

# RIGHT AMENABILITY IN SEMIGROUPS OF FORMAL POWER SERIES

FEDOR PAKOVICH

ABSTRACT. Let  $k$  be an algebraically closed field of characteristic zero, and  $k[[z]]$  the ring of formal power series over  $k$ . We provide several characterizations of right amenable finitely generated subsemigroups of  $z^2k[[z]]$  with the semigroup operation  $\circ$  being composition. In particular, we show that a subsemigroup  $S = \langle Q_1, Q_2, \dots, Q_k \rangle$  of  $z^2k[[z]]$  is right amenable if and only if there exists an invertible element  $\beta$  of  $zk[[z]]$  such that  $\beta^{-1} \circ Q_i \circ \beta = \omega_i z^{d_i}$ ,  $1 \leq i \leq k$ , for some integers  $d_i$ ,  $1 \leq i \leq k$ , and roots of unity  $\omega_i$ ,  $1 \leq i \leq k$ .

## 1. INTRODUCTION

Let  $R$  be a commutative ring with identity, and  $R[[z]]$  the ring of formal power series over  $R$ . For an element  $A(z) = \sum_{n \geq 0} c_n z^n$  of  $R[[z]]$ , its *order* is defined by the formula  $\text{ord } A = \min\{n \geq 0 \mid c_n \neq 0\}$ . If  $A$  and  $B$  are elements of  $R[[z]]$  with  $\text{ord } B \geq 1$ , then the operation  $A \circ B$  of composition of  $A$  and  $B$  is well defined and provides  $zR[[z]]$  with the structure of a semigroup. This semigroup contains a group  $\mathcal{J}(R)$  consisting of all formal power series of the form  $z + \sum_{n \geq 2} c_n z^n$ . The group  $\mathcal{J}(R)$  has been extensively studied (see survey [1] and references therein). In particular, it was established in [2] that  $\mathcal{J}(R)$  is amenable as a *topological* group. In this note, we study the right amenability of subsemigroups of  $z^2k[[z]]$ , where  $k$  denotes an algebraically closed field of characteristic zero. However, in distinction with [2], all studied semigroups are considered as *discrete*. This setting is different, and requires another approach. In a sense, the results of this note can be seen as analogues of the results of the recent papers [3], [4], [11] about right amenable semigroups of polynomials and rational functions over  $\mathbb{C}$ . Nevertheless, our methods are different.

Let us recall that a semigroup  $S$  is called *right amenable* if it admits a finitely additive probability measure  $\mu$  defined on all the subsets of  $S$  such that for all  $a \in S$  and  $T \subseteq S$  the equality

$$\mu(Ta^{-1}) = \mu(T)$$

holds, where the set  $Ta^{-1}$  is defined by the formula

$$Ta^{-1} = \{s \in S \mid sa \in T\}.$$

A semigroup  $S$  is called *right reversible* if for all  $a, b \in S$  the left ideals  $Sa$  and  $Sb$  have a non-empty intersection, that is, if for all  $a, b \in S$  there exist  $x, y \in S$  such that  $xa = yb$ . It is well-known and follows easily from the definition that any right amenable semigroup is right reversible.

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We denote by  $\mathcal{Z}^U$  the subsemigroup of  $z^2k[[z]]$  consisting of all monomials of the form  $\omega z^n$ ,  $n \geq 2$ , where  $\omega$  is a root of unity. We say that two subsemigroups  $S_1$  and  $S_2$  of  $z^2k[[z]]$  are *conjugate* if there exists  $\beta \in k[[z]]$  of order one such that

$$\beta^{-1} \circ S_1 \circ \beta = S_2.$$

In this notation, our main result is following.

