as $Q(z) = E(B(z)) + F(D(z))$ for some polynomials $E(z), F(z)$ condition (*) is satisfied with $q(z) = Q'(z)$. On the other hand, it was shown in [9] that if $\deg B(z)$ and $\deg D(z)$ are coprime then condition (***) is not satisfied already for $Q(z) = B(z) + D(z)$.

Note that double decompositions with $\deg A(z) = \deg D(z)$, $\deg B(z) = \deg C(z)$ and $\deg B(z), \deg D(z)$ coprime are described explicitly by Ritt’s theory of factorization of polynomials. They are equivalent to decompositions with $A(z) = z^n R^m(z)$, $B(z) = z^m$, $C(z) = z^m$, $D(z) = z^n R(z^m)$ for a polynomial $R(z)$ and $\text{GCD}(n, m) = 1$ or to decompositions with $A(z) = T_m(z)$, $B(z) = T_n(z)$, $C(z) = T_n(z)$, $D(z) = T_m(z)$ for Chebyshev polynomials $T_n(z)$, $T_m(z)$ and $\text{GCD}(n, m) = 1$ (see [11], [13]).

The counterexamples above suggest to weaken the composition conjecture as follows: polynomials $P(z), q(z)$ satisfy condition (*) if and only if $\int q(z) dz$ can be represented as a sum of polynomials $Q_j$ such that

\[
(***) \quad P(z) = \tilde{P}_j(W_j(z)), \quad Q_j(z) = \tilde{Q}_j(W_j(z)), \quad \text{and} \quad W_j(a) = W_j(b)
\]

for some $\tilde{P}_j(z), \tilde{Q}_j(z), W_j(z) \in \mathbb{C}[z]$. For the case when $P(z) = T_n(z)$ this statement was verified in [10]. Moreover, it was shown that for $P(z) = T_n(z)$ the number of terms in the representation $\int q(z) dz = \sum_j Q_j(z)$ can be reduced to two.

In this paper we give a solution of the polynomial moment problem in the case when $P(z)$ is indecomposable that is when $P(z)$ can not be represented as a composition $P(z) = P_1(P_2(z))$ with non-linear polynomials $P_1(z), P_2(z)$. In this case conditions (**) and (***) are equivalent and the composition conjecture reduces to the following statement.

**Theorem 1.** Let $P(z), Q(z) = \int q(z) dz$ be complex polynomials and let $a, b$ be distinct complex numbers such that $P(a) = P(b), Q(a) = Q(b)$, and

\[
\int_a^b P^i(z)q(z) dz = 0
\]

for $i \geq 0$. Suppose that $P(z)$ is indecomposable. Then there exists a polynomial $Q(z)$ such that $Q(z) = \tilde{Q}(P(z))$.

We also examine the following condition which is stronger than (*):

\[
\int_a^b P^i(z)Q^j(z)Q'(z) dz = 0
\]

for $i \geq 0, j \geq 0$. If $\gamma$ is a curve which is the image of the segment $[a, b]$ in $\mathbb{C}^2$ under the map $z \rightarrow (P(z), Q(z))$ then this condition is equivalent to the condition that $\int_{\gamma} \omega = 0$ for all global holomorphic 1-forms $\omega$ in $\mathbb{C}^2$ (“the moment condition”). For an oriented simple closed curve $\delta$ of class $C^2$ in $\mathbb{C}^2$ the moment condition is necessary and sufficient to be a boundary of a bounded analytic variety $\Sigma$.
in \( \mathbb{C}^2 \); it is a special case of the result of R. Harvey and B. Lawson [7]. The case when \( \delta \) is an image of \( S^1 \) under the map \( z \to (f(z), g(z)) \), where \( f(z), g(z) \) are functions analytic in an annulus containing \( S^1 \) was investigated earlier by J. Wermer [14]: in this case the moment condition is equivalent to the condition that there exists a finite Riemann surface \( \Sigma \) with border \( S^1 \) such that \( f(z), g(z) \) have an analytic extension to \( \Sigma \).

