TAME RATIONAL FUNCTIONS: DECOMPOSITIONS OF ITERATES AND ORBIT INTERSECTIONS

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Abstract. Let A be a rational function of degree at least two on the Riemann sphere. We say that A is tame if the algebraic curve $A(x) - A(y) = 0$ has no factors of genus zero or one distinct from the diagonal. In this paper, we show that if tame rational functions A and B have orbits with infinite intersection, then A and B have a common iterate. We also show that for a tame rational function A decompositions of its iterates $A^{\circ d}$, $d \geq 1$, into compositions of rational functions can be obtained from decompositions of a single iterate $A^{\circ N}$ for N big enough.

1. Introduction

Let A be a rational function of degree at least two on the Riemann sphere. For a point $z_1 \in \mathbb{CP}^1$ we denote by $O_A(z_1)$ the forward orbit of A, that is, the set $\{z_1, A(z_1), A^{\circ 2}(z_1), \dots\}$. In this paper, we address the following problem: *given two rational functions A and B of degree at least two, under what conditions do there exist orbits $O_A(z_1)$ and $O_B(z_2)$ having an infinite intersection?* We show that under a mild restriction on A and B this happens if and only if A and B have an iterate in common, that is, if and only if

$$A^{\circ k} = B^{\circ \ell} \tag{1}$$

for some $k, l \geq 1$. Put another way, unless rational functions A and B have the same global dynamics, an orbit of A may intersect an orbit of B at most at finitely many places.

In the particular case where A and B are *polynomials*, the problem under consideration was completely settled in the papers [8], [9], where it was shown that the above condition on orbits is equivalent to condition (1). An essential ingredient of the proof was a result of the paper [32], concerning functional decompositions of iterates of polynomials, which can be described as follows. Let

$$A^{\circ d} = X \circ Y \tag{2}$$

be a decomposition of an iterate $A^{\circ d}$ of a rational function A into a composition of rational functions X and Y. We say that this decomposition is *induced* by a decomposition $A^{\circ d'} = X' \circ Y'$, where $d' < d$, if there exist $k_1, k_2 \geq 0$ such that

$$X = A^{\circ k_1} \circ X', \qquad Y = Y' \circ A^{\circ k_2}.$$

In general, decompositions of $A^{\circ d}$ are not exhausted by decompositions induced by decompositions of smaller iterates. However, the main result of [32] states that if A is a polynomial of degree $n \geq 2$ not conjugate to z^n or to $\pm T_n$, where T_n stands for the Chebyshev polynomial, then there exists an integer $N \geq 1$ such that every

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decomposition of $A^{\circ d}$ with $d \geq N$ is induced by a decomposition of $A^{\circ N}$. Moreover, the number N depends on n only.

It seems highly likely that the result of [8], [9] about orbits intersections of polynomials remains true for all rational functions, while the result of [32] about decompositions of iterates of polynomials not conjugate to z^n or to $\pm T_n$ remains true for all *non-special* rational functions, where by a special function we mean a rational function A that is either a Lattès map or is conjugate to $z^{\pm n}$ or $\pm T_n$. However, the approach of the papers [32], [8], [9] cannot be extended to the general case, since it crucially depends on results of the Ritt theory of functional decompositions of polynomials ([26]), some of which have no analogues in the rational case while other are known not to be true. The result of the paper [32] was proved by a different method in the paper [16]. Nevertheless, the method of [16] does not extend to rational functions either.

A partial generalization of the result of [32] to rational functions was obtained in the paper [25]. Namely, it was shown in [25] that there exists a function with integer arguments $N = N(n, l)$ such that for every rational function A of degree $n \geq 2$ decompositions (2) with $\deg X \leq l$ and $d \geq N$ are induced by decompositions of $A^{\circ N}$. Other related results in the rational case were obtained in the papers [2], [3]. Specifically, it was shown in [2] that decompositions of iterates of a rational function A correspond to equivalence classes of certain analytic spaces defined in dynamical terms. On the other hand, in [3], an analogue of the problem about orbits was considered for *semigroups* of rational functions, and obtained results were formulated in terms of the *amenability* of corresponding semigroups. Giving a new look at the considered problems, the papers [2], [3], however, do not provide handy conditions on rational functions A and B under which the results of [32], [8], [9] remain true.

To formulate our results explicitly, we introduce the following definition. Let A be a rational function of degree at least two. We say that A is *tame* if the algebraic curve

$$A(x) - A(y) = 0$$

has no factors of genus zero or one distinct from the diagonal. Otherwise, we say that A is *wild*. By the Picard theorem, the condition that A is tame is equivalent to the condition that for any functions f and g meromorphic on $\mathbb{C}$ the equality

$$A \circ f = A \circ g \tag{3}$$

implies that $f \equiv g$. The problem of describing tame rational functions appears in holomorphic dynamics (see [10]). It is also closely related to the problem of describing rational functions sharing the measure of maximal entropy ([31], [22]).

It is easy to see that every rational function of degree two is wild. Consequently, a tame rational function has degree at least three. On the other hand, a *generic* rational function of degree at least four is tame. Specifically, a rational function of degree at least four is tame whenever it has only simple critical values ([15]). A comprehensive classification of wild rational functions is not known. The most complete result in this direction, obtained in the paper [1], is the classification of solutions of equation (3) under the assumption that A is a polynomial and f, g are rational functions. For an account of recent progress in the general case we refer the reader to [29].

Our first main result is a generalization of the result of [32] to tame rational functions.