**Theorem 1.1.** *Let  $k$  be an algebraically closed field of characteristic zero, and  $Q_1, Q_2, \dots, Q_k$  elements of  $z^2k[[z]]$ . Then for the semigroup  $S = \langle Q_1, Q_2, \dots, Q_k \rangle$  generated by  $Q_1, Q_2, \dots, Q_k$  the following conditions are equivalent:*

- 1) *The semigroup  $S$  is right amenable.*
- 2) *The semigroup  $S$  is right reversible.*
- 3) *The semigroup  $S$  contains no free subsemigroup of rank two.*
- 4) *The intersection of principal left ideals  $SQ_1 \cap SQ_2 \cap \dots \cap SQ_k$  is non-empty.*
- 5) *The semigroup  $S$  is conjugate to a subsemigroup of  $\mathcal{Z}^U$ .*

The rest of this note is organized as follows. In Section 2.1, we collect some auxiliary results that are used in the paper. In Section 2.2, we show that every subsemigroup  $S$  of  $\mathcal{Z}^U$ , not necessarily finitely generated, is right amenable and contains no free subsemigroup of rank two. Then, using the result of the paper [10], which is essentially equivalent to the equivalence 4)  $\Leftrightarrow$  5) in Theorem 1.1, we prove Theorem 1.1. Finally, in Section 2.3, we provide a class of examples showing that Theorem 1.1 is not true for *infinitely* generated subsemigroups of  $z^2k[[z]]$ .

## 2. PROOF OF THEOREM 1.1

**2.1. Auxiliary results.** Let us recall that a semigroup  $S$  is called *left cancellative* if the equality  $ab = ac$ , where  $a, b, c \in S$ , implies that  $b = c$ . Right cancellative semigroups are defined similarly. A semigroup  $S$  is called *cancellative* if it is left and right cancellative.

The following result relates left reversibility with the presence of free subsemigroup of rank two (see [6], Theorem 8.4, or [7], Corollary 4.2).

**Lemma 2.1.** *Let  $S$  be a right cancellative semigroup that contains no free subsemigroup of rank two. Then  $S$  is right reversible.* □

A subsemigroup of a right amenable semigroup is not necessarily right amenable. However, the following result holds (see [6], Theorem 8.5, or [5], Theorem 4).

**Theorem 2.2.** *Let  $S$  be a cancellative semigroup such that  $S$  contains no free subsemigroup on two generators. If  $S$  is right amenable, then every subsemigroup of  $S$  is right amenable.* □

For a semigroup  $U$ , we denote by  $\text{End}(U)$  the set of endomorphisms of  $U$ . Suppose that  $U$  and  $T$  are semigroups with a homomorphism  $\rho : T \rightarrow \text{End}(U)$ . Denoting for  $a \in T$  the endomorphism  $\rho(a)$  of  $U$  by  $\rho_a$ , we define the semidirect product of  $U$  and  $T$  as the semigroup  $F = U \times_{\rho} T$  of ordered pairs  $(u, a)$ , where  $u \in U$  and  $a \in T$ , with the operation

$$(u, a)(v, b) = (u\rho_a(v), ab).$$

An example of a semidirect product is provided by the subsemigroup  $\mathcal{Z}$  of  $z^2k[[z]]$  consisting of all monomials  $az^n$ ,  $n \geq 2$ , where  $a \in k^*$ . Indeed, for every integer

$n \geq 2$  the map  $\varphi_n : a \rightarrow a^n$  is an endomorphism of  $k^*$ . Moreover, the corresponding map  $n \rightarrow \varphi_n$  induces a homomorphism

$$(1) \quad \rho : \mathbb{N} \setminus \{1\} \rightarrow \text{End}(k^*),$$

where  $\mathbb{N} \setminus \{1\}$  stands for the corresponding subsemigroup of the multiplicative semigroup of natural numbers, identified with the subsemigroup  $z^n$ ,  $n \geq 2$ , of  $z^2k[[z]]$ . Thus, the semidirect product  $k^* \times_{\rho} \mathbb{N} \setminus \{1\}$  is well defined and can be identified with  $\mathcal{Z}$ . Similarly,  $\mathcal{Z}^U$  can be identified with the semidirect product  $\mu_{\infty} \times_{\rho} \mathbb{N} \setminus \{1\}$ , where  $\mu_{\infty}$  is the group of all roots of unity. More generally, we can define the semidirect product  $U \times_{\rho} N$ , where  $U$  is any subsemigroup of  $k^*$ ,  $N$  is any subsemigroup  $\mathbb{N} \setminus \{1\}$ , and  $\rho$  is the restriction of the homomorphism (1) on  $N$ . Below, we will omit  $\rho$  in the notation of such semigroups.