Unlike condition (*) the more restrictive moment condition imposed on polynomials \( P(z), Q(z) \) turns out to be equivalent to composition condition (**). We show that actually even a weaker condition is needed.

**Theorem 2.** Let \( P(z), Q(z) \) be complex polynomials and let \( a, b \) be distinct complex numbers such that \( P(a) = P(b), Q(a) = Q(b) \), and

\[
\int_a^b P^i(z)Q^j(z)Q'(z)dz = 0
\]

for \( 0 \leq i \leq \infty \), \( 0 \leq j \leq d_a + d_b - 2 \), where \( d_a \) (resp. \( d_b \)) is the multiplicity of the point \( a \) (resp. \( b \)) with respect to \( P(z) \). Then there exist polynomials \( \tilde{P}(z), \tilde{Q}(z), W(z) \) such that \( P(z) = \tilde{P}(W(z)), Q(z) = \tilde{Q}(W(z)), \) and \( W(a) = W(b) \).

Note that if \( a, b \) are not critical points of \( P(z) \) that is if \( d_a = d_b = 1 \) then conditions of the theorem reduce to condition (*) and therefore Theorem 2 includes as a particular case the result of C. Christopher.

## 2. Proofs

### 2.1. Lemmata about branches of \( Q(P^{-1}(z)) \)

Let \( P(z) \) and \( Q(z) \) be rational functions and let \( U \subset \mathbb{C} \) be a domain in which there exists a single-valued branch \( p^{-1}(z) \) of the algebraic function \( P^{-1}(z) \). Denote by \( Q(P^{-1}(z)) \) the complete algebraic function obtained by the analytic continuation of the functional element \( \{U, Q(p^{-1}(z))\} \). Since the monodromy group \( G(P^{-1}) \) of the algebraic function \( P^{-1}(z) \) is transitive this definition does not depend of the choice of \( p^{-1}(z) \). Denote by \( d(Q(P^{-1}(z))) \) the degree of the algebraic function \( Q(P^{-1}(z)) \) that is the number of its branches.

**Lemma 1.** Let \( P(z), Q(z) \) be rational functions. Then

\[
d(Q(P^{-1}(z))) = \deg P(z)/[\mathbb{C}(z) : \mathbb{C}(P,Q)].
\]

**Proof.** Since any algebraic relation over \( \mathbb{C} \) between \( Q(p^{-1}(z)) \) and \( z \) supplies an algebraic relation between \( Q(z) \) and \( P(z) \) and vice versa we see that \( d(Q(P^{-1}(z))) = [\mathbb{C}(P,Q) : \mathbb{C}(P)] \). As \( [\mathbb{C}(P,Q) : \mathbb{C}(P)] = [\mathbb{C}(z) : \mathbb{C}(P)]/ [\mathbb{C}(z) : \mathbb{C}(P,Q)] \) the lemma follows now from the observation that \( [\mathbb{C}(z) : \mathbb{C}(P)] = \deg P(z) \).

Recall that by Lüroth theorem each field \( k \) such that \( \mathbb{C} \subset k \subset \mathbb{C}(z) \) and \( k \neq \mathbb{C} \) is of the form \( k = \mathbb{C}(R), R \in \mathbb{C}(z) \setminus \mathbb{C} \). Therefore, the field \( \mathbb{C}(P,Q) \) is a proper subfield of \( \mathbb{C}(z) \) if and only if \( P(z) = \tilde{P}(W(z)), Q(z) = \tilde{Q}(W(z)) \) for some rational functions \( \tilde{P}(z), \tilde{Q}(z), W(z) \) with \( \deg W(z) > 1 \); in this case we
say that \( P(z) \) and \( Q(z) \) have a common right divisor in the composition algebra. The Lemma 1 implies the following explicit criterion which is essentially due to Ritt [11] (cf. also [6], [12]).