Theorem 1.1. *Let A be a tame rational function of degree n. Then there exists an integer N, depending on n only, such that every decomposition of $A^{\circ d}$ with $d \geq N$ is induced by a decomposition of $A^{\circ N}$.*

Our second main result is a similar generalization of the result of [8], [9].

Theorem 1.2. *Let A and B be tame rational functions such that an orbit of A has an infinite intersection with an orbit of B. Then A and B have a common iterate.*

Our proof of Theorem 1.1 is based on the result of [25] about decompositions of iterates cited above and the following statement of independent interest, providing lower bounds for genera of irreducible components of algebraic curves of the form

$$C_{A,B} : \, A(x) - B(y) = 0,$$

where A and B are rational functions.

Theorem 1.3. *Let A be a tame rational function of degree n, B a rational function of degree m, and C an irreducible component of the curve $C_{A,B}$. Then*

$$g(C) \geq \frac{m/n! - 84n + 168}{84}, \tag{4}$$

unless $B = A \circ S$ for some rational function S, and C is the graph $x - S(y) = 0$.

Since equality (2) implies that the curve $C_{A,X}$ has a factor of genus zero, it follows from Theorem 1.3 that if $\deg X$ is big enough, then $X = A \circ S$ for some $S \in \mathbb{C}(z)$, and the further analysis combined with the result of [25] permits to prove Theorem 1.1.

In turn, the proof of Theorem 1.2 goes as follows. First, using the theorem of Faltings, we conclude that if $O_A(z_1) \cap O_B(z_2)$ is infinite, then for every pair $(d, i) \in \mathbb{N} \times \mathbb{N}$ the algebraic curve

$$A^{\circ d}(x) - B^{\circ i}(y) = 0 \tag{5}$$

has a factor of genus zero or one. Then, using Theorem 1.3, we prove that each iterate of B is a *compositional left factor* of some iterate of A, where by a compositional left factor of a rational function f we mean any rational function g such that $f = g \circ h$ for some rational function h. Finally, we deduce Theorem 1.2 from the following result of independent interest.

Theorem 1.4. *Let A and B be tame rational functions. Then the following conditions are equivalent.*

1) *Each iterate of B is a compositional left factor of some iterate of A.*
2) *Each iterate of B is a compositional right factor of some iterate of A.*
3) *The functions A and B have a common iterate.*

In addition to Theorem 1.2, we prove two other results supporting the conjecture that existence of orbits with an infinite intersection is equivalent to condition (1). The first result states that for arbitrary rational functions A and B the existence of such orbits imposes strong restrictions on their degrees consistent with condition (1). Specifically, letting $\mathcal{P}(n)$ denote the set of prime divisors of a natural number n, we prove the following statement.

Theorem 1.5. *Let A and B be rational functions of degree at least two such that an orbit of A has an infinite intersection with an orbit of B. Then the set $\mathcal{P}(\deg A)$ coincides with the set $\mathcal{P}(\deg B)$.*

The second result states that special rational functions, which are the simplest examples of wild rational functions and for which Theorem 1.1 is not true, cannot serve as counterexamples to Theorem 1.2.

Theorem 1.6. *Let A and B be rational functions of degree at least two such that an orbit of A has an infinite intersection with an orbit of B. Assume that at least one of these functions is special. Then A and B have a common iterate.*

Besides the above results, we give new proofs of the main results of the papers [32], [8], [9], using instead of the Ritt theory results of the papers [19], [20] and the classification of commuting polynomials.

The rest of the paper is organized as follows. In the second section, we discuss tame and wild rational functions, and provide a sufficient condition for a rational function to be wild. In the third section, we prove Theorem 1.3. In the fourth section, we prove Theorem 1.1, Theorem 1.2, and Theorem 1.4. In the fifth section, we deduce Theorem 1.5 and Theorem 1.6 from results of the paper [20]. Specifically, we use a description of pairs of rational functions A and U such that for every $d \geq 1$ the algebraic curve

$$A^{\circ d}(x) - U(y) = 0 \tag{6}$$

has a factor of genus zero or one. Finally, in the sixth section, we reconsider the polynomial case and give new proofs of the main results of the papers [32], [8], [9].

2. Tameness and normalization

Let $f : S \to \mathbb{CP}^1$ be a holomorphic function on a compact Riemann surface S. Let us recall that the *normalization* of f is defined as a holomorphic function of the lowest possible degree between compact Riemann surfaces $\widetilde{f} : \widetilde{S}_f \to \mathbb{CP}^1$ such that $\widetilde{f}$ is a Galois covering and

$$\widetilde{f} = f \circ h \tag{7}$$

for some holomorphic map $h : \widetilde{S}_f \to S$. Equivalently, $\widetilde{f}$ can be defined as a Galois covering $\widetilde{f} : \widetilde{S}_f \to \mathbb{CP}^1$ of the form (7) such that

$$\deg \widetilde{f} = |Mon(f)|, \tag{8}$$

where $Mon(f)$ is the monodromy group of f (see e.g. [7], Proposition 2.72). We will denote by $\Sigma(f)$ the subgroup of $Aut(S)$ consisting of automorphisms σ satisfying the condition $f \circ \sigma = f$.

