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Commuting rational functions revisited

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Abstract. Let B be a rational function of degree at least two that is neither a Lattès map nor conjugate to $z^{\pm n}$ or $\pm T_n$. We provide a method for describing the set C_B consisting of all rational functions commuting with B. Specifically, we define an equivalence relation $\underset{B}{\sim}$ on C_B such that the quotient $C_B/\underset{B}{\sim}$ possesses the structure of a finite group G_B, and describe generators of G_B in terms of the fundamental group of a special graph associated with B.

Key words: commuting rational functions, the Ritt theorem, common iterates
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1. *Introduction*

In this paper, we study commuting rational functions, that is rational solutions of the functional equation

$$B \circ X = X \circ B. \tag{1}$$

More precisely, we fix a function $B \in \mathbb{C}(z)$ of degree at least two and study the set C_B consisting of all $X \in \mathbb{C}(z)$ such that (1) holds.

Functional equation (1) has been investigated previously by Julia [**3**] and Fatou [**2**]. In particular, they showed that commuting rational functions X and B of degree at least two have the same Julia set $J = J(X) = J(B)$. Using Poincaré functions, Julia and Fatou proved that if X and B have no iterate in common and $J \neq \mathbb{CP}^1$, then, up to a conjugacy, X and B are either powers or Chebyshev polynomials. The assumption $J \neq \mathbb{CP}^1$ was removed by Ritt [**13**], who used a topological-algebraic method. Ritt proved that solutions of (1) having no iterate in common reduce to powers, to Chebyshev polynomials, or to Lattès maps. A proof of the Ritt theorem based on modern dynamical methods was given by Eremenko [**1**].

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All the above results assume that X and B have no iterate in common. However, commuting rational functions X and B such that

$$B^{\circ l} = X^{\circ k} \tag{2}$$

for some $l, k \geq 1$ also exist. The simplest examples of such functions can be obtained by setting

$$X = R^{\circ l_1}, \quad B = R^{\circ l_2},$$

where R is an arbitrary rational function and $l_1, l_2 \geq 1$. More generally, denoting by $\mathrm{Aut}(R)$ the group of Möbius transformations commuting with R, we can set

$$X = \mu_1 \circ R^{\circ l_1}, \quad B = \mu_2 \circ R^{\circ l_2}, \tag{3}$$

where μ_1 and μ_2 are elements of $\mathrm{Aut}(R)$ commuting between themselves. However, it has been shown already by Ritt [**13**] that commuting rational functions satisfying (2) are not exhausted by functions of the form (3). Although Ritt's method provides some insight into the structure of commuting rational functions X and B satisfying (2), it does not permit the description of this class of functions in an explicit way, and Ritt concluded his paper by saying: 'we think that the example given above makes it conceivable that no great order may reign in this class'.

Functional equation (1) is a particular case of the functional equation

$$A \circ X = X \circ B, \tag{4}$$

where A and B are rational functions of degree at least two. In case that (4) is satisfied for some rational function X of degree at least two, the function B is called *semiconjugate* to the function A. Semiconjugate rational functions were investigated in the recent papers [**5**, **6**, **8–10**]. In particular, it was shown in [**6**] that solutions of (4) satisfying $\mathbb{C}(X, B) = \mathbb{C}(z)$, called *primitive*, can be described in terms of group actions on $\mathbb{CP}^1$ or $\mathbb{C}$, implying strong restrictions on a possible form of A, B and X. Any solution of (4) reduces to a primitive one by a certain iterative process, and the quantitative aspects of this reduction were studied in [**5**]. In particular, it was shown in [**5**] that if a rational function B is not *special*, that is, if B is neither a Lattès map nor conjugate to $z^{\pm n}$ or $\pm T_n$, then solutions of equations (1) and (4) obey some finiteness conditions.

Specifically, with regards to equation (1), it was shown in [**5**] that if B is not special, then there exist *finitely many* rational functions $X_1, X_2, \ldots, X_r$ such that X commutes with B if and only if

$$X = X_j \circ B^{\circ k}$$

for some j, $1 \leq j \leq r$, and $k \geq 0$. Moreover, the number r and the degrees of X_j, $1 \leq j \leq r$, can be bounded by numbers depending on $\deg B$ only. Note that this result immediately implies the Ritt theorem. Indeed, if X commutes with B, then any iterate $X^{\circ l}$, $l \geq 1$, does. Thus, by the Dirichlet box principle, there exist distinct l_1, l_2 such that

$$X^{\circ l_1} = X_j \circ B^{\circ k_1}, \quad X^{\circ l_2} = X_j \circ B^{\circ k_2}$$

for the same j and some $k_1, k_2 \geq 0$. Therefore, if, say, $l_2 > l_1$, then

$$X^{\circ l_2} = X^{\circ l_1} \circ B^{\circ k_2 - k_1},$$

implying that (2) holds for $l = l_2 - l_1$ and $k = k_2 - k_1$, since X and B commute.

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In this paper, we provide a method for describing the set C_B for non-special B. For such B, essentially all the information about C_B provided by the Ritt method reduces to the fact that any element of C_B has a common iterate with B. Thus, new approaches and techniques are needed, and we develop them in this paper. Our main results are as follows. First, for any non-special rational function B, we define an equivalence relation $\underset{B}{\sim}$ on the set C_B such that the quotient $C_B/\underset{B}{\sim}$ possesses the structure of a finite group G_B. Second, we describe generators of this group in terms of the fundamental group of a special graph associated with B, providing a method for describing C_B. Finally, we calculate G_B for several classes of rational functions. Note that our method of describing C_B reduces the problem to the easier problem of finding all functional decompositions $F = U \circ V$ for finitely many rational functions F.

In more detail, for a non-special rational function B, we define an equivalence relation $\underset{B}{\sim}$ on the set C_B, setting $A_1 \underset{B}{\sim} A_2$ if

$$A_1 \circ B^{\circ l_1} = A_2 \circ B^{\circ l_2}$$

for some $l_1 \geq 0$, $l_2 \geq 0$, and show that the multiplication of classes induced by the functional composition of their representatives provides $C_B/\underset{B}{\sim}$ with the structure of a finite group G_B. The group structure on $C_B/\underset{B}{\sim}$ offers a new look at the problem of describing C_B, and permits the characterization of properties of C_B in group theoretic terms. For example, the group G_B is trivial if and only if any element of C_B is an iterate of B, while G_B is isomorphic to $\mathrm{Aut}(B)$ if and only if any element of C_B can be represented in the form $X = \mu \circ B^k$, where $\mu \in \mathrm{Aut}(B)$ and $k \geq 0$.

We describe generators of G_B using a special finite graph Γ_B defined as follows. Let B be a rational function. We say that a rational function $\widehat{B}$ is an *elementary transformation* of B if there exist rational functions U and V such that $B = V \circ U$ and $\widehat{B} = U \circ V$. We say that rational functions B and A are *equivalent* and write $A \sim B$ if there exists a chain of elementary transformations between B and A (this equivalence relation should not be confused with the previous one where the subscript B is used). Since for any Möbius transformation μ the equality

$$B = (B \circ \mu^{-1}) \circ \mu$$

holds, the equivalence class $[B]$ of a rational function B is a union of conjugacy classes. Moreover, by the result of [**9**], the class $[B]$ consists of *finitely many* conjugacy classes, unless B is a flexible Lattès map. The graph Γ_B is defined as a multigraph whose vertices are in a one-to-one correspondence with some fixed representatives B_i of conjugacy classes in $[B]$, and whose multiple edges connecting the vertices corresponding to B_i to B_j are in a one-to-one correspondence with solutions of the system

$$B_i = V \circ U, \quad B_j = U \circ V$$

in rational functions. In these terms, the main result of the paper about the group G_B is a construction of a group epimorphism from the fundamental group of the graph Γ_B to the group G_B.

The paper is organized as follows. In §2, we describe the set C_B in terms of elementary transformations. In §3, we define the group G_B. In §§4 and 5, we define the graph Γ_B

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and construct a group epimorphism from $\pi_1(\Gamma_B)$ to G_B. We also show that if $A \sim B$, then the groups G_A and G_B are isomorphic. Note that this implies, in particular, that if A is a rational function such that the group $\mathrm{Aut}(A)$ is non-trivial, then for any rational function $B \sim A$ the group G_B is also non-trivial, even though $\mathrm{Aut}(B)$ can be trivial. In the last case, functions of degree one in C_A give rise to functions of higher degree in C_B through the isomorphism $G_A \cong G_B$.

In §6, we calculate the group G_B for certain classes of rational functions, and consider some examples. Specifically, we show that for a wide class of rational functions, which we call *generically decomposable*, G_B is isomorphic to $\mathrm{Aut}(B)$. We also show that for a polynomial B the group G_B is metacyclic. Finally, we discuss in detail the example of commuting rational functions B and X satisfying condition (2) from the paper of Ritt [**13**]. In particular, we calculate the group G_B that turns out to be a cyclic group of order three. We also provide a different example of this kind.

2. *The set C_B and elementary transformations*

Let B be a rational function of degree at least two. We denote by C_B the set of all rational functions commuting with B.