**Corollary 1.** Let \( P(z), Q(z) \) be rational functions. Then \( P(z) \) and \( Q(z) \) have a common right divisor in the composition algebra if and only if

\[
Q(p^{-1}(z)) = Q(\tilde{p}^{-1}(z))
\]

for two different branches \( p^{-1}(z), \tilde{p}^{-1}(z) \) of \( P^{-1}(z) \).

**Proof.** Indeed, by Lemma 1, the field \( \mathbb{C}(P, Q) \) is a proper subfield of \( \mathbb{C}(z) \) if and only if \( d(Q(P^{-1}(z))) < \deg P(z) \). On the other hand, the last inequality is clearly equivalent to condition (1).

\[\square\]

**Lemma 2.** Let \( P(z), Q(z) \) be rational functions, \( \deg P(z) = n \). Suppose that there exist \( a_i \in \mathbb{C}, 1 \leq i \leq n \), not all equal between themselves such that

\[
\sum_{i=1}^{n} a_i Q(p_i^{-1}(z)) = 0.
\]

If, in addition, the group \( G(P^{-1}) \) is doubly transitive then \( Q(z) = \tilde{Q}(P(z)) \) for some rational function \( \tilde{Q}(z) \).

**Proof.** Let \( G \subset S_n \) be a permutation group and let \( \rho_G : G \rightarrow GL(\mathbb{C}^n) \) be the permutation representation of \( G \) that is \( \rho_G(g), g \in G \) is the linear map which sends a vector \( \vec{a} = (a_1, a_2, ..., a_n) \) to the vector \( \vec{a}_g = (a_{g(1)}, a_{g(2)}, ..., a_{g(n)}) \). It is well known (see e.g. [15], Th. 29.9) that \( G \) is doubly transitive if and only if \( \rho_G \) is the sum of the identical representation and an absolutely irreducible representation. Clearly, the one-dimensional \( \rho_G \)-invariant subspace \( E \subset \mathbb{C}^n \) corresponding to the identity representation is generated by the vector \((1, 1, ..., 1)\). Therefore, since the Hermitian inner product \((\vec{a}, \vec{b}) = a_1 b_1 + a_2 b_2 + ... + a_n b_n\) is invariant with respect to \( \rho_G \), the group \( G \) is doubly transitive if and only if the subspace \( E \) and its orthogonal complement \( E^\perp \) are the only \( \rho_G \)-invariant subspaces of \( \mathbb{C}^n \).

Suppose that (2) holds. In this case also

\[
\sum_{i=1}^{n} a_i Q(p_{\sigma(i)}^{-1}(z)) = 0
\]

for all \( \sigma \in G(P^{-1}) \) by the analytic continuation. To prove the lemma it is enough to show that \( Q(p_i^{-1}(z)) = Q(p_j^{-1}(z)) \) for all \( i, j, 1 \leq i, j \leq n \); then by Lemma 1 \([\mathbb{C}(z) : \mathbb{C}(P, Q)] = \deg P(z) = [\mathbb{C}(z) : \mathbb{C}(P)] \) and therefore \( Q(z) = \tilde{Q}(P(z)) \) for some rational function \( \tilde{Q}(z) \). Assume the converse i.e. that there exists \( z_0 \in U \) such that not all \( Q(p_i^{-1}(z_0)) \), \( 1 \leq i \leq n \), are equal between themselves. Without loss of generality we can suppose that all \( Q(p_i^{-1}(z_0)), 1 \leq i \leq n \), are finite. Consider the subspace \( V \subset \mathbb{C}^n \) generated by the vectors \( \vec{v}_\sigma, \sigma \in G(P^{-1}) \),
where \( \tilde{\sigma} = (Q(p_{\sigma(1)}(z_0)), Q(p_{\sigma(2)}(z_0)), ..., Q(p_{\sigma(n)}(z_0))) \). Clearly, \( V \) is \( \rho_G(P-1) \)-invariant and \( V \neq E \). Moreover, it follows from (3) that \( V \) is contained in the orthogonal complement \( A^\perp \) of the subspace \( A \subset \mathbb{C}^n \) generated by the vector \((\bar{a}_1, \bar{a}_2, ..., \bar{a}_n)\). Since \( A \neq E \) we see that \( V \) is a proper \( \rho_G \)-invariant subspace of \( \mathbb{C}^n \) distinct from \( E \) and \( E^\perp \) that contradicts the assumption that the group \( G(P-1) \) is doubly transitive. \( \square \)