Theorem 2.1. *Let A be a rational function of degree at least two. Assume that there exist a compact Riemann surface S of genus zero or one, a holomorphic function $U : S \to \mathbb{CP}^1$, and a Galois covering $\Psi : S \to \mathbb{CP}^1$ such that $A \circ U$ is a rational function in Ψ, but U is not a rational function in Ψ. Then A is wild.*

Proof. Since conditions of the theorem imply that

$$A \circ U = A \circ (U \circ \alpha)$$

for every $\alpha \in \Sigma(\Psi)$, to prove that the algebraic curve

$$C_A : \frac{A(x) - A(y)}{x - y} = 0 \tag{9}$$

has a factor of genus zero or one, it is enough to show that there exists $\alpha \in \Sigma(\Psi)$ such that $U \circ \alpha \not\equiv U$. Assume in contrary that $U \circ \alpha \equiv U$ for any $\alpha \in \Sigma(\Psi)$. Since the equality $\Psi(x) = \Psi(y)$ holds for $x, y \in S$ if and only if $y = \sigma(x)$ for some $\sigma \in \Sigma(\Psi)$, in this case the algebraic function $S = U \circ \Psi^{-1}$ is single-valued and therefore rational. Thus, $U = S \circ \Psi$, in contradiction with the assumption. $\square$

Remark 2.2. We do not know whether *all* wild rational functions A can be obtained in the way described in Theorem 2.1. Nevertheless, the result of [22] (Theorem 3.1) implies that this is true if the curve C_A is irreducible. Moreover, in this case we can assume that Ψ has degree two.

Corollary 2.3. *Let A be a rational function of degree at least two. Assume that there exist a compact Riemann surface R and holomorphic functions $X : R \to \mathbb{CP}^1$, $Y : R \to \mathbb{CP}^1$, $B : \mathbb{CP}^1 \to \mathbb{CP}^1$ such that:*

1) The diagram

$$\begin{array}{ccc} R & \xrightarrow{Y} & \mathbb{CP}^1 \\ \big\downarrow{\scriptstyle X} & & \big\downarrow{\scriptstyle B} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

commutes,

2) The function X is not a rational function in Y,

3) For the normalization $\widetilde{Y} : \widetilde{S}_Y \to \mathbb{CP}^1$ the inequality $g(\widetilde{S}_Y) \le 1$ holds.

Then A is wild.

Proof. Let $H : \widetilde{S}_Y \to R$ be a holomorphic map such that $\widetilde{Y} = Y \circ H$. Then

$$A \circ (X \circ H) = B \circ \widetilde{Y}.$$

On the other hand, $X \circ H$ is not a rational function in $\widetilde{Y}$ for otherwise X would be a rational function in Y. Thus, conditions of Theorem 2.1 are satisfied for $S = \widetilde{S}_Y$, $U = X \circ H$, and $\Psi = \widetilde{Y}$. $\square$

Let $f : R_1 \to R_2$ be a holomorphic map between Riemann surfaces. We say that a holomorphic map $h : R_1 \to R'$ is *a compositional right factor* of f, if $f = g \circ h$ for some holomorphic map $g : R' \to R_2$. Compositional left factors are defined similarly.

Corollary 2.4. *Every rational function A that has a compositional right factor Y of degree at least two with $g(\widetilde{S}_Y) \le 1$ is wild. In particular, a rational function A of degree at least two is wild whenever $g(\widetilde{S}_A) \le 1$.*

Proof. Let B be a rational function such that $A = B \circ Y$. Then the conditions of Corollary 2.3 are satisfied for B, Y, and $X = z$. $\square$

Notice that rational functions A with $g(\widetilde{S}_A) = 0$ can be listed explicitly as compositional left factors of rational Galois coverings. On the other hand, functions with $g(\widetilde{S}_A) = 1$ admit a simple geometric description (see [18]).

Corollary 2.5. *Any special rational function is wild.*

Proof. The function $z^{\pm n}$ itself is a Galois covering. On the other hand, $\pm T_n$ is a compositional left factor of the Galois covering $z^n + \frac{1}{z^n}$, implying that $g(\widetilde{S}_{\pm T_n}) = 0$. Finally, every Lattès map A satisfies $g(\widetilde{S}_A) \leq 1$ (see [18]). $\square$

For a holomorphic function $f : S \to \mathbb{CP}^1$ the condition $g(\widetilde{S}_f) \leq 1$ can be expressed merely in terms of the ramification of f. The easiest way to formulate the corresponding criterion it is to use the notion of *Riemann surface orbifold* (see e.g. [20], Section 2.1 for basic definitions). Specifically, with each holomorphic function $f : S \to \mathbb{CP}^1$ one can associate in a natural way two orbifolds $\mathcal{O}_1^f = (S, \nu_1^f)$ and $\mathcal{O}_2^f = (\mathbb{CP}^1, \nu_2^f)$, setting $\nu_2^f(z)$ equal to the least common multiple of local degrees of f at the points of the preimage $f^{-1}\{z\}$, and

$$\nu_1^f(z) = \frac{\nu_2^f(f(z))}{\deg_z f}.$$

In these terms, the following statement holds.

Lemma 2.6. *Let S be a compact Riemann surface and $f : S \to \mathbb{CP}^1$ a holomorphic function. Then $g(\widetilde{S}_f) = 0$ if and only if $\chi(\mathcal{O}_2^f) > 0$, and $g(\widetilde{S}_f) = 1$ if and only if $\chi(\mathcal{O}_2^f) = 0$.*

Proof. In the case $S = \mathbb{CP}^1$ the proof can be found in [18] (Lemma 2.1), and this proof carries over verbatim to the case of arbitrary compact Riemann surface S. $\square$

By Corollary 2.4, any rational function A with $g(\widetilde{S}_A) \leq 1$ gives rise to the family of wild rational functions $f \circ A$, $f \in \mathbb{C}(z)$. However, other examples of wild rational functions also exist.