LEMMA 2.1. *The set C_B is closed with respect to the operation of composition, that is, $A_1, A_2 \in C_B$ implies $A_1 \circ A_2 \in C_B$. Furthermore, if $A \circ U \in C_B$ and $U \in C_B$, then $A \in C_B$.*

Proof. Indeed, if $A_1, A_2 \in C_B$, then

$$A_1 \circ A_2 \circ B = A_1 \circ B \circ A_2 = B \circ A_1 \circ A_2.$$

On the other hand, if $A \circ U \in C_B$ and $U \in C_B$, then

$$B \circ A \circ U = A \circ U \circ B = A \circ B \circ U,$$

implying that

$$B \circ A = A \circ B. \qquad \square$$

We emphasize that we allow to elements of C_B to have degree one, that is to be Möbius transformations. All Möbius transformations commuting with B obviously form a group denoted by $\mathrm{Aut}(B)$ and called the *symmetry group* of B. Since any $\mu \in \mathrm{Aut}(B)$ maps periodic points of B of order $l \geq 1$ to themselves, and any Möbius transformation is defined by its values at any three points, the symmetry group of any rational function is finite. In particular, $\mathrm{Aut}(B)$ is one of the five well-known finite rotation groups of the sphere: A_4, S_4, A_5, C_n, D_{2n}. Note that the property of $\mu \in \mathrm{Aut}(B)$ to map periodic points of B to periodic points can be used for a practical description of $\mathrm{Aut}(B)$.

Let B be a rational function. A rational function $\widehat{B}$ is called an *elementary transformation* of B if there exist rational functions U and V such that $B = V \circ U$ and $\widehat{B} = U \circ V$. We say that rational functions B and A are *equivalent* and write $A \sim B$ if there exists a chain of elementary transformations between B and A. Since for any Möbius transformation μ the equality

$$B = (B \circ \mu^{-1}) \circ \mu$$

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holds, the equivalence class $[B]$ of a rational function B is a union of conjugacy classes. Thus, the relation $\sim$ can be considered as a weaker form of the classical conjugacy relation. The equivalence class $[B]$ contains infinitely many conjugacy classes if and only if B is a flexible Lattès map [**9**].

The following lemma is obtained by a direct calculation (see [**10**, Lemma 3.1]).

LEMMA 2.2. *Let*

$$L : B \to B_1 \to B_2 \to \cdots \to B_s \tag{5}$$

be a sequence of elementary transformations, and U_i, V_i, $1 \le i \le s$, *rational functions such that*

$$B = V_1 \circ U_1, \quad B_i = U_i \circ V_i, \quad 1 \le i \le s,$$

and

$$U_i \circ V_i = V_{i+1} \circ U_{i+1}, \quad 1 \le i \le s-1.$$

Then the functions

$$U = U_s \circ U_{s-1} \circ \cdots \circ U_1, \quad V = V_1 \circ \cdots \circ V_{s-1} \circ V_s \tag{6}$$

make the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ U\downarrow & & \downarrow U \\ \mathbb{CP}^1 & \xrightarrow{B_s} & \mathbb{CP}^1 \\ V\downarrow & & \downarrow V \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \end{array}$$

commutative and satisfy the equalities

$$V \circ U = B^{\circ s}, \quad U \circ V = B_s^{\circ s}. \qquad \square$$

It follows from Lemma 2.2, that any sequence of elementary transformations (5) such that $B_s = B$ gives rise to a rational function U commuting with B, and the main result of this section states that for non-special B any element of C_B can be obtained in this way.

THEOREM 2.3. *Let B be a non-special rational function of degree at least two. Then a rational function X belongs to C_B if and only if there exists a sequence of elementary transformation (5) such that $B_s = B$ and $X = U_s \circ U_{s-1} \circ \cdots \circ U_1$.*

The proof of Theorem 2.3 uses the following two lemmas which are particular cases of [**6**, Lemma 2.1] and [**5**, Theorem 2.18], respectively. For the reader's convenience, we provide short independent proofs. We recall that a solution A, X, B of (4) is called *primitive* if $\mathbb{C}(X, B) = \mathbb{C}(z)$. We also mention that for an arbitrary solution A, X, B of (4) the equality

$$\deg A = \deg B \tag{7}$$

holds.

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LEMMA 2.4. *A solution A, X, B of (4) is primitive if and only if the algebraic curve*

$$A(x) - X(y) = 0 \tag{8}$$

is irreducible.

Proof. By the Lüroth theorem, there exists a rational function W such that $\mathbb{C}(X, B) = \mathbb{C}(W)$, implying that the equalities

$$X = X' \circ W, \quad B = B' \circ W \tag{9}$$

hold for some rational functions X' and B' with $\mathbb{C}(X', B') = \mathbb{C}(z)$. Clearly, $x = X'(t)$, $y = B'(t)$ is a generically one-to-one parametrization of some irreducible component

$$C: \ F(x, y) = 0$$

of (8). Furthermore, since the degree of the projection of C on x (respectively, y) is equal to $\deg X'$ (respectively, $\deg B'$) the equalities

$$\deg_x F = \deg B', \quad \deg_y F = \deg X' \tag{10}$$

hold. If $\mathbb{C}(X, B) = \mathbb{C}(z)$, then $\deg W = 1$, and it follows from equalities (9), (10), and (7) that the curve C coincides with curve (8), implying that (8) is irreducible. On the other hand, if $\mathbb{C}(X, B) \neq \mathbb{C}(z)$, then $\deg W > 1$, and equalities (9), (10), and (7) imply that C is a proper component of (8). □

LEMMA 2.5. *Let A, X, B be a primitive solution of (4). Then for any $l \geq 1$, the solution $A^{\circ l}$, X, $B^{\circ l}$ is also primitive.*

Proof. The proof is by induction on l. For $l = 1$, the lemma is trivially true. Assume that it is true for all $k \leq l$. By Lemma 2.4, this implies that the algebraic curve

$$C_k : A^{\circ k}(x) - X(y) = 0$$

is irreducible for all $k \leq l$, and

$$R_k : x = X(t), \quad y = B^{\circ k}(t)$$

is its generically one-to-one parametrization.

Let P_1, P_2 be arbitrary rational functions satisfying the equality

$$A^{\circ(l+1)} \circ P_1 = X \circ P_2. \tag{11}$$

Since the curve C_l is irreducible and R_l is its generically one-to-one parametrization, the equality

$$A^{\circ(l+1)} \circ P_1 = A^{\circ l} \circ (A \circ P_1) = X \circ P_2$$

implies that

$$A \circ P_1 = X \circ W, \quad P_2 = B^{\circ l} \circ W$$

for some $W \in \mathbb{C}(z)$. Furthermore, since the curve C_1 is also irreducible, it follows from the first of these equalities that

$$P_1 = X \circ U, \quad W = B \circ U$$

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for some $U \in \mathbb{C}(z)$. Thus, any pair of rational functions P_1, P_2 satisfying (11) has the form

$$P_1 = X \circ U, \quad P_2 = B^{\circ(l+1)} \circ U$$

for some $U \in \mathbb{C}(z)$. In particular, this implies that if the equalities

$$X = P_1 \circ W, \quad B^{\circ(l+1)} = P_2 \circ W \tag{12}$$

hold for some P_1, P_2, $W \in \mathbb{C}(z)$, then $\deg W = 1$, since P_1, P_2 in (12) satisfy (11). Therefore, $\mathbb{C}(X, B^{\circ(l+1)}) = \mathbb{C}(z)$, that is, $A^{\circ(l+1)}$, X, $B^{\circ(l+1)}$ is a primitive solution. □

Proof of Theorem 2.3. The sufficiency follows from Lemma 2.2. In the other direction, assume that $X \in C_B$. If X is a Möbius transformation, then the sequence

$$B = (B \circ X^{-1}) \circ X \to X \circ (B \circ X^{-1}) = B$$

is as required. Thus, assume that $\deg X \geq 2$.

We observe first that there exist a sequence (5) and a commutative diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ U\downarrow & & \downarrow U \\ \mathbb{CP}^1 & \xrightarrow{B_s} & \mathbb{CP}^1 \\ X_0\downarrow & & \downarrow X_0 \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \end{array}$$

such that U is defined by (6), the equality $X = X_0 \circ U$ holds, and the triple B, X_0, B_s is a primitive solution of (4). Indeed, if B, X, B is a primitive solution of (4), we can set $U = z$, $X_0 = X$, and $B_s = B$. Otherwise, $\mathbb{C}(X, B) = \mathbb{C}(W)$ for some W with $\deg W > 1$, and substituting equalities (9) in (4) we see that the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ W\downarrow & & \downarrow W \\ \mathbb{CP}^1 & \xrightarrow{W \circ B'} & \mathbb{CP}^1 \\ X'\downarrow & & \downarrow X' \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \end{array}$$

commutes. If the solution B, X', $W \circ B'$ of (4) is primitive, we are done. Otherwise, we can apply the above transformation to this solution. Since $\deg X' < \deg X$, it is clear that after a finite number of steps we obtain a sequence of elementary transformations (5) and functions U, X_0, and B_s as required.