The following result was proved in [9].

**Theorem 2.3.** *If  $U$  and  $T$  are right amenable semigroups with a homomorphism  $\rho : T \rightarrow \text{End}(U)$ , then  $F = U \times_{\rho} T$  is right amenable.  $\square$*

Let us recall that a *congruence* on a semigroup  $S$  is an equivalence relation on  $S$  compatible with the structure of semigroup, that is, an equivalence relation such that  $x \sim y$  and  $x' \sim y'$  implies that  $xx' \sim yy'$ . If  $S$  is a semigroup and  $\sim$  is a congruence on  $S$ , then one can define the *quotient semigroup*  $S/\sim$ , whose elements are the equivalence classes of  $\sim$ , and for  $a, b \in S$  the operation on the corresponding classes is defined by  $[a] * [b] = [ab]$ .

Congruences correspond to homomorphic images of  $S$  in the following sense. If  $\varphi : S \rightarrow T$  is a homomorphism of semigroups, then the equivalence relation  $\sim_{\varphi}$ , defined by  $x \sim_{\varphi} y$  if and only if  $\varphi(x) = \varphi(y)$ , is a congruence on  $S$  and the isomorphism

$$(2) \quad \varphi(T) \cong S/\sim_{\varphi}$$

holds.

Let  $S$  be a right reversible semigroup, and let  $\sim$  be the relation on  $S$ , defined by  $x \sim y$  if and only if there exists  $s \in S$  such that

$$(3) \quad s \circ x = s \circ y.$$

In this notation, the following criterion for the right amenability holds (see [12], Proposition 1.24 and Proposition 1.25).

**Theorem 2.4.** *Let  $S$  be a right reversible semigroup. Then the relation  $\sim$  is a congruence on  $S$ , and the semigroup  $S/\sim$  is left cancellative. Moreover,  $S$  is right amenable if and only if  $S/\sim$  is right amenable.  $\square$*

Let  $A \in z^2k[[z]]$  be a formal power series of order  $n$ . We recall that a *Böttcher function* associated with  $A$  is a formal series  $\beta_A$  of order one such that the equality

$$A \circ \beta_A = \beta_A \circ z^n$$

holds. It is known that such a function exists and is defined in a unique way up to the change  $\beta_A(z) \rightarrow \beta_A(\varepsilon z)$ , where  $\varepsilon^{n-1} = 1$  (see [8], Hilfsatz 4). Among other things, it follows from the existence of Böttcher functions that the semigroup  $z^2k[[z]]$  is right cancellative. Indeed, if

$$(4) \quad A_1 \circ X = A_2 \circ X,$$

then conjugating (4) by  $\beta_X$  we obtain the equality

$$\beta_X^{-1} \circ A_1 \circ \beta_X \circ z^n = \beta_X^{-1} \circ A_2 \circ \beta_X \circ z^n,$$

which implies that

$$\beta_X^{-1} \circ A_1 \circ \beta_X = \beta_X^{-1} \circ A_2 \circ \beta_X$$

and  $A_1 = A_2$ .

Finally, we need the following result.

**Theorem 2.5.** *Let  $k$  be an algebraically closed field of characteristic zero,  $Q_1, Q_2, \dots, Q_k$  elements of  $z^2k[[z]]$ , and  $S = \langle Q_1, Q_2, \dots, Q_k \rangle$  the semigroup generated by  $Q_1, Q_2, \dots, Q_k$ . Assume that  $Q_1$  is contained in  $\mathcal{Z}^U$ . Then*

$$(5) \quad SQ_1 \cap SQ_2 \cap \dots \cap SQ_k \neq \emptyset$$

if and only if every  $Q_i$ ,  $2 \leq i \leq k$ , is contained in  $\mathcal{Z}^U$ .  $\square$

In case  $k = \mathbb{C}$ , Theorem 2.5 was proved in [10] (Theorem 2.3), and the proof carries over verbatim to the case of an arbitrary algebraically closed field  $k$  of characteristic zero.