Example 2.7. Let us consider the family of polynomials

$$A_{l,m} = z^l (z+1)^m,$$

where l, m are coprime and $l + m \geq 3$, found in [1]. It was shown in [1] that the corresponding curve $C_{A_{l,m}}$ defined by (9) is irreducible and has the parametrization $z \to (X(z), Z(z))$, where

$$X = \frac{1 - z^l}{z^{l+m} - 1}, \qquad Z = z^m X.$$

Moreover, $A_{l,m}$ is an *indecomposable* rational function, that is, $A_{l,m}$ has no decompositions into a composition of rational functions of degree at least two. Thus, any compositional right factor of $A_{l,m}$ of degree at least two has the form $\mu \circ A_{l,m}$ for some $\mu \in Aut(\mathbb{CP}^1)$. On the other hand, it is easy to see that if $l + m > 4$, then $\chi\left(\mathcal{O}_2^{\mu \circ A_{l,m}}\right) < 0$, implying that $g(\widetilde{S}_f) > 1$. Indeed, $A_{l,m}$ has three critical values $\infty, 0, \frac{(-l)^l m^m}{(l+m)^m}$, and the signature of the orbifold $\mathcal{O}_2^{\mu \circ A_{l,m}}$ is $(l + m, \mathrm{lcm}(l, m), 2)$. Thus, for $l + m > 4$, we have:

$$\begin{aligned} \chi\left(\mathcal{O}_2^{\mu \circ A_{l,m}}\right) &= 2 + \left(\frac{1}{l+m} - 1\right) + \left(\frac{1}{\mathrm{lcm}(l,m)} - 1\right) + \left(\frac{1}{2} - 1\right) = \\ &= -\frac{1}{2} + \frac{1}{l+m} + \frac{1}{\mathrm{lcm}(l,m)} < -\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 0. \end{aligned}$$

Let us notice however that although the family $A_{l,m}$ for $l+m>4$ does not satisfy the condition of Corollary 2.4, it *does* satisfy the conditions of Theorem 2.1. Indeed, one can check that $Z = X \circ \frac{1}{z}$, implying that the function

$$A_{l,m} \circ X = A_{l,m} \circ Z$$

is invariant with respect to the transformation $z \to \frac{1}{z}$. Therefore,

$$A_{l,m} \circ X = B \circ \left(z + \frac{1}{z}\right)$$

for some rational function B and the Galois covering $Y = z + \frac{1}{z}$. On the other hand, X is not a rational function in Y, since X is not invariant with respect to $z \to \frac{1}{z}$.

3. Bounds for genera of components of $A(x) - B(y) = 0$

3.1. **Fiber products.** Let $f : C_1 \to C$ and $g : C_2 \to C$ be holomorphic maps between compact Riemann surfaces. The collection

$$(C_1, f) \times_C (C_2, g) = \bigcup_{j=1}^{n(f,g)} \{R_j, p_j, q_j\},$$

where $n(f,g)$ is an integer positive number and R_j are compact Riemann surfaces provided with holomorphic maps

$$p_j : R_j \to C_1, \quad q_j : R_j \to C_2, \quad 1 \le j \le n(f,g),$$

is called the *fiber product* of f and g if

$$f \circ p_j = g \circ q_j, \quad 1 \le j \le n(f,g),$$

and for any holomorphic maps $p : R \to C_1$, $q : R \to C_2$ between compact Riemann surfaces satisfying

$$f \circ p = g \circ q$$

there exist a uniquely defined index j and a holomorphic map $w : R \to R_j$ such that

$$p = p_j \circ w, \quad q = q_j \circ w.$$

The fiber product exists and is defined in a unique way up to natural isomorphisms.

Notice that the universality property yields that the holomorphic maps p_j and q_j, $1 \le j \le n(f,g)$, have no *non-trivial compositional common right factor* in the following sense: the equalities

$$p_j = \widetilde{p} \circ w, \quad q_j = \widetilde{q} \circ w,$$

where

$$w : R_j \to \widetilde{R}, \quad \widetilde{p} : \widetilde{R} \to C_1, \quad \widetilde{q} : \widetilde{R} \to C_2$$

are holomorphic maps between compact Riemann surface, imply that $\deg w = 1$. In particular, this implies that

$$\deg q_j \le \deg f, \quad \deg p_j \le \deg g, \quad 1 \le j \le n(f,g).$$

Another corollary is that p_j, $1 \le j \le n(f,g)$, is a rational function in q_j if and only if $\deg q_j = 1$.

In practical terms, the fiber product is described by the following algebro-geometric construction. Let us consider the algebraic curve

$$L = \{(x, y) \in C_1 \times C_2 \mid f(x) = g(y)\}.$$

Let us denote by L_j, $1 \le j \le n(f, g)$, irreducible components of L, by R_j, $1 \le j \le n(f, g)$, their desingularizations, and by

$$\pi_j : R_j \to L_j, \quad 1 \le j \le n(f, g),$$

the desingularization maps. Then the compositions

$$x \circ \pi_j : L_j \to C_1, \quad y \circ \pi_j : L_j \to C_2, \quad 1 \le j \le n(f, g),$$

extend to holomorphic maps

$$p_j : R_j \to C_1, \quad q_j : R_j \to C_2, \quad 1 \le j \le n(f, g),$$

and the collection $\bigcup_{j=1}^{n(f,g)} \{R_j, p_j, q_j\}$ is the fiber product of f and g. Abusing notation we call the Riemann surfaces R_j, $1 \le j \le n(f, g)$, irreducible components of the fiber product of f and g.