2.2. Lemma about preimages of domains. For a polynomial \( P(z) \) denote by \( c(P) \) the set of finite critical values of \( P(z) \).

**Lemma 3.** Let \( P(z) \) be a polynomial and let \( V \subset \mathbb{C}P^1 \) be a simply connected domain containing infinity such that \( c(P) \cap V = \emptyset \). Then \( P^{-1}(V) \) is conformally equivalent to the unit disk and \( P^{-1}(\partial V) \) is connected.

**Proof.** Indeed, by the Riemann theorem \( V \) is conformally equivalent to the unit disk \( \mathbb{D} \) whenever \( \partial V \) contains more than one point. It follows from \( c(P) \cap V = \emptyset \) that \( \partial V \) contains a unique point if and only if \( P(z) \) has a unique finite critical value \( c \) and \( \partial V = \{ c \} \); in this case there exist linear functions \( \sigma_1, \sigma_2 \) such that \( \sigma_1(P(\sigma_2(z))) = z^n, n \in \mathbb{N} \) and the lemma is obvious. Therefore, we can suppose that \( V \cong \mathbb{D} \). Since \( c(P) \cap V = \emptyset \) the restriction of the map \( P(z) : \mathbb{C}P^1 \to \mathbb{C}P^1 \) on \( P^{-1}(V) \setminus P^{-1}(\{ \infty \}) \) is a covering map. As \( V \setminus \{ \infty \} \) is conformally equivalent to the punctured unit disc \( \mathbb{D}^* \) it follows from covering spaces theory that \( P^{-1}(V) \setminus P^{-1}(\{ \infty \}) \) is a disjoint union of domains \( \cup V_i \) conformally equivalent to \( \mathbb{D}^* \) such that all induced maps \( f_i : \mathbb{D}^* \to \mathbb{D}^* \) are of the form \( z \to z^{l_i}, l_i \in \mathbb{N} \). But, as \( P^{-1}(\{ \infty \}) = \{ \infty \} \), there may be only one such a domain. Therefore, the preimage \( P^{-1}(V) \) is conformally equivalent to the unit disk. In particular, since \( P^{-1}(\partial V) = \partial P^{-1}(V) \) we see that \( P^{-1}(\partial V) \) is connected. \( \square \)

2.3. Proof of Theorem 2: the case of a regular value. In this section we investigate the case when \( t_0 = P(a) = P(b) \) is not a critical value of the polynomial \( P(z) \). For a simple closed curve \( M \subset \mathbb{C} \) denote by \( D^+_M \) (resp. \( D^-_M \)) the domain that is interior (resp. exterior) with respect to \( M \).