To prove Theorem 2.3, we only must show that $\deg X_0 = 1$. Indeed, in this case changing U_s to $X_0 \circ U_s$ and B_s to $X_0 \circ B_s \circ X_0^{-1}$, without loss of generality we may assume that $X_0 = z$, so that $B_s = B$ and (5) is the sequence required. Assume, in contrast, that $\deg X_0 > 1$. By Lemma 2.5, for any $l \geq 1$ the triple $B^{\circ l}$, X_0, $B_s^{\circ l}$ is a primitive solution

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of (4). On the other hand, by the Ritt theorem, there exist k and l such that equality (2) holds. Thus,

$$B^{\circ l} = X^{\circ k} = X_0 \circ (U \circ X^{\circ k-1}),$$

implying that the curve

$$(U \circ X^{\circ k-1})(x) - y = 0$$

is a component of the curve

$$B^{\circ l}(x) - X_0(y) = 0.$$

Moreover, this component is proper because $\deg X_0 > 1$. Since, by Lemma 2.4, this contradicts the fact that $B^{\circ l}, X_0, B_s^{\circ l}$ is a primitive solution of (4), we conclude that $\deg X_0 = 1$. □

3. *The group G_B*

Define an equivalence relation $\underset{B}{\sim}$ on the set C_B, setting $A_1 \underset{B}{\sim} A_2$ if

$$A_1 \circ B^{\circ l_1} = A_2 \circ B^{\circ l_2} \tag{13}$$

for some $l_1 \geq 0$, $l_2 \geq 0$ (in order to distinguish this relation from the relation $\sim$ introduced in the previous section we use the subscript B). It is easy to see that $\underset{B}{\sim}$ is really an equivalence relation. Indeed, $\underset{B}{\sim}$ is clearly reflexive and symmetric. Furthermore, if equalities (13) and

$$A_2 \circ B^{\circ n_1} = A_3 \circ B^{\circ n_2}$$

hold, and $n_1 \geq l_2$, then

$$A_1 \circ B^{\circ(l_1+n_1-l_2)} = A_2 \circ B^{\circ n_1} = A_3 \circ B^{\circ n_2},$$

implying that $A_1 \underset{B}{\sim} A_3$. Similarly, if $l_2 \geq n_1$, then

$$A_3 \circ B^{\circ(n_2+l_2-n_1)} = A_2 \circ B^{\circ l_2} = A_1 \circ B^{\circ l_1}.$$

LEMMA 3.1. *Let* $\mathbf{A}$ *be an equivalence class of* $\underset{B}{\sim}$. *For any* $n \geq 1$, *the class* $\mathbf{A}$ *contains at most one rational function of degree* n. *Furthermore, if* $A_0 \in \mathbf{A}$ *is a function of minimal possible degree, then any* $A \in \mathbf{A}$ *has the form* $A = A_0 \circ B^{\circ l}$, $l \geq 1$. *Alternatively, the function* A_0 *can be described as a unique function in* $\mathbf{A}$ *that is not a rational function in* B.

Proof. If $\deg A_1 = \deg A_2$ in (13), then $l_1 = l_2$, implying that $A_1 = A_2$. Furthermore, if

$$A \circ B^{\circ l_1} = A_0 \circ B^{\circ l_2} \tag{14}$$

and $l_1 > l_2$, then

$$A_0 = A \circ B^{\circ(l_1-l_2)},$$

implying that $\deg A < \deg A_0$ in contradiction with the assumption. Therefore, $l_1 \leq l_2$ and, hence,

$$A = A_0 \circ B^{\circ(l_2-l_1)}.$$

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Moreover, A_0 is not a rational function in B, since if $A_0 = A' \circ B$, then A' commutes with B by Lemma 2.1, implying that $A' \underset{B}{\sim} A_0$ and $\deg A' < \deg A_0$. On the other hand, if A is an other function in the class $\mathbf{A}$ that is not a rational function in B, then (14) implies that $l_1 = l_2$ and $A = A_0$. □

For a rational function B, we denote by G_B the set of equivalence classes of $\underset{B}{\sim}$ on C_B. We define a binary operation on the set G_B as follows. If $\mathbf{A}_1$ and $\mathbf{A}_2$ are equivalence classes of $\underset{B}{\sim}$, and $A_1 \in \mathbf{A}_1$ and $A_2 \in \mathbf{A}_2$ are their representatives, then $\mathbf{A}_1 \cdot \mathbf{A}_2$ is defined as the equivalence class containing $A_1 \circ A_2$. It is easy to see that this operation is well defined. Indeed, assume that $A_1 \underset{B}{\sim} A_1'$ and $A_2 \underset{B}{\sim} A_2'$. Then

$$A_1 \circ B^{\circ l_1} = A_1' \circ B^{\circ l_1'}$$

and

$$A_2 \circ B^{\circ l_2} = A_2' \circ B^{\circ l_2'},$$

implying that

$$A_1 \circ B^{\circ l_1} \circ A_2 \circ B^{\circ l_2} = A_1' \circ B^{\circ l_1'} \circ A_2' \circ B^{\circ l_2'}. \tag{15}$$

Since $A_1, A_2 \in C_B$, equality (15) implies that

$$A_1 \circ A_2 \circ B^{\circ (l_1 + l_2)} = A_1' \circ A_2' \circ B^{\circ (l_1' + l_2')},$$

and, hence,

$$A_1 \circ A_2 \underset{B}{\sim} A_1' \circ A_2'.$$

THEOREM 3.2. *The set G_B equipped with the operation $\cdot$ is a finite group.*

Proof. By definition, if $A_i \in \mathbf{A}_i$, $1 \leq i \leq 3$, then $(\mathbf{A}_1 \cdot \mathbf{A}_2) \cdot \mathbf{A}_3$ and $\mathbf{A}_1 \cdot (\mathbf{A}_2 \cdot \mathbf{A}_3)$ are classes containing the functions $(A_1 \circ A_2) \circ A_3$ and $A_1 \circ (A_2 \circ A_3)$, respectively. On the other hand,

$$(A_1 \circ A_2) \circ A_3 = A_1 \circ (A_2 \circ A_3),$$

since $\circ$ is an associative operation on the set of rational functions. Therefore, the classes $(\mathbf{A}_1 \cdot \mathbf{A}_2) \cdot \mathbf{A}_3$ and $\mathbf{A}_1 \cdot (\mathbf{A}_2 \cdot \mathbf{A}_3)$ coincide, and, hence, the operation $\cdot$ satisfies the associativity axiom.

Clearly, the class $\mathbf{e}$ containing the function z and consisting of all iterates of B serves as the unit element. Moreover, for any class $\mathbf{X}$ there exists a class $\mathbf{X}^{-1}$ such that

$$\mathbf{X} \cdot \mathbf{X}^{-1} = \mathbf{X}^{-1} \circ \mathbf{X} = \mathbf{e}. \tag{16}$$

Indeed, by Theorem 2.3, for any $X \in \mathbf{X}$ there exists a sequence of elementary transformation (5) such that

$$X = U_s \circ U_{s-1} \circ \cdots \circ U_1.$$

Further, it follows from Lemma 2.2 that the function

$$Y = V_s \circ V_{s-1} \circ \cdots \circ V_1$$

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belongs to C_B, and the functions X and Y satisfy

$$X \circ Y = Y \circ X = B^{\circ s}.$$

Therefore, condition (16) holds for $\mathbf{X}^{-1}$ defined as the class containing the rational function Y.

Finally, by the result of [**5**] cited in the introduction, there exist at most finitely many rational functions $A \in C_B$ which are not rational functions in B, implying by Lemma 3.1 that the group G_B is finite. □

Note that the above proof provides a method for the actual finding $\mathbf{X}^{-1}$. On the other hand, merely the existence of the inverse element follows from the Ritt theorem. Indeed, since for any $X \in \mathbf{X}$ there exist $l, k \geq 1$ such that (2) holds, for any class $\mathbf{X}$ there exists k such that $\mathbf{X}^k = \mathbf{e}$, implying that (16) holds for $\mathbf{X}^{-1} = \mathbf{X}^{k-1}$. Note also that the Ritt theorem by itself does not imply that the group G_B is finite, although it does imply that any its element has finite order.

For $X \in C_B$, we denote by $\boldsymbol{X}$ the element of G_B corresponding to the equivalence class of $\underset{B}{\sim}$ containing X.

LEMMA 3.3. *The map* $\mu \to \boldsymbol{\mu}$ *is a group monomorphism from the group* Aut(B) *to the group* G_B.

Proof. Since functions from Aut(B) have degree one, it follows from Lemma 3.1 that $\boldsymbol{\mu_1} = \boldsymbol{\mu_2}$ if and only if $\mu_1 = \mu_2$. Therefore, the map $\tau : \mu \to \boldsymbol{\mu}$ is injective, and it is easy to see that τ is a homomorphism of groups. □

We denote the image of Aut(B) in G_B under the group monomorphism $\mu \to \boldsymbol{\mu}$ by Aut$_G(B)$.

LEMMA 3.4. *The following conditions are equivalent.*
(1) *Any* $X \in C_B$ *has the form* $X = \mu \circ B^{\circ l}$ *for some* $\mu \in$ Aut(B) *and* $l \geq 0$.
(2) *Any* $X \in C_B$ *of degree at least two is a rational function in* B.
(3) *The group* G_B *coincides with* Aut$_G(B)$.