**2.2. Right amenability of subsemigroups  $\mathcal{Z}^U$ .** Let us denote by  $\mu_n$  the group of  $n$ th roots of unity.

**Theorem 2.6.** *Every subsemigroup  $S$  of  $\mathcal{Z}^U$  is right reversible and contains no free subsemigroup of rank two.*

*Proof.* Let us show that  $S$  contains no free subsemigroup of rank two. Let

$$F_1 = \varepsilon_1 z^{d_1}, \quad F_2 = \varepsilon_2 z^{d_2}, \quad \varepsilon_1, \varepsilon_2 \in \mu_\infty, \quad d_1, d_2 \geq 2,$$

be elements of  $S$ , and  $n \geq 1$  an integer such that  $\varepsilon_1, \varepsilon_2 \in \mu_n$ . Assume first that  $d_1 = d_2$ . Then for every  $j \geq 1$  the equality

$$F_1^{\circ j} = \omega_j F_2^{\circ j}$$

holds for some  $\omega_j \in \mu_n$ , implying by the pigeonhole principle that there exist  $j_1 \neq j_2$  such that

$$F_1^{\circ j_1} = \varepsilon F_2^{\circ j_1}, \quad F_1^{\circ j_2} = \varepsilon F_2^{\circ j_2}$$

for the same  $\varepsilon \in \mu_n$ . Assuming that  $j_2 > j_1$ , this yields that

$$(6) \quad F_1^{\circ j_2} = F_1^{\circ j_1} \circ F_2^{\circ(j_2-j_1)},$$

and hence the semigroup  $\langle F_1, F_2 \rangle$  generated by  $F_1$  and  $F_2$  is not free.

In case  $d_1 \neq d_2$ , let us consider the elements

$$F'_1 = F_1 \circ F_2, \quad F'_2 = F_2 \circ F_1$$

of  $S$ . If  $F'_1 = F'_2$ , then the semigroup  $\langle F_1, F_2 \rangle$  obviously is not free. On the other hand, if  $F'_1 \neq F'_2$ , then, since  $F'_1$  and  $F'_2$  have the same order, we can apply the above reasoning to  $F'_1$  and  $F'_2$ , and find  $j_2 > j_1$  such that

$$(7) \quad (F_1 \circ F_2)^{\circ j_2} = (F_1 \circ F_2)^{\circ j_1} \circ (F_2 \circ F_1)^{\circ(j_2-j_1)}.$$

To prove that  $S$  is right reversible it is enough to observe that equalities (6) and (7) provide solutions of the equation

$$X \circ F_1 = Y \circ F_2$$

in  $X, Y \in S$ .  $\square$

For  $\varepsilon \in \mu_\infty$ , we denote by  $|\varepsilon|$  the order of  $\varepsilon$  in the semigroup  $\mu_\infty$ . With every subsemigroup  $S$  of  $\mathcal{Z}^U$  we associate several objects. First, we define  $U(S)$  as the subsemigroup of  $\mu_\infty$  generated by all roots of unity  $\varepsilon$  such that  $\varepsilon z^d \in S$  for some  $d \geq 2$ . Notice that since  $\varepsilon^{-1} = \varepsilon^{|\varepsilon|-1}$  belongs to  $U(S)$  whenever  $\varepsilon$  belongs to  $U(S)$ , the semigroup  $U(S)$  is a group. Second, we define  $N(S)$  as the subsemigroup of the multiplicative semigroup  $\mathbb{N} \setminus \{1\}$  consisting of all  $d \geq 2$  such that  $\varepsilon z^d \in S$  for some  $\varepsilon \in \mu_\infty$ . Notice that by construction the semigroup  $S$  is a subsemigroup of the semigroup  $U(S) \times N(S)$ .