Let \( L \subset \mathbb{C} \) be a simple closed curve such that \( t_0 \in L \) and \( c(P) \subset D^+_L \). Denote by \( \tilde{L} \) the same curve considered as an oriented graph embedded into the complex plane. By definition, the graph \( \tilde{L} \) has one vertex \( t_0 \) and one counter-clockwise oriented edge \( l \). Let \( \tilde{\Omega} = P^{-1}(\tilde{L}) \) be an oriented graph which is the preimage of the graph \( \tilde{L} \) under the mapping \( P(z) : \mathbb{C} \to \mathbb{C} \), i.e. vertices of \( \tilde{\Omega} \) are preimages of \( t_0 \) and oriented edges of \( \tilde{\Omega} \) are preimages of \( l \). As \( L \cap c(P) = \emptyset \) the graph \( \tilde{\Omega} \) has \( n = \deg P(z) \) vertices and \( n \) edges. Furthermore, by Lemma 3 the graph \( \tilde{\Omega} = P^{-1}(\partial L) \) is connected. Therefore, as a point set in \( \mathbb{C} \) the graph \( \tilde{\Omega} \) is a simple closed curve. Let \( l_j, 1 \leq j \leq n \), be oriented edges of \( \tilde{\Omega} \) and let \( a_j \) (resp. \( b_j \)) be the starting (resp. ending) point of \( l_j \). We will suppose that edges of \( \tilde{\Omega} \) are numerated by such a way that \( a_1 = a \) and that under a moving around the domain \( P^{-1}(D^-_L) \) along its boundary \( \tilde{\Omega} \) the edge \( l_i, 1 \leq i \leq n - 1 \), is followed by the edge \( l_{i+1} \) (see fig. 1).
Let \( U \subset \mathbb{C} \) be a simply connected domain such that \( U \cap c(P) = \emptyset \) and \( L \setminus \{t_0\} \subset U \). By the monodromy theorem, in such a domain there exist \( n \) single-valued branches of \( P^{-1}(t) \). Denote by \( p_j^{-1}(t) \), \( 1 \leq j \leq n \), the single-valued branch of \( P^{-1}(t) \) defined in \( U \) by the condition \( p_j^{-1}(t) \setminus \{t_0\} = l_j \setminus \{a_j, b_j\} \); such a numeration of branches of \( P^{-1}(t) \) means that the analytic continuation of the functional element \( \{U, p_j^{-1}(t)\}, 1 \leq j \leq n - 1 \), along \( L \) is the functional element \( \{U, p_j^{-1}(t)\} \). Let \( l_k, k < n \), be the edge of \( \Omega \) such that \( b_k = b \) and let \( \Gamma = \{l_1, l_2, ..., l_k\} \) be the oriented path in the graph \( \Omega \) joining the vertices \( a_1 = a \) to \( b_k = b \). For \( t \in U \) set \( \varphi(t) = \sum_{j=1}^{k} Q(p_j^{-1}(t)) \).

Consider an analytic function on \( \mathbb{CP}^1 \setminus L \)

\[
I(\lambda) = \oint_{\Gamma} \frac{\varphi(t)}{t - \lambda} \, dt = \int_{\Gamma} \frac{Q(z)P'(z) \, dz}{P(z) - \lambda}.
\]

More precisely, the integral above defines two analytic functions: one of them \( I^+(\lambda) \) is analytic in \( D_L^+ \) and the other one \( I^-(\lambda) \) is analytic in \( D_L^- \). Furthermore, calculating the Taylor expansion of \( I^-(\lambda) \) at infinity and using integration by part we see that condition (*) reduces to the condition that \( I^-(\lambda) \equiv 0 \) in \( D_L^- \).