Proof. It is easy to see that (1) and (3) are equivalent, and that (1) implies (2). Assume now that (2) holds, and let $X \in C_B$ be a function of degree at least two. By the assumption, $X = R_1 \circ B$ for some $R \in \mathbb{C}(z)$. Moreover, since by Lemma 2.1 the function R_1 belongs to C_B, using (2) again we conclude that either $R_1 \in$ Aut(B), or there exists $R_2 \in \mathbb{C}(z)$ such that $R_1 = R_2 \circ B$ and $R_2 \in C_B$. It is clear that continuing this process we will eventually obtain a representation $X = \mu \circ B^l$ for some $\mu \in$ Aut(B) and $l \geq 1$. □

4. *The graph* Γ_B

Let B be a rational function of degree at least two. Define Γ_B as a multigraph whose vertices are in a one-to-one correspondence with some fixed representatives of conjugacy classes in $[B]$, and whose multiple edges connecting vertices corresponding to representatives B_i and B_j are in a one-to-one correspondence with solutions of the system

$$B_i = V \circ U, \quad B_j = U \circ V \tag{17}$$

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in rational functions. Note that Γ_B have loops. They correspond to solutions of

$$B_i = U \circ V = V \circ U.$$

LEMMA 4.1. *The graph Γ_B does not depend on the choice of representatives of conjugacy classes in $[B]$.*

Proof. Indeed, for any Möbius transformations α and β, to a solution U, V of system (17) corresponds a solution

$$U' = \beta \circ U \circ \alpha^{-1}, \quad V' = \alpha \circ V \circ \beta^{-1} \tag{18}$$

of the system

$$\alpha \circ B_i \circ \alpha^{-1} = V' \circ U', \quad \beta \circ B_j \circ \beta^{-1} = U' \circ V'. \tag{19}$$

Furthermore, it is easy to see that formulas (18) provide a one-to-one correspondence between solutions of (17) and (19). □

THEOREM 4.2. *Let B a rational function of degree at least two. Then the graph Γ_B is finite, unless B is a flexible Lattès map.*

Proof. By the main result of the paper [**9**], the class $[B]$ contains infinitely many conjugacy classes if and only if B is a flexible Lattès map. Therefore, if B is not such a map, the graph Γ_B contains only finitely many vertices.

Let us show now that the number of edges connecting two vertices is finite. Recall that two decompositions

$$B = V \circ U, \quad B = V' \circ U' \tag{20}$$

of a rational function B into compositions of rational functions are called *equivalent* if there exists a Möbius transformation μ such that

$$V' = V \circ \mu^{-1}, \quad U' = \mu \circ U. \tag{21}$$

It is well known that equivalence classes of decompositions of B are in one-to-one correspondence with imprimitivity systems of the monodromy group $\mathrm{Mon}(B)$ of B. In particular, there exist at most finitely many such classes. Therefore, to prove the finiteness of the number of edges adjacent to the vertices corresponding to B_i and B_j it is enough to show that for any fixed solution U, V of (17) there exist only finitely many solutions U', V' of (17) such that decompositions (20) are equivalent. Since equalities (21) combined with the equality

$$U \circ V = U' \circ V'$$

imply the equality

$$U \circ V = \mu \circ U \circ V \circ \mu^{-1},$$

the last statement follows from the finiteness of the group $\mathrm{Aut}(U \circ V)$. □

Since in this paper we consider only non-special rational functions B, the corresponding graphs Γ_B are always finite by Theorem 4.2. Note that the results of [**5**] imply that the number of vertices of Γ_B can be bounded by a number depending on $\deg B$ only (see [**5**, Remark 5.2]). Nevertheless, there exists no absolute bound for the number of vertices

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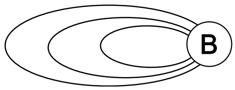

FIGURE 1. The form of Γ_B in Example 1.

of Γ_B, and it is easy to construct rational functions B of degree n for which the graph Γ_B contains $\approx \log_2 n$ vertices (see [**6**, p. 1241]).

We always assume that the representative of the conjugacy class of the function B in Γ_B is the function B itself. Abusing notation, in the following we call the functions B_j simply 'vertices' of Γ_B. Note that for each vertex B_j of Γ_B there exists at least one loop starting and ending at B that corresponds to the solution

$$B = B \circ z = z \circ B \tag{22}$$

of (17). More generally, the solutions

$$B = (\mu^{-1} \circ B) \circ \mu = \mu \circ (\mu^{-1} \circ B), \quad \mu \in \text{Aut}(B), \tag{23}$$

give rise to $|\text{Aut}(B)|$ loops.

Example 1. Assume that B is an *indecomposable* rational function. By definition, this means that the equality $B = V \circ U$ implies that at least one of the functions U and V has degree one. In this case, the equivalence class $[B]$ obviously consists of a unique conjugacy class. Thus, Γ_B has a unique vertex, and all edges of Γ_B are loops corresponding to solutions of

$$B = U \circ V = V \circ U$$

such that one of the functions U, V has degree one. Assuming without loss of generality that $\deg U = 1$, we see that

$$B \circ U = U \circ V \circ U = U \circ B,$$

implying that $U \in \text{Aut}(B)$. Therefore, Γ_B has the form shown in Figure 1, and the number of loops of Γ_B is equal to $|\text{Aut}(B)|$.

Example 2. Assume now that a rational function B has, up to equivalency (21), a unique decomposition $B = V \circ U$ into a composition of rational functions of degree at least two, and that the same is true for the function $B_1 = U \circ V$. In this case, graph Γ_B may have two distinct forms. Namely, if B_1 and B are not conjugate, then Γ_B has the form shown in Figure 2, where all loops correspond to some automorphisms. Note that for such B and B_1 the groups $\text{Aut}(B)$ and $\text{Aut}(B_1)$ are isomorphic (see Lemma 6.3), implying that B and B_1 have the same number of attached loops.

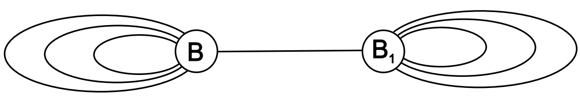

FIGURE 2. The form of Γ_B in Example 2.

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On the other hand, if B_1 is conjugate to B, then without loss of generality we may assume that $B_1 = B$, so that

$$B = V \circ U = U \circ V. \tag{24}$$

In this case, the graph Γ_B has one vertex and $|\mathrm{Aut}(B)| + 1$ loops corresponding to (23) and (24). Note that since by the assumption the decompositions in (24) are equivalent, the equalities

$$U = V \circ \mu^{-1}, \quad V = \mu \circ U$$

hold for some Möbius transformation μ, implying that

$$B = V \circ U = \mu \circ U^{\circ 2}.$$

Thus, up to a composition with a Möbius transformation μ, the function B is the second iterate of some rational function U. Moreover, since

$$U = V \circ \mu^{-1} = \mu \circ U \circ \mu^{-1},$$

the transformation μ belongs to $\mathrm{Aut}(U)$.

Example 3. Set

$$B = -\frac{2z^2}{z^4+1} = -\frac{2}{z^2 + 1/z^2}.$$

The function B is an invariant for the finite automorphism group of $\mathbb{CP}^1$ generated by the transformations

$$z \to \frac{1}{z}, \quad z \to -z,$$

and its monodromy group $\mathrm{Mon}(B)$ is the Klein four group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ having three proper imprimitivity systems. The corresponding decompositions of B are

$$B = -\frac{2}{z^2-2} \circ \frac{z^2+1}{z}, \quad B = -\frac{2}{z^2+2} \circ \frac{z^2-1}{z},$$

and

$$B = \frac{z^2-1}{z^2+1} \circ \frac{z^2-1}{z^2+1}. \tag{25}$$

Using, for example, the 'Maple' system, one can check that the function

$$B_1 = \frac{z^2+1}{z} \circ -\frac{2}{z^2-2} = -\frac{1}{2}\,\frac{z^4 - 4\,z^2 + 8}{z^2-2} \tag{26}$$

has three critical values in $\mathbb{CP}^1$, and the corresponding permutations in $\mathrm{Mon}(B_1)$ can be identified with the permutations $(12)(34)$, (1243), and (14) in S_4. On the other hand, the function

$$B_2 = \frac{z^2-1}{z} \circ -\frac{2}{z^2+2} = \frac{1}{2}\,\frac{z^2(z^2+4)}{z^2+2} \tag{27}$$

has four critical values, and the corresponding permutations in $\mathrm{Mon}(B_2)$ can be identified with $(12)(34)$, (23), $(12)(34)$, and (14). Since B_1 and B_2 have a different number of critical values, they are not conjugate. Furthermore, it is easy to see that the both groups $\mathrm{Mon}(B_1)$ and $\mathrm{Mon}(B_2)$ have a unique proper imprimitivity system $\{1, 4\}$, $\{2, 3\}$,

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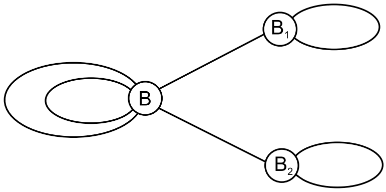

FIGURE 3. The form of Γ_B in Example 3.

corresponding to decompositions (26) and (27), implying, in particular, that B is not conjugate to B_1 or B_2. Finally, one can check by a direct calculations, solving the system

$$\frac{az+b}{cz+d} \circ B = B \circ \frac{az+b}{cz+d}$$

in a, b, c, d, that the functions B, B_1, B_2 have no automorphisms. Summing up, we conclude that the graph Γ_B has the form shown on Figure 3.

5. *The epimorphism* $\pi_1(\Gamma_B) \to G_B$

Considering the graph Γ_B as a one-dimensional CW complex in $\mathbb{R}^3$, we can provide each edge of Γ_B, including loops, with two opposite *orientations*. With each oriented edge e of Γ_B, we associate a rational function $\mathcal{F}(e)$ as follows. Assume first that e corresponds to solution (17) with *different* B_i and B_j. Then we set $\mathcal{F}(e) = U$, if the initial point of e is B_i and the final point is B_j, and $\mathcal{F}(e) = V$, if the orientation is opposite. For a loop, we simply set the value of $\mathcal{F}$ equal to U for one of the two corresponding oriented edges, and equal to V for the opposite oriented edge. For an oriented *path*

$$l = e_n e_{n-1} \dots e_1,$$

set

$$\mathcal{F}(l) = \mathcal{F}(e_n) \circ \mathcal{F}(e_{n-1}) \circ \cdots \circ \mathcal{F}(e_1).$$

We emphasize that since we always compose functions from right to left, we follow this convention also for a concatenation of paths. Thus, a path obtained by a concatenation of the paths l_1 and l_2 is denoted by

$$l = l_2 l_1,$$

and the above definition implies that

$$\mathcal{F}(l) = \mathcal{F}(l_2) \circ \mathcal{F}(l_1). \tag{28}$$

As usual, we denote the path l traversed in the opposite direction by l^{-1}.