Below we will use the following results, describing the fiber product of maps f and $g \circ u$ through the fiber product of maps f and g (see [20], Theorem 2.8 and Corollary 2.9). For better understanding, see diagram (10).

Theorem 3.1. *Let $f : C_1 \to C$, $g : C_2 \to C$, and $u : C_3 \to C_2$ be holomorphic maps between compact Riemann surfaces. Assume that*

$$(C_1, f) \times_C (C_2, g) = \bigcup_{j=1}^{n(f,g)} \{R_j, p_j, q_j\}$$

and

$$(R_j, q_j) \times_{C_2} (C_3, u) = \bigcup_{i=1}^{n(u,q_j)} \{R_{ij}, p_{ij}, q_{ij}\}, \qquad 1 \le j \le n(f, g).$$

Then

$$(C_1, f) \times_C (C_3, g \circ u) = \bigcup_{j=1}^{n(f,g)} \bigcup_{i=1}^{n(u,q_j)} \{R_{ij}, p_j \circ p_{ij}, q_{ij}\}. \qquad \square$$

Corollary 3.2. *In the above notation, the fiber products $(C_1, f) \times_C (C_2, g)$ and $(C_1, f) \times_C (C_3, g \circ u)$ have the same number of irreducible components if and only if for every j, $1 \le j \le n(f, g)$, the fiber product $(R_j, q_j) \times_{C_2} (C_3, u)$ has a unique irreducible component.* $\square$

$$\begin{array}{ccccc} R_{ij} & \xrightarrow{p_{ij}} & R_j & \xrightarrow{p_j} & C_1 \\ \downarrow{\scriptstyle q_{ij}} & & \downarrow{\scriptstyle q_j} & & \downarrow{\scriptstyle f} \\ C_3 & \xrightarrow{u} & C_2 & \xrightarrow{g} & C. \end{array} \tag{10}$$

3.2. **Proof of Theorem 1.3.** The proof of Theorem 1.3 uses two results. The first result is the following statement (see [20], Theorem 3.1), generalizing an earlier result from [17].

Theorem 3.3. *Let T, R be compact Riemann surfaces and $W : T \to \mathbb{CP}^1$ a holomorphic map of degree n. Then for any holomorphic map $P : R \to \mathbb{CP}^1$ of degree m such that the fiber product of P and W consists of a unique component E, the inequality*

$$\chi(E) \leq \chi(R)(n-1) - \frac{m}{42} \tag{11}$$

holds, unless $g(\widetilde{S}_W) \leq 1$.[1] □

Since $\chi(E) = 2 - 2g(E)$ and $\chi(R) = 2 - 2g(R) \leq 2$, inequality (11) implies the inequality

$$g(E) \geq \frac{m - 84n + 168}{84}. \tag{12}$$

In particular, Theorem 3.3 implies the following result proved in [17]: if A and B are rational functions of degrees n and m such that $g(\widetilde{S}_A) > 1$ and the curve $C_{A,B}$ is irreducible, then $g(C_{A,B})$ satisfies inequality (12). Theorem 1.3 can be considered as an analogue of the last result for reducible curves $C_{A,B}$, with the condition $g(\widetilde{S}_A) > 1$ replaced by the stronger condition that A is tame.

The second result we need is the following result of Fried (see [6], Proposition 2, or [14], Theorem 3.5).

Theorem 3.4. *Let A and B be rational functions such that $n(A,B) > 1$. Then there exist rational functions A_1, B_1, U, V such that*

$$A = A_1 \circ U, \qquad B = B_1 \circ V,$$

and the equalities $\widetilde{A}_1 = \widetilde{B}_1$ and $n(A,B) = n(A_1, B_1)$ hold. □

Proof of Theorem 1.3. Let E be the desingularization of C, and $\{E, X, Y\}$ the corresponding component of $(\mathbb{CP}^1, A) \times (\mathbb{CP}^1, B)$. Assume first that $n(A,B) = 1$, and hence $C = C_{A,B}$. Since A is tame, $g(\widetilde{S}_A) > 1$, by Corollary 2.4. Therefore, by Theorem 3.3, inequality (12) holds, implying that (4) also holds. Thus, in this case the theorem is true.

Assume now that $n(A,B) > 1$, and let A_1, B_1, U, V be the rational functions provided by Theorem 3.4. By Theorem 3.1, the component $\{E, X, Y\}$ of the fiber product $(\mathbb{CP}^1, A) \times (\mathbb{CP}^1, B)$ factors through some component of $(\mathbb{CP}^1, A) \times (\mathbb{CP}^1, B_1)$, that is, there exist a compact Riemann surface R and holomorphic maps between

[1]In the paper [20], instead of the condition $g(\widetilde{S}_W) \leq 1$ the equivalent condition $\chi(\mathcal{O}_2^W) \geq 0$ is used.

compact Riemann surfaces X_1, F, H such that $X = X_1 \circ H$ and the diagram

$$\begin{tikzcd}
E \arrow[r, "Y"] \arrow[d, "H"] & \mathbb{CP}^1 \arrow[d, "V"] \\
R \arrow[r, "F"] \arrow[d, "X_1"] & \mathbb{CP}^1 \arrow[d, "B_1"] \\
\mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1
\end{tikzcd} \tag{13}$$

commutes. Moreover, the maps X_1 and F have no common non-trivial compositional right factor, and the inequalities

$$\deg X_1 \leq \deg B_1, \qquad \deg F \leq \deg A \tag{14}$$

hold. Finally, since

$$n(A, B) \geq n(A, B_1) \geq n(A_1, B_1),$$

it follows from $n(A, B) = n(A_1, B_1)$ that $n(A, B) = n(A, B_1)$. Therefore, by Corollary 3.2, the equality $n(F, V) = 1$ holds.