By a well-known result about integrals of the Cauchy type (see e.g. [8]) the last condition implies that \( \varphi(t) \) is the boundary value on \( L \) of the analytic function \( I^+(\lambda) \) in \( D_L^+ \). It follows from the uniqueness theorem for boundary values of analytic functions that the functional element \( \{U, \varphi(t)\} \) can be analytically continued along any curve \( M \subset D_L^+ \). As \( c(P) \subset D_L^+ \) this fact implies that \( \{U, \varphi(t)\} \) can be analytically continued along any curve \( M \subset \mathbb{C} \). Therefore, by the monodromy theorem, the element \( \{U, \varphi(t)\} \) extends to a single-valued analytic function in the whole complex plane. In particular, the analytic continuation of \( \{U, \varphi(t)\} \) along any closed curve coincides with \( \{U, \varphi(t)\} \). On the other hand, by construction the analytic continuation of \( \{U, \varphi(t)\} \) along the curve \( L \) is \( \{U, \varphi_L(t)\} \), where \( \varphi_L(t) = \sum_{j=1}^{k+1} Q(p_j^{-1}(t)) \). It follows from \( \varphi(t) = \varphi_L(t) \) that \( Q(p_1^{-1}(t)) = Q(p_{k+1}^{-1}(t)) \) and by Corollary 1 we conclude that \( P(z) \) and \( Q(z) \) have a common right divisor in the composition algebra.
As the field \( \mathbb{C}(P,Q) \) is a proper subfield of \( \mathbb{C}(z) \) and \( P(z) \), \( Q(z) \) are polynomials it is easy to prove that \( \mathbb{C}(P,Q) = \mathbb{C}(W) \) for some polynomial \( W(z) \), \( \deg W(z) > 1 \). It means that \( P(z) = \tilde{P}(W(z)), Q(z) = \tilde{Q}(W(z)) \) for some polynomials \( \tilde{P}(z), \tilde{Q}(z) \) such that \( \tilde{P}(z) \) and \( \tilde{Q}(z) \) have no a common right divisor in the composition algebra. Let us show that \( W(a) = W(b) \). Since \( t_0 \) is not a critical value of the polynomial \( P(z) = \tilde{P}(W(z)) \) the chain rule implies that \( t_0 \) is not a critical value of the polynomial \( \tilde{P}(z) \). Therefore, if \( W(a) \neq W(b) \) then after the change of variable \( z \to W(z) \) in the same way as above we find that \( \tilde{P}(z) = \tilde{P}(U(z)), \tilde{Q}(z) = \tilde{Q}(U(z)) \) for some polynomials \( \tilde{P}(z), \tilde{Q}(z), U(z) \) with \( \deg U(z) > 1 \) that contradicts the fact that \( \tilde{P}(z), \tilde{Q}(z) \) have no a common right divisor in the composition algebra. This completes the proof in the case when \( z_0 \) is not a critical value of \( P(z) \).

2.4. Proof of Theorem 2: the case of a critical value. Assume now that \( t_0 = P(a) = P(b) \) is a critical value of \( P(z) \). In this case let \( L \) be a simple closed curve such that \( t_0 \in L \) and \( c(P) \setminus t_0 \subset D_L^+ \). Consider again a graph \( \tilde{\Omega} = P^{-1}\{\tilde{L}\} \). Since \( P^{-1}\{D_L^-\} \) is still conformally equivalent to the unit disk by Lemma 3, we see that the graph \( \tilde{\Omega} \) topologically is the boundary of a disc although it is not a simple closed curve any more. Let \( l_j, 1 \leq j \leq n, \) be oriented edges of \( \tilde{\Omega} \) and let \( a_j \) (resp. \( b_j \)) be the starting (resp. the ending) point of \( l_j \). Let us fix again such a number of edges of \( \tilde{\Omega} \) that \( a_1 = a \) and that under a moving around the domain \( P^{-1}\{D_L^-\} \) along its boundary \( \tilde{\Omega} \) the edge \( l_i, 1 \leq i \leq n - 1, \) is followed by the edge \( l_{i+1} \). As above denote by \( U \) a domain in \( \mathbb{C} \) such that \( U \cap c(P) = \emptyset, L \setminus \{t_0\} \subset U \) and let \( p_j^{-1}(t), 1 \leq j \leq n, \) be the single-valued branch of \( P^{-1}(t) \) defined in \( U \) by the condition \( p_j^{-1}\{l \setminus t_0\} = l_j \setminus \{a_j, b_j\} \). If \( k < n \) is a number such that \( b_k = b \) then for the same reason as above the function \( \varphi(t) = \sum_{j=1}^{k} Q(p_j^{-1}(t)) \) extends to an analytic function in \( U \cup D_L^+ \) but this fact does not imply now that \( \varphi(t) \) extends to an analytic function in the whole complex plane since \( D_L^+ \) does not contain \( t_0 \in c(P) \). Nevertheless, if \( V \) is a simply connected domain such that \( U \subset V \) and \( t_0 \notin V \) then \( \varphi(t) \) still extends to a single-valued analytic function in \( V \). In particular, the analytic continuation of \( \{U, \varphi(t)\} \) along any simple closed curve \( M \) such that \( t_0 \subset D_M^- \) coincides with \( \{U, \varphi(t)\} \).