By construction, oriented paths from B to B_s correspond to sequences of elementary transformation (5). Furthermore, in the notation of Lemma 2.2, if

$$\mathcal{F}(l) = U_s \circ U_{s-1} \circ \cdots \circ U_1,$$

then

$$\mathcal{F}(l^{-1}) = V_1 \circ \cdots \circ V_{s-1} \circ V_s.$$

In particular, Lemma 2.2 implies the following statement.

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LEMMA 5.1. *Let l be an oriented path in Γ_B from the vertex B to a vertex B_s consisting of k oriented edges. Then*

$$B_s \circ \mathcal{F}(l) = \mathcal{F}(l) \circ B, \tag{29}$$

and

$$\mathcal{F}(l^{-1}) \circ \mathcal{F}(l) = B^{\circ k}, \quad \mathcal{F}(l) \circ \mathcal{F}(l^{-1}) = B_s^{\circ k}. \qquad \square \tag{30}$$

If l is a closed path in Γ_B starting and ending at B, then (29) implies that the function $\mathcal{F}(l)$ commutes with B, while equalities (30) reduce to the equalities

$$\mathcal{F}(l^{-1}) \circ \mathcal{F}(l) = \mathcal{F}(l) \circ \mathcal{F}(l^{-1}) = B^{\circ k}. \tag{31}$$

Thus, we obtain a map $\phi_B : l \to \mathcal{F}(l)$ from the set of closed paths starting and ending at B to the set C_B.

THEOREM 5.2. *The map $\phi_B : l \to \mathcal{F}(l)$ descends to an epimorphism of groups $\Phi_B : \pi_1(\Gamma_B, B) \to G_B$.*

Proof. Let Γ be a graph. Recall that an oriented path l in Γ is called *reduced* if no two successive oriented edges in l are opposite orientations of the same edge. Paths of the form $e^{-1}e$, where e is an oriented edge are called *spurs*. Paths l and l' are called *equivalent* if l' is obtained from l by a finite number of insertions and removals of spurs between successive oriented edges or at the endpoints. In these terms, the fundamental group $\pi_1(\Gamma, V)$ of the graph Γ can be defined as the set of equivalence classes of paths that begin and end at some fixed vertex V of Γ, equipped with the product of classes defined in an obvious way (see e.g. [**14**, §2.1.6]).

To prove that the map ϕ_B descends to a map from $\pi_1(\Gamma_B, B)$ to G_B, we must show that whenever closed paths l and l' in Γ_B that start and end at B are equivalent, the rational functions $\mathcal{F}(l)$ and $\mathcal{F}(l')$ are in the same equivalence class of C_B. Since any path is equivalent to a path with no spurs, for this purpose it is enough to show that if l' is obtained from l by an insertion of a spur, then $\mathcal{F}(l) \underset{B}{\sim} \mathcal{F}(l')$. Assume that

$$l' = l_2 e^{-1} e l_1,$$

where l_1 is a path from B to B_s, and l_2 is a path from B_s to B (one of the paths l_1 and l_2 can be empty in which case $B_s = B$). Then

$$\mathcal{F}(l') = \mathcal{F}(l_2) \circ B_s \circ \mathcal{F}(l_1),$$

by (28) and (31). It follows now from (29) that

$$\mathcal{F}(l') = \mathcal{F}(l_2) \circ \mathcal{F}(l_1) \circ B = \mathcal{F}(l) \circ B,$$

implying that $\mathcal{F}(l) \underset{B}{\sim} \mathcal{F}(l')$. Thus, ϕ_B descends to a map $\Phi_B : \pi_1(\Gamma_B, B) \to G_B$, and (28) implies that Φ_B is a homomorphism of groups.

Finally, it follows from Theorem 2.3 that Φ_B is an epimorphism. Indeed, by Theorem 2.3, any $X \in C_B$ can be obtained from a sequence of elementary transformations (5). Moreover, we can change if necessary each of rational functions B_i, $1 \le i \le s-1$, appearing in (5) to any desired representative of its conjugacy class, consecutively

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changing the function U_i to $\alpha_i \circ U_i$, the function B_i to $\alpha_i \circ B_i \circ \alpha_i^{-1}$, and the function U_{i+1} to $\alpha_i^{-1} \circ U_{i+1}$ for a convenient Möbius transformation α_i. Therefore, for any $X \in C_B$, there exists a closed path l starting and ending at B such that $\mathcal{F}(l) = X$, implying that $\Phi_B : \pi_1(\Gamma_B, B) \to G_B$ is an epimorphism. □

THEOREM 5.3. *Let A and B be equivalent rational functions. Then $G_B \cong G_A$.*

Proof. Assuming that A and B are vertices of Γ_B, take a path s from A to B in Γ_B. Since the map $\psi : l \to s^{-1}ls$, from the set of closed paths starting and ending at B to the set of closed paths starting and ending at A, descends to an isomorphism of the fundamental groups

$$\Psi : \pi_1(\Gamma_B, B) \to \pi_1(\Gamma_B, A),$$

it follows from Theorem 5.2 that we only need to prove the equality

$$\Psi(\text{Ker } \Phi_B) = \text{Ker } \Phi_A. \tag{32}$$

Let l_0 be a path starting and ending at B such that $\mathcal{F}(l_0) = B^{\circ k}$, $k \geq 1$, and let $k_0 = \psi(l_0)$. Then

$$\mathcal{F}(k_0) = \mathcal{F}(s^{-1}) \circ \mathcal{F}(l_0) \circ \mathcal{F}(s) = \mathcal{F}(s^{-1}) \circ B^{\circ k} \circ \mathcal{F}(s),$$

implying by (29) and (30) that

$$\mathcal{F}(k_0) = \mathcal{F}(s^{-1}) \circ \mathcal{F}(s) \circ A^{\circ k} = A^{\circ l} \circ A^{\circ k} = A^{\circ(k+l)}$$

for some $k, l \geq 1$. This implies that

$$\Psi(\text{Ker } \Phi_B) \subseteq \text{Ker } \Phi_A.$$

Similarly, considering the isomorphism inverse to Ψ we obtain that

$$\Psi^{-1}(\text{Ker } \Phi_A) \subseteq \text{Ker } \Phi_B.$$

This proves equality (32). □

6. *Examples of groups G_B*

6.1. *Functions with $G_B = \text{Aut}_G(B)$.* The simplest application of Theorem 5.2 is the following result.

THEOREM 6.1. *Let B be an indecomposable non-special rational function of degree at least two. Then $G_B = \text{Aut}_G(B)$. Equivalently, $X \in C_B$ if and only if $X = \mu \circ B^l$ for some $\mu \in \text{Aut}(B)$ and $l \geq 1$.*

Proof. Since Γ_B has a unique vertex and $|\text{Aut}(B)|$ loops corresponding to automorphisms of B (see Example 1), it follows easily from Theorem 5.2 that G_B is generated by $\boldsymbol{\mu}$, $\mu \in \text{Aut}(B)$. Thus, $G_B = \text{Aut}_G(B)$. The second statement follows from Lemma 3.4. □

Note that Theorem 6.1 implies that for a 'random' rational function B, the group G_B is trivial, since such a function is indecomposable and has no automorphisms.

Theorem 6.1 can be extended to a wide class of decomposable rational functions. Recall that a functional decomposition

$$B = U_r \circ U_{r-1} \circ \cdots \circ U_1 \tag{33}$$

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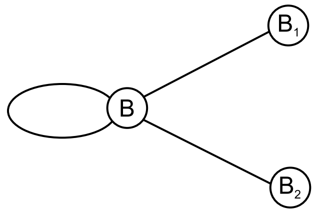

FIGURE 4. The form of Γ_B^0 in Example 3.

of a rational function B is called *maximal* if all $U_1, U_2, \ldots, U_r$ are indecomposable and of degree greater than one. The number r is called the length of the maximal decomposition (33). Two decompositions (maximal or not) having an equal number of terms

$$F = F_r \circ F_{r-1} \circ \cdots \circ F_1 \quad \text{and} \quad F = G_r \circ G_{r-1} \circ \cdots \circ G_1$$

are called equivalent if either $r = 1$ and $F_1 = G_1$ or $r \geq 2$ and there exist Möbius transformations μ_i, $1 \leq i \leq r-1$, such that

$$F_r = G_r \circ \mu_{r-1}, \quad F_i = \mu_i^{-1} \circ G_i \circ \mu_{i-1}, \quad 1 < i < r, \quad \text{and} \quad F_1 = \mu_1^{-1} \circ G_1.$$

Note that all maximal decompositions of a polynomial have the same length [**11**], but this is not true for arbitrary rational functions (see e.g. [**4**]).

We say that a rational function B having a maximal decomposition (33) is *generically decomposable* if the following conditions are satisfied:

- each of the functions

$$B_i = (U_i \circ \cdots \circ U_2 \circ U_1) \circ (U_r \circ U_{r-1} \circ \cdots \circ U_{i+1}), \quad 0 \leq i \leq r-1,$$

  has a unique equivalence class of maximal decompositions;
- the functions B_i, $0 \leq k \leq r-1$, are pairwise not conjugate.

