IMRN International Mathematics Research Notices
1996, No. 14

A Remark on Complex Polynomials of Least Deviation

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In [Pa1] the second author gave a solution of a problem posed by C.-C. Yang [Ya], using a description of the polynomials of least deviation on certain compacta in $\mathbb{C}$. Namely, it was proven that, up to a sign, a complex polynomial of a given degree is determined uniquely by the preimage of the two-point subset $\{1, -1\} \subset \mathbb{C}$. Here we generalize these results along the same line of ideas. For another generalization, concerning meromorphic functions on Riemann surfaces, see [Pa2].

Recall that, for a given compact $K \subset \mathbb{C}$, a monic polynomial $p(z) \in \mathbb{C}[z]$ of degree $n > 0$ is called *the nth polynomial of least deviation (from zero)* if $\|p\|_K \leq \|q\|_K$ for any monic polynomial $q(z) \in \mathbb{C}[z]$ of degree n, where $\|p\|_K := \max_{z\in K}\{|p(z)|\}$. It is known[1] [To], [VP], [Wa], [Ko] that such a polynomial is unique as soon as $\operatorname{card} K \geq n$.

Recall the following.

Definition 1. Given a compact $K \subset \mathbb{C}$, the closed disc $\Delta = \Delta_{a,r}$ of the smallest radius which contains K is called the *Chebyshev disc* of K. Its center a is called the *Chebyshev center* of K, and its radius r is called the *Chebyshev radius* of K.

Some properties of Chebyshev discs are mentioned in Lemma 2 below.

The next result generalizes Theorem 1 of [Pa1].

Theorem 1. Let a compact $K \subset \mathbb{C}$ with $\operatorname{card} K \geq 2$ have the origin as its Chebyshev center. For a monic polynomial $p \in \mathbb{C}[z]$ of degree n, set $K_p = p^{-1}(K)$. Then p is the unique nth polynomial of least deviation on K_p. □

Received 31 May 1996.
Communicated by G. A. Margulis.

[1] A unicity theorem for Chebyshev approximation in a complex domain was proven by Tonelli (1908). In the case of a general plane compact K with $\operatorname{card} K \geq n$, another proof was given by Vallée Poussin (1911). Walsh (1930) obtained a more general result on rational approximations via the Tonelli approach. The Kolmogorov unicity theorem (1948), based on his criterion for the best approximation polynomials, deals more generally with Chebyshev systems.

In the next corollary we describe a class of plane compacta for which a given monic polynomial p serves as the polynomial of least deviation. The maximal such class is called the *Chebyshev cluster* of p.

Corollary (cf. [Pa, Theorem 1]). Denote $\Delta_r = \Delta_{0,r}$ and $S_r = \partial\Delta_r$. Let Δ_r be the Chebyshev disc of a compact[2] $K \subset S_r$. Let $p \in \mathbb{C}[z]$ be a monic polynomial of degree $n > 0$. Then p is the nth polynomial of least deviation on any compact T such that $K_p \subset T \subset p^{-1}(\Delta_r)$. $\square$

In the proof of Theorem 1 we use the following averaging projection.

Definition 2. Let $p, q \in \mathbb{C}[z]$ and $\deg p = n > 0$. We call the *average of* q *over* p the transform $q \longmapsto \sigma_p(q) = \hat{q} \circ p$, where

$$\hat{q}(z) = \frac{1}{n} \sum_{p^{-1}(z)=\{\xi_1,\ldots,\xi_n\}} q(\xi_j). \tag{1}$$

The summation is over all the roots of the polynomial $p(\xi) - z$, and a root of multiplicity m is repeated m times.

Lemma 1. (a) $\sigma_p\colon \mathbb{C}[z] \to \mathbb{C}[p]$ is a linear projection. Moreover, it is a homomorphism of $\mathbb{C}[p]$-moduli, i.e., $\sigma_p(\varphi(p) \cdot q) = \varphi(p) \cdot \sigma_p(q)$ for any $q, \varphi \in \mathbb{C}[z]$.