Let \( t_1 \in U \) be a point and let \( M_1 \) (resp. \( M_2 \)) be a simple closed curve such that \( t_1 \in M_1, M_1 \cap c(P) = \emptyset \) and \( D_{M_1}^+ \cap c(P) = t_0 \) (resp. \( t_1 \in M_2, M_2 \cap c(P) = \emptyset \) and \( D_{M_2}^+ \cap c(P) = c(P) \setminus t_0 \)). Define a permutation \( \rho_1 \in S_n \) (resp. \( \rho_2 \in S_n \)) by the condition that the functional element \( \{U, p_1^{-1}(t)\} \) (resp. \( \{U, p_2^{-1}(t)\} \)) is the result of the analytic continuation of the functional element \( \{U, p_j^{-1}(t)\}, 1 \leq j \leq n, \) from \( t_1 \) along the curve \( M_1 \) (resp. \( M_2 \)). Having in mind the identification of the set of elements \( \{U, p_j^{-1}(t)\}, 1 \leq j \leq n, \) with the set of oriented edges of the graph \( \tilde{\Omega} \) the permutations \( \rho_1, \rho_2 \) can be described as follows: \( \rho_1 \) cyclically permutes the edges of \( \tilde{\Omega} \) around the vertices from which they go.
\[ \rho_1 = (28)(467), \quad \rho_2 = (18)(237)(45) \]

**Figure 2**

while cycles \((j_1, j_2, \ldots, j_k)\) of \(\rho_2\) correspond to simple cycles \((l_{j_1}, l_{j_2}, \ldots, l_{j_k})\) of the graph \(\bar{\Omega}\) and \(\rho_1 \rho_2 = (12\ldots n)\) (see fig. 2).

To unload notation denote temporarily the element \(\{U, Q(p_i^{-1}(t))\}\), \(1 \leq i \leq n\), by \(s_i\). Since \(t_0 \subset D_{M_2}\) we have:

\[
0 = \sum_{j=1}^{k} s_{\rho_2(j)} - \sum_{j=1}^{k} s_j = s_{\rho_2(k)} + \sum_{j=1}^{k-1} [s_{\rho_2(j)} - s_{j+1}] - s_1.
\]

Using \(\rho_1 \rho_2 = (12\ldots n)\) we can rewrite (4) as

\[
s_{\rho_1^{-1}(k+1)} - s_1 + \sum_{j=1}^{k-1} [s_{\rho_2(j)} - s_{\rho_1 \rho_2(j)}] = 0.
\]

Therefore, by the analytic continuation

\[
(5) \quad s_{\rho_1^{-1}(k+1)} - s_{\rho_1(1)} + \sum_{j=1}^{k-1} [s_{\rho_1 \rho_2(j)} - s_{\rho_1 \rho_2(j)}] = 0
\]

for \(f \geq 0\). Summing equalities (5) from \(f = 1\) to \(f = o(\rho_1)\), where \(o(\rho_1)\) is the order of the permutation \(\rho_1\), changing the order of summing, and observing that

\[
\sum_{f=0}^{o(\rho_1)-1} \left[ s_{\rho_1 \rho_2(j)} - s_{\rho_1 \rho_2(j)} \right] = s_{\rho_2(j)} - s_{\rho_1 \rho_2(j)} = 0
\]