For a graph Γ_B, define Γ_B^0 as a graph obtained from Γ_B by removing all loops that correspond to automorphisms. For example, for the graph Γ_B from Example 3 the graph Γ_B^0 is shown in Figure 4. Recall that a *complete graph* is a graph in which every pair of distinct vertices is connected by a unique edge. The complete graph on n vertices is denoted by K_n.

LEMMA 6.2. *Assume that a non-special rational function B having a maximal decomposition of length r is generically decomposable. Then Γ_B^0 is the complete graph K_r.*

Proof. Let (33) be a maximal decomposition of B. Since all the functions B_i, $0 \leq i \leq r-1$, are equivalent and pairwise not conjugate, the graph Γ_B contains at least r vertices. Observe now that any decomposition $B = V \circ U$ of B into a composition of two rational functions of degree at least two has the form

$$V = (U_r \circ U_{r-1} \circ \cdots \circ U_{i+1}) \circ \mu, \quad U = \mu^{-1} \circ (U_i \circ \cdots \circ U_2 \circ U_1), \quad 0 \leq i \leq r-1, \tag{34}$$

where μ is a Möbius transformation. Indeed, concatenating arbitrary maximal decompositions of U and V we must obtain a maximal decomposition equivalent to

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(33), implying that (34) holds. Therefore, any edge of Γ_B adjacent to $B_0 = B$ and not corresponding to an automorphism of B is adjacent to one of the vertices B_i, $1 \leq k \leq r-1$, and there exists exactly one edge connecting B_0 and B_i, $1 \leq k \leq r-1$. Since the same argument holds for any B_i, $0 \leq k \leq r-1$, we conclude that Γ_B^0 is the complete graph K_r. □

LEMMA 6.3. *Assume that a non-special rational function B is generically decomposable, and let l be an oriented path from a vertex B_{i_1} to a vertex B_{i_2} in Γ_B. Then for any $\mu \in \mathrm{Aut}(B_{i_1})$ there exists $\alpha(\mu) \in \mathrm{Aut}(B_{i_2})$ such that*

$$\mathcal{F}(l) \circ \mu = \alpha(\mu) \circ \mathcal{F}(l). \tag{35}$$

Furthermore, the map $\mu \to \alpha(\mu)$ is an isomorphism of the groups $\mathrm{Aut}(B_{i_1})$ and $\mathrm{Aut}(B_{i_2})$. In particular, the same number of loops is attached to each vertex of Γ_B.

Proof. In view of formula (28), it is enough to prove the lemma for the case where l is an oriented edge. If l is a loop, then by Lemma 6.2, it corresponds to a solution of (17) of the form

$$B_{i_1} = (\mu_0^{-1} \circ B_{i_1}) \circ \mu_0 = \mu_0 \circ (\mu_0^{-1} \circ B_{i_1}), \quad \mu_0 \in \mathrm{Aut}(B_{i_1}).$$

Thus, either $\mathcal{F}(l) = \mu_0$ or $\mathcal{F}(l) = \mu_0^{-1} \circ B_{i_1}$, and it is easy to see that in these cases equality (35) holds for the automorphisms

$$\alpha(\mu) = \mu_0 \circ \mu \circ \mu_0^{-1}, \quad \alpha(\mu) = \mu_0^{-1} \circ \mu \circ \mu_0,$$

respectively.

Assume now that l is an oriented edge from a vertex $B_{i_1} = V \circ U$ to a different vertex $B_{i_2} = U \circ V$. Let us observe that for any $\mu \in \mathrm{Aut}(B_{i_1})$ the decompositions $B_{i_1} = V \circ U$ and

$$B_{i_1} = (\mu^{-1} \circ V) \circ (U \circ \mu)$$

are equivalent, since for arbitrary maximal decompositions of U and V the corresponding induced maximal decompositions of B_{i_1} are equivalent. Therefore, for any $\mu \in \mathrm{Aut}(B_{i_1})$, there exists a Möbius transformation $\alpha = \alpha(\mu)$ such that

$$\mu^{-1} \circ V = V \circ \alpha(\mu)^{-1}, \quad U \circ \mu = \alpha(\mu) \circ U.$$

Furthermore, since

$$B_{i_2} = U \circ V = U \circ \mu \circ \mu^{-1} \circ V = \alpha(\mu) \circ U \circ V \circ \alpha(\mu)^{-1},$$

the transformation $\alpha(\mu)$ belongs to $\mu \in \mathrm{Aut}(B_{i_2})$, and it is easy to see that $\mu \to \alpha(\mu)$ is a group homomorphism from $\mathrm{Aut}(B_{i_1})$ to $\mathrm{Aut}(B_{i_2})$.

Finally, if

$$\nu \to \beta(\nu)$$

is a homomorphism from $\mathrm{Aut}(B_{i_2})$ to $\mathrm{Aut}(B_{i_1})$, defined by the conditions

$$\nu^{-1} \circ U = U \circ \beta(\nu)^{-1}, \quad V \circ \nu = \beta(\nu) \circ V,$$

and $\mu \in \mathrm{Aut}(B_{i_1})$, then

$$V \circ U \circ \mu = V \circ \alpha(\mu) \circ U = \beta(\alpha(\mu)) \circ V \circ U.$$

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Since

$$V \circ U \circ \mu = \mu \circ V \circ U,$$

this implies that $\beta \circ \alpha$ is the identical mapping of $\mathrm{Aut}(B_{i_1})$, and hence $\mu \to \alpha(\mu)$ is an isomorphism. □

THEOREM 6.4. *Let B be a non-special generically decomposable rational function. Then $G_B = \mathrm{Aut}_G(B)$. Equivalently, $X \in C_B$ if and only if $X = \mu \circ B^l$ for some $\mu \in \mathrm{Aut}(B)$ and $l \geq 1$.*

Proof. Let (33) be a maximal decomposition of B. For convenience, define rational functions U_i for $i \geq r$ setting $U_i = U_{i'}$, where $i \equiv i' \mod r$. Let us recall that any decomposition $B = V \circ U$, where U and V are functions of degree at least two, has the form (34), and a similar statement holds for all B_i, $0 \leq i \leq r-1$. Therefore, for the oriented edge e from a vertex B_{i_1} to a different vertex B_{i_2} the equality

$$\mathcal{F}(e) = U_{i_2} \circ \cdots \circ U_{i_1+2} \circ U_{i_1+1}$$

holds, implying inductively by (28) that for an arbitrary path l with no loops from B_{i_1} to B_{i_2} the equality

$$\mathcal{F}(l) = U_{i_2+rk} \circ \cdots \circ U_{i_1+2} \circ U_{i_1+1} = B_{i_2}^{\circ k} \circ U_{i_2} \circ \cdots \circ U_{i_1+2} \circ U_{i_1+1}$$

holds for some $k \geq 1$. In particular, if l is a closed path starting and ending at B and containing no loops, then $\mathcal{F}(l) = B^{\circ k}$, $k \geq 1$, implying that the image of l under the homomorphism Φ_B from Theorem 5.2 is the unit element. Further, if l contains a loop, then either

$$\mathcal{F}(l) = U_{kr} \circ \cdots \circ U_{i+1} \circ \nu \circ U_i \circ \cdots \circ U_1,$$

or

$$\mathcal{F}(l) = U_{kr} \circ \cdots \circ U_{i+1} \circ (\nu^{-1} \circ B_i) \circ U_i \circ \cdots \circ U_1$$

for some $k \geq 1$, $0 \leq i \leq r-1$, and $\nu \in \mathrm{Aut}(B_i)$. Therefore, by Lemmas 6.3 and 5.1, either

$$\mathcal{F}(l) = \mu \circ B^{\circ k},$$

or

$$\mathcal{F}(l) = \mu \circ B^{\circ (k+1)}$$

for some $\mu \in \mathrm{Aut}(B)$. Finally, if l contains several loops, then repeatedly using Lemmas 6.3 and 5.1, we conclude that

$$\mathcal{F}(l) = \mu \circ B^{\circ s}$$

for some $\mu \in \mathrm{Aut}(B)$ and $s \geq 1$. Thus, $G_B = \mathrm{Aut}_G(B)$. □

COROLLARY 6.5. *Let B be a non-special rational function of degree at least two such that G_B is strictly larger than $\mathrm{Aut}_G(B)$. Then there exists $A \sim B$ such that either A can be represented as a composition of two commuting rational functions of degree at least two, or A has more than one class of maximal decompositions.*

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Proof. By Theorem 6.4, it is enough to show that if any $A \sim B$ has a unique equivalence class of maximal decompositions and cannot be represented as a composition of two commuting rational functions of degree at least two, then for the function B the both conditions defining generically decomposable rational functions are satisfied. For the first condition, this is obvious. For the second condition, this is also true. Indeed, if say $B_0 = B$ is conjugate to B_i and μ is a Möbius transformation such that

$$(U_r \circ \cdots \circ U_{i+1}) \circ (U_i \circ \cdots \circ U_1) = \mu \circ (U_i \circ \cdots \circ U_1) \circ (U_r \circ \cdots \circ U_{i+1}) \circ \mu^{-1},$$

then for the functions

$$N = \mu \circ (U_i \circ \cdots \circ U_1), \qquad M = (U_r \circ \cdots \circ U_{i+1}) \circ \mu^{-1}$$

the equality

$$B = M \circ N = N \circ M \tag{36}$$

holds. □

Note that whenever B *is* a composition of two commuting rational functions of degree at least two, the group G_B is strictly larger than $\mathrm{Aut}_G(B)$. Indeed, equality (36) implies easily that the functions N and M belong to C_B. Moreover, their images in G_B are not trivial and do not belong to $\mathrm{Aut}_G(B)$, since

$$1 < \deg M < \deg B, \qquad 1 < \deg N < \deg B.$$

In particular, if $B = T^{\circ s}$, where $s > 1$, the group G_B contains a cyclic group of order s whose intersection with $\mathrm{Aut}_G(B)$ is trivial.