(b) $\deg \hat{q} = [(\deg q)/n]$. In particular, $\deg \hat{q} = 1$ if $\deg q = n$. Furthermore, if both p and q are monic polynomials of degree n, then $\sigma_p(q) = p + c$, where $c \in \mathbb{C}$. $\square$

Proof. By definition, the function $\sigma_p(q)$ is constant on each fibre $p^{-1}(z)$ of p, and $\sigma_p(p) = p$. Let $q(z) = \sum_k b_k(z)p^k$, where $b_k \in \mathbb{C}[z]$ and $\deg b_k < n$ for all k, be the p-adic decomposition[3] of q. Then, clearly, $\sigma_p(q) = \sum_k \delta_k p^k$, where $\delta_k = \sigma_p(b_k)$. Here δ_k are constants, since $\sigma_p(q)$ is constant for any polynomial q of degree $m < n$. Indeed, this is enough to check for the monomials $q_m(z) = z^m$, $0 \leq m < n$. But the Newton sum of the roots of $p - z$

$$\hat{q}_m(z) = \frac{1}{n} \sum_{p^{-1}(z)=\{\xi_1,\ldots,\xi_n\}} \xi_j^m$$

is a polynomial in the coefficients $a_{n-1}, \ldots, a_{n-m}$ of $p(\xi) = \xi^n + \sum_{i=0}^{n-1} a_i \xi^i$, and therefore, it is constant. The lemma easily follows from these observations. $\blacksquare$

Remark. The logarithmic residue formula yields an alternative proof of the lemma. Indeed, by this formula we have

$$\sigma_p(q)(z) = \frac{1}{2\pi i n} \int_{|u|=R} \frac{q(u)p'(u)du}{p(u) - p(z)} = \sum_{k=0}^{\infty} \frac{p^k(z)}{2\pi i n} \int_{|u|=R} \frac{q(u)p'(u)du}{p^{k+1}(u)},$$

[2]See Lemma 2 (iv) below for a particular choice of such a compact K.

[3]We are thankful to D. N. Akhiezer, who proposed the use of the p-adic decomposition.

where $R = R(z)$ is sufficiently large and where all the members of the series with $k > (\deg q)/n$ are zero.

Proof of Theorem 1. Let $q \in \mathbb{C}[z]$ be a monic polynomial of the same degree n as p. Since the compact $K_p = p^{-1}(K)$ is saturated by the fibres of p, from Definition 2 it easily follows that

$$\|\sigma_p(q)\|_{K_p} \leq \|q\|_{K_p}. \tag{2}$$

By Lemma 1, $\sigma_p(q) = p + c$, where $c \in \mathbb{C}$. By our assumption, K has its Chebyshev center at the origin. Denote by $\Delta_r = \Delta_{0,r}$ the Chebyshev disc of K. We have

$$r = \|p\|_{K_p} \leq \|p + c\|_{K_p} = \|\sigma_p(q)\|_{K_p}. \tag{3}$$

From (2) and (3) we obtain

$$\|p\|_{K_p} \leq \|q\|_{K_p}. \tag{4}$$

This proves that p is, indeed, a polynomial of least deviation on the compact K_p. Note (see, e.g., [Pa1]) that the geometric preimage of two distinct values a_1, a_2 of a polynomial of degree n consists of at least $n + 1$ distinct points, so that $\operatorname{card} K_p \geq n + 1$. Now the unicity of such a polynomial follows from the classical theorem cited above. ■

Remark. Here we provide an easy proof of the unicity (cf. [VP]). Assume that in (4) the equality holds. This implies the equality signs also in (2) and (3). The latter is possible only if $c = 0$. Therefore, $\sigma_p(q) = p$, i.e., $\hat{q}(z) = z$. Choose two arbitrary distinct points $a_1, a_2 \in K \cap S_r$, where $S_r = \partial\Delta_r$. Then

$$\hat{q}(a_i) = 1/n \sum_{p^{-1}(a_i)=\{\xi_{i1},\ldots,\xi_{in}\}} q(\xi_{ij}) = a_i, \quad i = 1, 2. \tag{5}$$