Further, we associate with  $S$  two subsets  $P_1(S)$  and  $P_2(S)$  of the set of prime numbers as follows. For an integer  $l \geq 2$ , we define  $\mathcal{P}(l)$  as the set of prime divisors of  $l$ . For an element  $Q = \varepsilon z^d$  of  $S$ , where  $\varepsilon \in \mu_\infty$  and  $d \geq 2$ , we set

$$p_1(Q) = \mathcal{P}(|\varepsilon|), \quad p_2(Q) = \mathcal{P}(d)$$

(in case  $|\varepsilon| = 1$ , we set  $p_1(Q) = \emptyset$ ). Finally, we set

$$P_1(S) = \bigcup_{Q \in S} p_1(Q), \quad P_2(S) = \bigcup_{Q \in S} p_2(Q).$$

Notice that by construction

$$(8) \quad P_1(S) = P_1(U(S) \times N(S)), \quad P_2(S) = P_2(U(S) \times N(S)).$$

**Theorem 2.7.** *Every subsemigroup  $S$  of  $\mathcal{Z}^U$  is right amenable.*

*Proof.* We start by observing that any semigroup of the form  $U \times N$ , where  $U$  is a subsemigroup of  $k^*$  and  $N$  is a subsemigroup  $\mathbb{N} \setminus \{1\}$  is right amenable. Indeed, it is well known that every commutative semigroup is right amenable. Therefore,  $U$  and  $N$  are right amenable, implying by Theorem 2.3 that  $U \times N$  is also right amenable.

Let us show first that the statement of the theorem holds if

$$(9) \quad P_1(S) \cap P_2(S) = \emptyset.$$

Since  $S$  is a subsemigroup of the amenable semigroup  $U(S) \times N(S)$  and the last group is right cancellative and contains no free subsemigroup of rank two by Theorem 2.6, it follows from Theorem 2.2 that to prove the amenability of  $S$  it is enough to show that the semigroup  $U(S) \times N(S)$  is left cancellative. Let us assume that

$$(10) \quad A \circ F_1 = A \circ F_2$$

for some  $A, F_1, F_2 \in U(S) \times N(S)$ . Clearly, (10) implies that  $\text{ord } F_1 = \text{ord } F_2$ . Thus,

$$A = \varepsilon z^m, \quad F_1 = \varepsilon_1 z^k, \quad F_2 = \varepsilon_2 z^k$$

for some  $m, k \in N(S)$  and  $\varepsilon, \varepsilon_1, \varepsilon_2 \in U(S)$ , and (10) implies the equality

$$(11) \quad \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^m = 1.$$

Set  $n = |\varepsilon_1/\varepsilon_2|$ . Since (8) and (9) yield that  $\text{gcd}(n, m) = 1$ , equality (11) is possible only if  $n = 1$ . Therefore,  $\varepsilon_1 = \varepsilon_2$  and  $F_1 = F_2$ .

Let us prove now the theorem in the general case. Since for any integers  $n, m_1, m_2 \geq 1$  such that  $n = m_1 m_2$  and  $\text{gcd}(m_1, m_2) = 1$  the equality

$$\mu_n = \mu_{m_1} \times \mu_{m_2}$$

holds, any element  $Q$  of  $S$  has a unique representation in the form

$$(12) \quad Q = \varepsilon_1 \varepsilon_2 z^d, \quad \varepsilon_1, \varepsilon_2 \in \mu_\infty, \quad d \geq 2,$$

where

$$\mathcal{P}(|\varepsilon_1|) \cap P_2(S) = \emptyset, \quad \mathcal{P}(|\varepsilon_2|) \subseteq P_2(S).$$