Finally, note that the group G_B can be strictly larger than $\mathrm{Aut}_G(B)$ even if B is not a composition of commuting functions, and that the relation $A \sim B$ does not imply, in general, the equality $\mathrm{Aut}_G(A) \cong \mathrm{Aut}_G(B)$ (see §6.3).

6.2. *The group G_B for polynomial B.* Before stating the theorem describing groups G_B for polynomial B let us recall several results.

First, for a non-special polynomial B of degree at least two, the set C_B consists of *polynomials*. Indeed, (1) yields that

$$B^{-1}(X^{-1}\{\infty\}) = X^{-1}\{\infty\}, \tag{37}$$

implying that $X^{-1}\{\infty\}$ contains at most two points. Furthermore, considering instead of B and X the functions

$$X \to \mu \circ X \circ \mu^{-1}, \qquad B \to \mu \circ B \circ \mu^{-1}$$

for a convenient Möbius transformation μ, without loss of generality one can assume that either $X^{-1}\{\infty\} = \{\infty\}$ or $X^{-1}\{\infty\} = \{\infty, 0\}$. In the first case, X is a polynomial. On the other hand, in the second case, (37) implies that B is conjugate to z^n, contradicting the assumption that B is not special.

Second, the symmetry group $\mathrm{Aut}(B)$ of a non-special polynomial B of degree at least two is cyclic. Indeed, unless B is conjugate to z^n, for any $\mu \in \mathrm{Aut}(B)$ necessarily $\mu^{-1}\{\infty\} = \{\infty\}$, implying that μ is a polynomial. By a polynomial conjugation, we can

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always assume that the coefficient of $z^{\deg B-1}$ is zero, and it is clear that $\mu = az + b$ may commute with such B only if $b = 0$. Furthermore, it is easy to see that $\mathrm{Aut}(B)$ is a cyclic rotation group of order n, where n is the maximal number such that

$$B = zR(z^n)$$

for some polynomial R.

Third, a polynomial B is special if and only if B is conjugate to z^n or $\pm T_n$, since it is well known that a polynomial cannot be a Lattès map.

In addition, we need the following result (see [**7**, Theorem 1.3]).

THEOREM 6.6. *Let A and B be fixed non-special polynomials of degree at least two, and let $\mathcal{E}(A, B)$ be the set of all polynomials of degree at least two X such that $A \circ X = X \circ B$. Then, either $\mathcal{E}(A, B)$ is empty, or there exists $X_0 \in \mathcal{E}(A, B)$ such that a polynomial X belongs to $\mathcal{E}(A, B)$ if and only if $X = \widehat{A} \circ X_0$ for some polynomial $\widehat{A}$ commuting with A.* □

Recall that a group G is called *metacyclic* if it has a normal cyclic subgroup H such that G/H is a cyclic group.

THEOREM 6.7. *Let B be a polynomial of degree at least two not conjugate to z^n or $\pm T_n$, $n \geq 2$. Then the group G_B is metacyclic.*

Proof. Applying Theorem 6.6 for $A = B$ and arguing as in Lemma 3.4, we see that any rational function X that belongs to $C_B = \mathcal{E}(B, B)$ has the form $X = \mu \circ X_0^{\circ l}$, where $\mu \in \mathrm{Aut}(B)$ and $l \geq 1$. In particular, $B = \mu \circ X_0^{l_0}$ for some $l_0 \geq 1$ and $\mu \in \mathrm{Aut}(B)$. Moreover, the degree of any element of C_B is a power of $d_0 = \deg X_0$, and for $l \geq 0$ the subset of elements of degree d_0^l coincides with the set $S_{1,l} = \{\mu \circ X_0^l \mid \mu \in \mathrm{Aut}(B)\}$.

Let us observe now that if

$$X_0^{\circ l} \circ \mu_1 = X_0^{\circ l} \circ \mu_2, \tag{38}$$

where $\mu_1, \mu_2 \in \mathrm{Aut}(B)$, then $\mu_1 = \mu_2$. Indeed, (38) implies that

$$X_0^{\circ l} \circ (\mu_1 \circ \mu_2^{-1}) = X_0^{\circ l}.$$

Therefore, since $B^{\circ l} = \nu \circ X_0^{\circ(l_0 l)}$ for some $\nu \in \mathrm{Aut}(B)$,

$$B^{\circ l} \circ (\mu_1 \circ \mu_2^{-1}) = B^{\circ l},$$

implying that $\mu_1 = \mu_2$. Thus, for $l \geq 0$ the set $S_{2,l} = \{X_0^l \circ \mu \mid \mu \in \mathrm{Aut}(B)\}$ has the same cardinality as the set $S_{1,l}$. Since $S_{2,l}$ is contained in C_B, this implies that $S_{1,l} = S_{2,l}$.

The above analysis shows that the right cosets of $\mathrm{Aut}_G(B)$ in G have the form

$$\boldsymbol{X}_0^l \mathrm{Aut}_G(B), \quad 0 \leq l < l_0,$$

the left cosets have the form

$$\mathrm{Aut}_G(B)\boldsymbol{X}_0^l, \quad 0 \leq l < l_0,$$

and any right coset of $\mathrm{Aut}_G(B)$ in G is a left coset. Thus, $\mathrm{Aut}_G(B)$ is a normal subgroup in G_B, and the group $G_B/\mathrm{Aut}_G(B)$ is a cyclic group of order l_0 generated by $\boldsymbol{X}_0$. Since $\mathrm{Aut}(B)$ is also a cyclic group, we conclude that the group G_B is metacyclic. □

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Note that Theorem 6.7 can be deduced from the Ritt theorem [**12**, **13**] saying that any commuting non-special polynomials X and B can be represented in the form (3). Nevertheless, the Ritt theorem does not imply Theorem 6.7 immediately, since R in (3) *a priori* depends on X, and the further analysis is needed.

6.3. *The group G_B for the Ritt example.* Let B be a rational function of degree at least two. Denote by $\widehat{\mathrm{Aut}}(B)$ the group consisting of Möbius transformations μ such that

$$B \circ \mu = \nu \circ B$$

for some Möbius transformations ν. Like the group $\mathrm{Aut}(B)$, the group $\widehat{\mathrm{Aut}}(B)$ is a finite rotation group of the sphere (see [**5**, §4]). More generally, denote by $\widehat{C}_B$ the set of rational functions X such that

$$B \circ X = Y \circ B$$

for some rational function Y. Clearly, $\mathrm{Aut}(B)$ is a subgroup of $\widehat{\mathrm{Aut}}(B)$, and $C_B \subseteq \widehat{C}_B$.

Let

$$V = \frac{z^2+2}{z+1}, \quad U = \frac{z^2-4}{z-1}, \quad \mu = \varepsilon z,$$

where $\varepsilon^3 = 1$. In [**13**], Ritt showed that the rational functions

$$B = V \circ U, \quad X = V \circ \mu \circ U$$

commute but no one of them is a rational function of the other. In particular, this implies that there is no R such that

$$B = \mu_1 \circ R^{\circ l_1}, \quad X = \mu_2 \circ R^{\circ l_2}$$

for some Möbius transformations μ_1, μ_2, and $l_1, l_2 \geq 1$. More generally, for any function C such that $C(\varepsilon z) = \varepsilon C(z)$, the functions

$$B' = V \circ C \circ U, \quad X' = V \circ \mu \circ C \circ U$$

commute, but no one of them is a rational function of the other.

The Ritt statement follows from the following more general observation.

LEMMA 6.8. *Let $W \in C_{U \circ V}$, but $W \notin \widehat{C}_V$. Then the functions $V \circ U$ and $V \circ W \circ U$ commute but the latter is not a rational function of the former. Furthermore, the same conclusion holds for the functions $V \circ C \circ U$ and $V \circ W \circ C \circ U$, where C is any function commuting with W.*

Proof. Indeed, we have

$$\begin{aligned}(V \circ C \circ U) \circ (V \circ W \circ C \circ U) &= V \circ C \circ (U \circ V \circ W) \circ C \circ U \\ &= V \circ C \circ (W \circ U \circ V) \circ C \circ U = (V \circ C \circ W \circ U) \circ (V \circ C \circ U) \\ &= (V \circ W \circ C \circ U) \circ (V \circ C \circ U).\end{aligned}$$

On the other hand, if

$$V \circ W \circ C \circ U = R \circ V \circ C \circ U$$

for some rational function R, then

$$V \circ W = R \circ V,$$

contradicting the assumption that $W \notin \widehat{C}_V$. □

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The Ritt statement is obtained from Lemma 6.8 for $W = \mu$. Indeed,

$$U \circ V = \frac{z(z^3 - 8)}{(z^3 + 1)},$$

implying that $\mu \in \mathrm{Aut}(U \circ V)$. On the other hand, the assumption that

$$V \circ \mu = \nu \circ V \tag{39}$$

for some Möbius transformation ν leads to a contradiction. Namely, (39) implies that $\nu(\infty) = \infty$. Therefore, $\nu = az + b$, $a, b \in \mathbb{C}$, and, hence, if (39) holds, then the functions V and

$$V \circ \mu = \frac{\varepsilon^2 z^2 + 2}{\varepsilon z + 1}$$

have the same set of poles. However, this is not true.