Hence

$$\left| 1/n \sum_{p^{-1}(a_i)=\{\xi_{i1},\ldots,\xi_{in}\}} q(\xi_{ij}) \right| = r = \|q\|_{K_p}. \tag{6}$$

Since $\xi_{ij} \in K_p$, we have $|q(\xi_{ij})| \leq r$ for all $j = 1, \ldots, n$. This holds only if $q(\xi_{ij}) = a_i = p(\xi_{ij})$, $i = 1, 2,\ j = 1, \ldots, n$. Thus, the polynomial $p - q$ of degree at most $n - 1$ vanishes in at least $n + 1$ distinct points[4] ξ_{ij}, $i = 1, 2,\ j = 1, \ldots, n$. This proves that $p = q$.

[4]See the remark at the end of the proof of Theorem 1.

The proof of the next lemma is a simple exercise.

Lemma 2. Let $S = \partial\Delta$ denote the boundary circle of a disc $\Delta \subset \mathbb{C}$. Let $K \subset \Delta$ be a compact. The following conditions are equivalent:

(i) Δ is the Chebyshev disc of K.

(ii) Δ is the Chebyshev disc of $K \cap S \subset S$.

(iii) $K \cap S \subset S$ is not contained in an open half-circle (or, what is the same, in an open half-plane with the boundary line passing through the center of Δ).

(iv) $K \cap S \subset S$ contains either a pair of symmetric points of S, or a triple of points $b, c, d \in S$ such that the center a of Δ is an inner point of the triangle bcd. $\square$

Theorem 2 below shows that, given a plane compact saturated by fibres of a polynomial of a given degree, there exists a unique such saturation, so that the fibres are uniquely determined by the compact itself. It follows from Theorem 1 in the same way as Corollary 1 in [Pa1] follows from Theorem 1 in [Pa1]. For the reader's convenience we repeat the arguments.

Theorem 2. Let a compact $K \subset \mathbb{C}$ contain at least two points. Suppose that $p, q \in \mathbb{C}[z]$ are two polynomials of the same degree n such that $p^{-1}(K) = q^{-1}(K)$. Then $p = \alpha(q)$, where $\alpha(z)$ is a rotation of $\mathbb{C}$ with center at the Chebyshev center of K which preserves K. $\square$

Proof. Let $\Delta = \Delta_{z_0, r}$ be the Chebyshev disc of K. Replacing K, p, q, respectively, by $K - z_0, p - z_0, q - z_0$, we may assume that $z_0 = 0$, so that $\Delta = \Delta_r = \Delta_{0,r}$ for some $r > 0$. Set $\tilde{p} = (1/a_n)p$, $\tilde{q} = (1/b_n)q$, where $p(z) = a_n z^n + \cdots$, $q(z) = b_n z^n + \cdots$. Then we have

$$K_p := p^{-1}(K) = q^{-1}(K) = \tilde{p}^{-1}((1/a_n)K) = \tilde{q}^{-1}((1/b_n)K).$$

Since $\Delta_{r/|a_n|}$ (resp. $\Delta_{r/|b_n|}$) is the Chebyshev disc of the compact $(1/a_n)K$ (resp. $(1/b_n)K$), by Theorem 1, $\tilde{p}$ (resp. $\tilde{q}$) is the unique nth monic polynomial of least deviation on the compact

$$K_p = ((1/a_n)K)_{\tilde{p}} = ((1/b_n)K)_{\tilde{q}}.$$

It follows that $|a_n| = |b_n|$ and $\tilde{p} = \tilde{q}$, so that $p = e^{i\varphi}q$ for some $\varphi \in \mathbb{R}$. Since $K_p = p^{-1}(K) = q^{-1}(K)$, we have

$$K = p(K_p) = e^{i\varphi}q(K_p) = e^{i\varphi}q(K_q) = e^{i\varphi}K.$$

This shows that K is stable under the rotation $z \longmapsto e^{i\varphi}z$. $\blacksquare$

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