Now we consider the cases $g(\widetilde{S}_F) > 1$ and $g(\widetilde{S}_F) \leq 1$ separately. In the first case, applying Theorem 3.3 to the fiber product of F and V, we see that

$$g(E) \geq \frac{\deg V - 84 \deg F + 168}{84}.$$

Since the order of the monodromy group of a rational function A does not exceed the order of the full symmetric group on $n = \deg A$ symbols, it follows from the equalities (8) and $\widetilde{A}_1 = \widetilde{B}_1$ that

$$\deg B_1 \leq \deg \widetilde{B}_1 = \deg \widetilde{A}_1 \leq (\deg A_1)! \leq (\deg A)! = n!,$$

implying that

$$\deg V = \frac{\deg B}{\deg B_1} \geq m/n!.$$

Taking into account the second equality in (14), we conclude that if $g(\widetilde{S}_F) > 1$, then

$$g(E) > \frac{\deg V - 84 \deg F + 168}{84} \geq \frac{m/n! - 84\, n + 168}{84}.$$

Assume now that $g(\widetilde{S}_F) \leq 1$. Since X_1 and F have no common non-trivial compositional right factor, X_1 is not a rational function in F, unless the equality $\deg F = 1$ holds. Therefore, if $\deg F > 1$, we can apply Corollary 2.3 to the bottom square in diagram (13), concluding that A is wild, in contradiction with the assumption. Thus, $\deg F = 1$, implying that $R = \mathbb{CP}^1$ and

$$B = B_1 \circ V = A \circ X_1 \circ F^{-1} \circ V,$$

$$X = X_1 \circ H = X_1 \circ F^{-1} \circ V \circ Y.$$

Thus, if $g(\widetilde{S}_F) \leq 1$, the equalities

$$B = A \circ S, \qquad X = S \circ Y \tag{15}$$

hold for

$$S = X_1 \circ F^{-1} \circ V.$$

Since X and Y have no non-trivial compositional common right factor, the second equality in (15) implies that $\deg Y = 1$ and $E = \mathbb{CP}^1$. Finally, C is the graph

$x - S(y) = 0$. Indeed, C is the image of $\mathbb{CP}^1$ under the map $t \to (X(t), Y(t))$. On the other hand, since $X = S \circ Y$, this image coincides with the image of $\mathbb{CP}^1$ under the map $t \to (S(t), t)$, which is equal to $x - S(y) = 0$. □

Theorem 1.3 implies two important corollaries. The first corollary concerns compositional left factors of iterates of a tame rational function A. We recall that a tame rational function has degree at least three.

Corollary 3.5. *Let A be a tame rational function, and X and Y rational functions such that*

$$A^{\circ s} = X \circ Y \tag{16}$$

for some $s \geq 1$. Then there exists a rational function X_0 such that

$$\deg X_0 \leq 84(\deg A - 2)(\deg A)!$$

and the equalities

$$X = A^{\circ \ell} \circ X_0, \qquad A^{\circ (s-\ell)} = X_0 \circ Y$$

hold for some $l \geq 1$.

Proof. Equality (16) implies that the curve $C_{A,X}$ has a factor of genus zero C, parametrized by the map

$$t \to (A^{\circ (s-1)}(t), Y(t)). \tag{17}$$

On the other hand, if

$$\deg X > 84(\deg A - 2)(\deg A)!,$$

then

$$\frac{(\deg X)/(\deg A)! - 84 \deg A + 168}{84} > \frac{84(\deg A - 2) - 84 \deg A + 168}{84} = 0,$$

implying by Theorem 1.3 that $X = A \circ X'$ and C is the graph $x - X'(y) = 0$ for some rational function X'. Since C is parametrized by the map (17), this implies that

$$A^{\circ (s-1)} = X' \circ Y.$$

Applying this reasoning recursively, we obtain the required statement. □

The second corollary is the following.

Corollary 3.6. *Let A and B be rational functions such that the curve $C_{A^{\circ s}, B}$, $s \geq 1$, has an irreducible factor C of genus zero or one. Assume in addition that B is tame,* $\deg A \geq 2$, *and*

$$s > \log_2 \left[84(\deg B - 1)(\deg B)!\right]. \tag{18}$$

Then $A^{\circ s} = B \circ Q$ for some rational function Q, and C is the graph $Q(x) - y = 0$.

Proof. Inequality (18) implies that

$$\deg A^{\circ s} = (\deg A)^s \geq 2^s > 84(\deg B - 1)(\deg B)!.$$

Thus,

$$\frac{(\deg A^{\circ s})/(\deg B)! - 84 \deg B + 168}{84} > \frac{84(\deg B - 1) - 84 \deg B + 168}{84} = 1,$$

and the corollary follows from Theorem 1.3. □

4. Proofs of Theorem 1.1, Theorem 1.2, and Theorem 1.4

Theorem 1.1 follows from Theorem 1.3 combined with the following result proved in [25].