Let us define a map

$$\varphi : S \rightarrow \mathcal{Z}^U,$$

setting for  $Q$ , defined by (12),

$$\varphi(Q) = \varepsilon_1 z^d.$$

It is easy to see that  $\varphi$  is a semigroup homomorphism. Indeed, for

$$\widehat{Q} = \widehat{\varepsilon}_1 \widehat{\varepsilon}_2 z^{\widehat{d}}, \quad \widehat{\varepsilon}_1, \widehat{\varepsilon}_2 \in \mu_\infty, \quad \widehat{d} \geq 2,$$

we have:

$$Q \circ \widehat{Q} = \varepsilon_1 \varepsilon_2 z^d \circ \widehat{\varepsilon}_1 \widehat{\varepsilon}_2 z^{\widehat{d}} = \varepsilon_1 \widehat{\varepsilon}_1^d \varepsilon_2 \widehat{\varepsilon}_2^d z^{d\widehat{d}},$$

where obviously

$$\mathcal{P}(|\varepsilon_1 \widehat{\varepsilon}_1^d|) \cap P_2(S) = \emptyset, \quad \mathcal{P}(|\varepsilon_2 \widehat{\varepsilon}_2^d|) \subseteq P_2(S).$$

Thus,

$$\varphi(Q \circ \widehat{Q}) = \varepsilon_1 \widehat{\varepsilon}_1^d z^{d\widehat{d}} = \varphi(Q) \circ \varphi(\widehat{Q}).$$

By construction, the image  $\varphi(S)$  of  $S$  under the homomorphism  $\varphi$  satisfies the condition

$$P_1(\varphi(S)) \cap P_2(\varphi(S)) = \emptyset.$$

Therefore, by what is proved above, the semigroup  $\varphi(S)$  is cancellative and right amenable. It follows now from Theorem 2.4 taking into account the isomorphism (2) that to prove the theorem it is enough to show that the congruence defined by the homomorphism  $\varphi$  coincides with the congruence (3).

Let us assume that

$$(13) \quad \varphi(Q_1) = \varphi(Q_2).$$

Then

$$Q_1 = \varepsilon \varepsilon_1 z^d, \quad Q_2 = \varepsilon \varepsilon_2 z^d, \quad \varepsilon, \varepsilon_1, \varepsilon_2 \in \mu_\infty, \quad d \geq 2,$$

where

$$\mathcal{P}(|\varepsilon|) \cap P_2(S) = \emptyset, \quad \mathcal{P}(|\varepsilon_1|) \subseteq P_2(S), \quad \mathcal{P}(|\varepsilon_2|) \subseteq P_2(S).$$

Let

$$(14) \quad |\varepsilon_1| = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}, \quad |\varepsilon_2| = q_1^{b_1} q_2^{b_2} \dots q_l^{b_l}$$

be the canonical decompositions of  $|\varepsilon_1|$  and  $|\varepsilon_2|$  into products of primes. Then for each  $i$ ,  $1 \leq i \leq r$ , there exists  $K_i \in S$  of order divisible by  $p_i$ , and for each  $j$ ,  $1 \leq j \leq l$ , there exists  $L_j \in S$  of order divisible by  $q_j$ . Obviously, this implies that the equality

$$(15) \quad A \circ Q_1 = A \circ Q_2$$

holds for the element

$$A = K_1^{\circ a_1} \dots K_r^{\circ a_r} L_1^{\circ b_1} \dots L_l^{\circ b_l}$$

of  $S$  (formulas (14) assume that  $|\varepsilon_1| > 1$ ,  $|\varepsilon_2| > 1$ , however, in case one of these numbers equals one the proof can be modified in an obvious way).

In the other direction, if (15) holds, then

$$\varphi(A) \circ \varphi(Q_1) = \varphi(A) \circ \varphi(Q_2),$$

implying that (13) holds since  $\varphi(S)$  is cancellative. Thus, the congruence defined by the homomorphism  $\varphi$  coincides with the congruence (3).  $\square$