Let us calculate the group G_B. Again using the assistance of a computer one can check that the function

$$B = V \circ U = \frac{z^4 - 6\,z^2 - 4\,z + 18}{(z^2 + z - 5)(z - 1)}$$

has four critical values and the corresponding permutations in $\mathrm{Mon}(B)$ can be identified with the permutations (13), $(12)(34)$, (13), and $(12)(34)$ in S_4, while the function

$$B_1 = U \circ V = \frac{z(z^3 - 8)}{(z^3 + 1)}$$

has three critical values and the corresponding permutations in $\mathrm{Mon}(B_1)$ can be identified with $(12)(34)$, $(13)(24)$, and $(14)(23)$. In particular, B_1 and B are not conjugate since they have a different number of critical values. Moreover, one can check that the group $\mathrm{Aut}(B)$ is trivial while $\mathrm{Aut}(B_1)$ is a cyclic group of order three generated by μ.

It is easy to see that $\mathrm{Mon}(B)$ has a unique imprimitivity system $\{1, 3\}, \{2, 4\}$, corresponding to the decomposition $B = V \circ U$ while $\mathrm{Mon}(B_1)$ has three imprimitivity systems

$$\{1, 3\}, \{2, 4\}, \quad \{1, 2\}, \{3, 4\}, \quad \{1, 4\}, \{2, 3\},$$

corresponding to the decompositions

$$B_1 = U \circ V, \quad B_1 = (\mu^{-1} \circ U) \circ (V \circ \mu), \quad B_1 = (\mu^{-2} \circ U) \circ (V \circ \mu^2).$$

Summing up, we see that the graph Γ_B has the form shown in Figure 5, where the edges connecting B and B_1 correspond to the solutions

$$B = (V \circ \mu^{i-1}) \circ (\mu^{-(i-1)} \circ U), \quad B_1 = (\mu^{-(i-1)} \circ U) \circ (V \circ \mu^{i-1}), \quad 1 \le i \le 3,$$

of system (17), the loops attached to B_1 correspond to the solutions

$$B_1 = (\mu^{-(i-1)} \circ B_1) \circ \mu^{i-1} = \mu^{i-1} \circ (\mu^{-(i-1)} \circ B_1), \quad 1 \le i \le 3,$$

and the loop attached to B corresponds to the solution (22).

The fundamental group of Γ_B can be easily calculated by the well-known method using the spanning tree (see e.g. [**14**, §4.1.2]). Namely, choosing a fixed orientation on each of edges of Γ_B as shown in Figure 6, and considering the edge l_1 together with vertices B and

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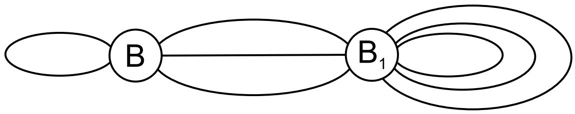

FIGURE 5. The form of Γ_B in the Ritt example.

B_1 as the spanning tree, we see that $\pi_1(\Gamma_B, B)$ is a free group of rank six generated by the paths

$$c, \quad l_1^{-1}l_i, \quad 2 \le i \le 3, \quad l_1^{-1}e_jl_1, \quad 1 \le j \le 3,$$

implying that the group G_B is generated by the images of these paths under the map Φ_B. Assuming that

$$\mathcal{F}(c) = z, \quad \mathcal{F}(e_i) = \mu^{i-1}, \quad 1 \le i \le 3,$$

we obtain

$$\mathcal{F}(l_1^{-1}l_i) = V \circ \mu^{-(i-1)} \circ U, \quad 2 \le i \le 3, \quad \mathcal{F}(l_1^{-1}e_jl_1) = V \circ \mu^{j-1} \circ U, \quad 1 \le j \le 3,$$

implying that the images of the functions

$$g_0 = z, \quad g_1 = V \circ \mu \circ U, \quad g_2 = V \circ \mu^2 \circ U \tag{40}$$

in the group G_B generate G_B. Since

$$\deg g_1 = \deg g_2 = \deg B, \tag{41}$$

and

$$g_1 \ne B, \quad g_2 \ne B, \quad g_1 \ne g_2,$$

it follows from Lemma 3.1 that g_1, g_2, g_3 represent different classes in $C_B/\underset{B}{\sim}$, so that G_B has at least three elements. On the other hand, we have

$$g_1^{\circ 2} = g_2 \circ B, \quad g_2^{\circ 2} = g_1 \circ B, \quad g_1^{\circ 3} = g_2^{\circ 3} = B^{\circ 3}, \quad g_1 \circ g_2 = g_2 \circ g_1 = B^{\circ 2}.$$

Therefore, $G_B = \mathbb{Z}/3\mathbb{Z}$.

In turn, the set C_B can be described as follows: $X \in C_B$ if and only if

$$X = B^{\circ j}, \quad j \ge 0,$$

$$X = V \circ \mu \circ U \circ B^{\circ j}, \quad j \ge 0,$$

or

$$X = V \circ \mu^2 \circ U \circ B^{\circ j}, \quad j \ge 0.$$

Indeed, by Lemma 3.1, it is enough to check that the functions (40) are not rational functions in B. Assume say that $g_1 = R \circ B$. Then it follows from (41) that R is a Möbius transformation. Moreover, $R \in \operatorname{Aut}(B)$ by Lemma 2.1. However, since $\operatorname{Aut}(B)$ is trivial and $g_1 \ne B$, this is impossible.

Note that since $G_B \cong G_{B_1}$ by Theorem 5.3 and $\operatorname{Aut}_G(B_1) = \mathbb{Z}/3\mathbb{Z}$, we have

$$G_{B_1} = \operatorname{Aut}_G(B_1) = \mathbb{Z}/3\mathbb{Z}.$$

Note also that since $G_B \cong G_{B_1}$, the non-triviality of $\operatorname{Aut}(B_1)$ already implies the non-triviality of G_B. Moreover, since B has no automorphisms, we can conclude that the set C_B contains functions of degree greater than one that are not iterates of B.

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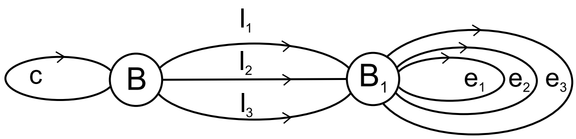

FIGURE 6. The form of Γ_B in the Ritt example with oriented edges.

6.4. *The group* G_B *for* $B = -2z^2/(z^4+1)$. Since equality (25) implies that the function

$$W = \frac{z^2-1}{z^2+1}$$

commutes with B, the group G_B clearly contains a cyclic group of order two generated by $\boldsymbol{W}$. Moreover, it is easy to see that in fact $G_B = \mathbb{Z}/2\mathbb{Z}$. Indeed, providing edges of the graph Γ_B with orientations shown in Figure 7, we see that $\pi_1(\Gamma_B, B)$ is a free group of rank four with generators

$$c, \quad t, \quad l_i^{-1}e_il_i, \quad i = 1, 2,$$

and assuming that

$$\mathcal{F}(c) = \mathcal{F}(e_1) = \mathcal{F}(e_2) = z, \quad \mathcal{F}(t) = W,$$

we see that G_B is generated by the $\boldsymbol{W}$. Similarly, one can conclude that G_{B_1} is generated by $\boldsymbol{X}$, where

$$X = \mathcal{F}(l_1tl_1^{-1}) = \frac{z^2+1}{z} \circ \frac{z^2-1}{z^2+1} \circ -\frac{2}{z^2-2}.$$

The above functions B_1 and X provide an example of commuting rational functions similar to that constructed by Ritt. Namely, set

$$V = \frac{z^2+1}{z}, \quad U = -\frac{2}{z^2-2}.$$

Then W commutes with $U \circ V = W^{\circ 2}$, but $W \notin \widehat{C}_V$. Indeed, assume the inverse, and let S be the rational function defined by any of the sides of the equality

$$\frac{z^2+1}{z} \circ \frac{z^2-1}{z^2+1} = R \circ \frac{z^2+1}{z}, \tag{42}$$

where $R \in \mathbb{C}(z)$. Then substituting z by $1/z$ in the right-hand side of (42), we obtain that $S \circ 1/z = S$. However, substituting z by $1/z$ in the left-hand side, we obtain

$$S \circ \frac{1}{z} = \frac{z^2+1}{z} \circ -\frac{z^2-1}{z^2+1} = -S.$$

The contradiction obtained shows that $W \notin \widehat{C}_V$. Therefore, by Lemma 6.8, the rational function

$$X = V \circ W \circ U$$

commutes with $B_1 = V \circ U$, but is not a rational function in B_1. Note that in distinction with the Ritt example, the non-triviality of G_{B_1} is explained by the existence in the class $[B_1]$ of a function that is an iterate.

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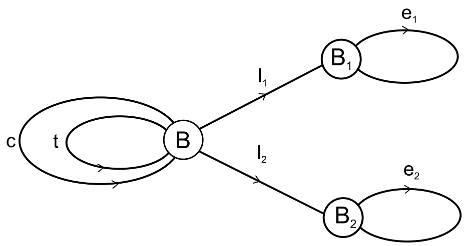

FIGURE 7. The form of Γ_B in Example 3 with oriented edges.

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