Theorem 4.1. *There exists a function $\varphi : \mathbb{N}\times\mathbb{N}\to\mathbb{R}$ with the following property. For any rational functions A and X such that the equality*

$$A^{\circ d} = X\circ R \tag{19}$$

holds for some rational function R and $d\geq 1$, there exists $N\leq \varphi(\deg A, \deg X)$ and a rational function R' such that

$$A^{\circ N} = X\circ R'$$

and $R = R'\circ A^{\circ(d-N)}$, if $d>N$. □

Proof of Theorem 1.1. By Corollary 3.5, for any decomposition

$$A^{\circ d} = X\circ Y \tag{20}$$

we can find X' and $l\geq 0$ such that

$$\deg X' \leq 84(n-2)n! \tag{21}$$

and the equalities

$$X = A^{\circ \ell}\circ X'$$

and

$$A^{\circ(d-\ell)} = X'\circ Y \tag{22}$$

hold. On the other hand, it follows from Theorem 4.1 that there exists N, which depends on n only, such that for any decomposition (22) with $d-l>N$ satisfying (21), there exists a rational function Y' such that

$$A^{\circ N} = X'\circ Y', \qquad Y = Y'\circ A^{\circ(d-\ell-N)}.$$

The above implies that any decomposition of $A^{\circ d}$ with $d\geq N$ is induced by a decomposition of $A^{\circ N}$. Indeed, if $d-l\leq N$, then decomposition (20) is induced by the decomposition

$$A^{\circ N} = (A^{\circ(N-d+\ell)}\circ X')\circ Y,$$

while if $d-l>N$, it is induced by the decomposition $A^{\circ N} = X'\circ Y'$. □

Let F be a rational function of degree at least two. We define $G(F)$ as the group of Möbius transformations σ such that

$$F\circ\sigma = \nu_\sigma\circ F$$

for some Möbius transformations ν_σ. Below we need the following result (see [21], Theorem 4.2).

Theorem 4.2. *Let F be a rational function of degree $d\geq 2$. Then the group $G(F)$ is one of the five finite rotation groups of the sphere A_4, S_4, A_5, C_n, D_{2n}, unless $F=\theta_1\circ z^d\circ\theta_2$ for some Möbius transformations θ_1 and θ_2.* □

Proof of Theorem 1.4. We recall that functional decompositions $F=U\circ V$ of a rational function F into compositions of rational functions U and V, considered up to the equivalence

$$U\to U\circ\mu, \qquad V\to\mu^{-1}\circ V, \qquad \mu\in Aut(\mathbb{CP}^1), \tag{23}$$

are in a one-to-one correspondence with imprimitivity systems of the monodromy group of F. In particular, the number of such classes is finite. Therefore, if for every $i \geq 1$ there exist $s_i \geq 1$ and $R_i \in \mathbb{C}(z)$ such that

$$A^{\circ s_i} = B^{\circ i} \circ R_i, \tag{24}$$

then Theorem 1.1 implies that there exist a rational function U and increasing sequences of non-negative integers f_k, $k \geq 0$, and v_k, $k \geq 0$, such that

$$B^{\circ f_k} = A^{\circ v_k} \circ U \circ \eta_k, \qquad k \geq 0,$$

for some $\eta_k \in Aut(\mathbb{CP}^1)$. In turn, this implies that there exists an increasing sequence of non-negative integers r_k, $k \geq 1$, such that

$$B^{\circ f_k} = A^{\circ r_k} \circ B^{\circ f_0} \circ \mu_k, \qquad k \geq 1, \tag{25}$$

for some $\mu_k \in Aut(\mathbb{CP}^1)$. Furthermore, since (25) implies that for every $k \geq 1$ the function $B^{\circ f_0} \circ \mu_k$ is a compositional right factor of an iterate of B, there exist a rational function V and an increasing sequence of non-negative integers k_l, $l \geq 0$, such that

$$B^{\circ f_0} \circ \mu_{k_l} = \theta_l \circ V, \qquad l \geq 0,$$

for some $\theta_l \in Aut(\mathbb{CP}^1)$, implying that

$$B^{\circ f_0} \circ \mu_{k_l} = \delta_l \circ B^{\circ f_0} \circ \mu_{k_0}, \qquad l \geq 1,$$

for some $\delta_l \in Aut(\mathbb{CP}^1)$.

Clearly, the Möbius transformations $\mu_{k_l} \circ \mu_{k_0}^{-1}$, $l \geq 1$, belong to the group $G(B^{\circ f_0})$. On the other hand, since the function B is tame, the function $B^{\circ f_0}$ is also tame and hence, by Corollary 2.4, it is not of the form $B^{\circ f_0} = \theta_1 \circ z^d \circ \theta_2$, where $\theta_1, \theta_2 \in Aut(\mathbb{CP}^1)$. Therefore, by Theorem 4.2,

$$\mu_{k_{l_2}} \circ \mu_{k_0}^{-1} = \mu_{k_{l_1}} \circ \mu_{k_0}^{-1}$$

for some $l_2 > l_1$, implying that $\mu_{k_{l_2}} = \mu_{k_{l_1}}$. It follows now from (25) that

$$B^{\circ f_{k_{l_2}}} = A^{\circ (r_{k_{l_2}} - r_{k_{l_1}})} \circ B^{\circ f_{k_{l_1}}},$$

implying that

$$B^{\circ (f_{k_{l_2}} - f_{k_{l_1}})} = A^{\circ (r_{k_{l_2}} - r_{k_{l_1}})}. \tag{26}$$

Since $l_2 > l_1$ and the sequences k_l, $l \geq 1$, and f_k, $k \geq 1$, are increasing, the inequality $f_{k_{l_2}} > f_{k_{l_1}}$ holds, and therefore A and B have a common iterate. This proves the implication 1) $\Rightarrow$ 3).