we conclude that

\[
(6) \quad \sum_{s=0}^{o(\rho_1)-1} Q(p_1^{-1}(t)) = \sum_{s=0}^{o(\rho_1)-1} Q(p_{\rho_1(1)}^{-1}(t))
\]

in \(U\). Note that if \(a, b\) are regular points of \(P(z)\) then \(\rho_1(1) = 1, \rho_1(k+1) = k+1\) and (6) reduces to the equality \(Q(p_{k+1}^{-1}(t)) = Q(p_1^{-1}(t))\).
Since (6) holds for any polynomial $Q(z)$ such that $q(z) = Q'(z)$ satisfies (*), substituting in (6) $Q^j(z)$, $2 \leq j \leq d_a + d_b - 1$, instead of $Q(z)$ we see that

$$
\sum_{s=0}^{o(p_i)-1} Q^j(p_i^{-1}(k+1)(t)) = \sum_{s=0}^{o(p_i)-1} Q^j(p_i^{-1}(1)(t))
$$

for all $j$, $1 \leq j \leq d_b + d_b - 1$. Consider a Vandermonde determinant $D = \| d_{j,i} \|$, where $d_{j,i} = Q^j(p_i^{-1}(t))$, $0 \leq j \leq d_a + d_b - 1$ and $i$ ranges the set of different indices from the cycles of $p_i$ containing 1 and $k + 1$. Since (7) implies that $D = 0$ we conclude again that $Q(p_i^{-1}(t)) = Q(p_j^{-1}(t))$ for some $i \neq j$, $1 \leq i,j \leq n$. Therefore, $P(z)$ and $Q(z)$ have a common right divisor in the composition algebra and we can finish the proof by the same argument as in section 2.3 taking into account that the multiplicity of a point $c \in \mathbb{C}$ with respect to $P(z) = \hat{P}(W(z))$ is greater or equal than the multiplicity of the point $W(c)$ with respect to $\hat{P}(z)$.

2.5. Proof of Theorem 1. Suppose at first that $n = \deg P(z)$ is a prime number. In this case the degree of the algebraic function $Q(P^{-1}(t))$ equals either $n$ or 1 since $d(Q(P^{-1}(t)))$ divides $\deg P(z)$. If $d(Q(p^{-1}(t))) = n$ then Puiseux expansions at infinity

$$
Q(p_i^{-1}(t)) = \sum_{k \leq k_0} a_k \varepsilon^k t^{k/n},
$$

$1 \leq i \leq n$, $a_k \in \mathbb{C}$, $\varepsilon = \exp(2\pi i/n)$, contain a coefficient $a_k \neq 0$ such that $k$ is not a multiple of $n$. Substituting (8) in the equality obtained by the analytic continuation of (6) along a curve going to the domain where series (8) converge, we conclude that $\varepsilon^k$ is a root of a polynomial with integer coefficients distinct from the $n$-th cyclotomic polynomial $\Phi_n(z) = 1 + z + \ldots + z^{n-1}$. Since $\varepsilon^k$ is a primitive $n$-th root of unity it is a contradiction. Therefore, $d(Q(p^{-1}(t))) = 1$ and $Q(z) = \hat{Q}(P(z))$ for some polynomial $\hat{Q}(z)$.

Suppose now that $n$ is composite. Since $P(z)$ is indecomposable the group $G(P^{-1})$ is primitive by the Ritt theorem [11]. By the Schur theorem (see e.g. [15], Th. 25.3) a primitive permutation group of composite degree $n$ which contains an $n$-cycle is doubly transitive. Therefore, by Lemma 2 equality (6) implies that $Q(z) = \hat{Q}(P(z))$ for some polynomial $\hat{Q}(z)$.

Acknowledgements

I am grateful to Y. Yomdin for drawing my attention to the polynomial moment problem and for stimulating discussions.

References


Max-Planck-Institut für Mathematik, P.O. Box 7280, D-53072 Bonn, Germany.
E-mail address: pakovich@mpim-bonn.mpg.de