*Proof of Theorem 1.1.* We first prove the chain of implications  $1) \Rightarrow 2) \Rightarrow 4) \Rightarrow 5) \Rightarrow 1)$ . It is well known (see [12], Proposition 1.23) that every right amenable semigroup is right reversible. Thus,  $1) \Rightarrow 2)$ . The implication  $2) \Rightarrow 4)$  is proved by induction on  $k$ . For  $k = 2$ , the condition 2) coincides with the right reversibility condition. On the other hand, if

$$F = A_1 \circ Q_1 = A_2 \circ Q_2 = \cdots = A_{k-1} \circ Q_{k-1}$$

for some  $A_1, A_2, \dots, A_{k-1} \in S$ , then applying the condition of right reversibility to  $F$  and  $Q_k$ , we can find  $G, Y_1, Y_2 \in S$  such that the equality

$$G = Y_1 \circ F = Y_2 \circ Q_k$$

holds. Thus,

$$G = (Y_1 \circ A_1) \circ Q_1 = (Y_1 \circ A_2) \circ Q_2 = \cdots = (Y_1 \circ A_{k-1}) \circ Q_{k-1} = Y_2 \circ Q_k.$$

The implication  $4) \Rightarrow 5)$  follows from Theorem 2.5. Indeed, if  $\beta_{Q_1}$  is a Böttcher function corresponding to  $Q_1$ , then  $\beta_{Q_1}^{-1} \circ S \circ \beta_{Q_1}$  is a semigroup satisfying the conditions  $Q_1 \in \mathcal{Z}^U$  and (5). Thus, by Theorem 2.5, every  $Q_i$ ,  $2 \leq i \leq k$ , is contained in  $\mathcal{Z}^U$ , and hence  $S$  is conjugate to a subsemigroup of  $\mathcal{Z}^U$ . Finally, the implication  $(5) \Rightarrow (1)$  follows from Theorem 2.7.

To finish the proof it is enough to prove the implications  $5) \Rightarrow 3)$  and  $3) \Rightarrow 2)$ . The first implication follows from Theorem 2.6. On the other hand, since  $z^2k[[z]]$  is right cancellative, the second implication follows from Lemma 2.1.  $\square$

**2.3. Infinitely generated semigroups.** Theorem 1.1 is not true for infinitely generated subsemigroups of  $z^2k[[z]]$ . A class of counterexamples is provided by Theorem 2.8 below.

We recall that an element of a semigroup  $S$  is called *indecomposable* if it belongs to  $S \setminus SS$ , where

$$SS = \{st : s, t \in S\}.$$

**Theorem 2.8.** *Let  $U$  be a subsemigroup of  $k^*$  such that  $U \not\subseteq \mu_\infty$  and  $1 \in U$ . Then for any subsemigroup  $N$  of  $\mathbb{N} \setminus \{1\}$  the semigroup  $S = U \times N$  satisfies the following conditions:*

- 1) *The semigroup  $S$  is right amenable.*
- 2) *The semigroup  $S$  is not finitely generated.*
- 3) *The semigroup  $S$  is not conjugate to a subsemigroup of  $\mathcal{Z}^U$ .*

*Proof.* The right amenability of  $S$  was established in the proof of Theorem 2.7. Further, since indecomposable elements of  $S$  must belong to any generating set of  $S$ , to prove that  $S$  is not finitely generated it is enough to find an infinite subset of indecomposable elements of  $S$ . An example of such a set is the set  $a^k z^d$ ,  $k \geq 1$ , where  $a$  is an arbitrary element of  $U$  that does not belong to  $\mu_\infty$  and  $d$  is an indecomposable element of the semigroup  $N$ .

Finally, the proof of Theorem 2.6 shows that to prove that  $S$  is not conjugate to a subsemigroup of  $\mathcal{Z}^U$  it is enough to find  $F_1, F_2 \in S$  such that  $\text{ord } F_1 = \text{ord } F_2$  but equality (6) does not hold for any choice of  $j_1, j_2$ . Since  $1 \in U$ , we can take

$$F_1 = z^d, \quad F_2 = az^d,$$

where  $d$  is an arbitrary element of  $N$ , and  $a$  is an element of  $U$  that does not belong to  $\mu_\infty$ .  $\square$

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DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, ISRAEL  
Email address: pakovich@math.bgu.ac.il