Similarly, if for every $i \geq 1$ there exist $s_i \geq 1$ and $R_i \in \mathbb{C}(z)$ such that

$$A^{\circ s_i} = R_i \circ B^{\circ i},$$

we conclude that there exist increasing sequences f_k, $k \geq 0$, and r_k, $k \geq 1$, such that

$$B^{\circ f_k} = \mu_k \circ B^{\circ f_0} \circ A^{\circ r_k}, \qquad k \geq 1, \tag{27}$$

for some $\mu_k \in Aut(\mathbb{CP}^1)$. Moreover, there exists an increasing sequence k_l, $l \geq 0$, such that

$$\mu_{k_l} \circ B^{\circ f_0} = \mu_{k_0} \circ B^{\circ f_0} \circ \delta_l, \qquad l \geq 1,$$

for some $\delta_l \in Aut(\mathbb{CP}^1)$. Finally, for some $l_2 > l_1$ the equality $\delta_{l_2} = \delta_{l_1}$ holds, implying that $\mu_{k_{l_2}} = \mu_{k_{l_1}}$. Now (27) implies that

$$B^{\circ f_{k_{l_2}}} = B^{\circ f_{k_{l_1}}} \circ A^{\circ (r_{k_{l_2}} - r_{k_{l_1}})}.$$

Since B is tame, the last equality in turn implies (26). This proves the implication 2) $\Rightarrow$ 3). Finally, it is clear that 3) implies 1) and 2). □

Remark 4.3. Theorem 1.4 is not true for *all* rational functions. For example, it is easy see that for the functions z^6 and z^{12} conditions 1) and 2) are satisfied, while the condition 3) does not. Nevertheless, one can expect that conditions 1) and 3) are equivalent for non-special functions. On the other hand, there exist non-special rational functions for which conditions 2) and 3) are not equivalent. Specifically, using wild rational functions one can construct A and B such that

$$A^{\circ 2} = A \circ B, \tag{28}$$

but A and B have no common iterate (see [31], [22]). Since (28) implies that

$$A^{\circ 2k} = A^{\circ k} \circ B^{\circ k},$$

for such A and B any iterate of B is a compositional right factor of an iterate of A.

Our starting point in the proof of Theorem 1.2, Theorem 1.5, and Theorem 1.6 is the following lemma.

Lemma 4.4. *Let A and B be rational functions of degree at least two such that an orbit of A has an infinite intersection with an orbit of B. Then for every pair $(d, i) \in \mathbb{N} \times \mathbb{N}$ the algebraic curve $A^{\circ d}(x) - B^{\circ i}(y) = 0$ has a factor of genus zero or one.*

Proof. Recall that by the theorem of Faltings ([5]) if an irreducible algebraic curve C defined over a finitely generated field K of characteristic zero has infinitely many K-points, then $g(C) \leq 1$. On the other hand, it is easy to see that if $O_A(z_1) \cap O_B(z_2)$ is infinite, then for every pair $(d, i) \in \mathbb{N} \times \mathbb{N}$ the algebraic curve (5) has infinitely many points $(x, y) \in O_A(z_1) \times O_B(z_2)$. Defining now K as the field generated over $\mathbb{Q}$ by z_1, z_2, and the coefficients of A, B, and observing that the orbits $O_A(z_1)$ and $O_B(z_2)$ belong to K, we conclude that for every pair $(d, i) \in \mathbb{N} \times \mathbb{N}$ curve (5) has a factor of genus zero or one. □

Proof of Theorem 1.2. Since $B^{\circ i}$, $i \geq 1$, is tame whenever B is tame, it follows from Lemma 4.4 and Corollary 3.6 that for every $i \geq 1$ there exist $s_i \geq 1$ and $R_i \in \mathbb{C}(z)$ such that equality (24) holds. Therefore, by Theorem 1.4, A and B have a common iterate. □

5. Proofs of Theorem 1.5 and Theorem 1.6

5.1. **Proof of Theorem 1.5.** We start by recalling the results of the paper [20], describing pairs of rational functions A and U of degree at least two such that for every $d \geq 1$ the algebraic curve (6) has an irreducible factor of genus zero or one. In case A is non-special, the main result of [20] in a slightly simplified form can be formulated as follows (see [20], Theorem 1.2).

Theorem 5.1. *Let A be a non-special rational function of degree at least two. Then there exist a rational Galois covering X_A and a rational function F such that the diagram*

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{F} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle X_A} & & \downarrow{\scriptstyle X_A} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{29}$$

commutes, and for a rational function U of degree at least two the algebraic curve $C_{A^{\circ d},U}$ has a factor of genus zero or one for every $d \geq 1$ if and only if U is a compositional left factor of $A^{\circ \ell} \circ X_A$ for some $l \geq 0$. □

The Galois covering X_A in Theorem 6.4 can be described explicitly (see [20], Theorem 3.4). However, we do not need this more explicit description to prove Theorem 1.5 in the case where both functions A and B are non-special. Indeed, since by Lemma 4.4 for every pair $(d,i) \in \mathbb{N} \times \mathbb{N}$ algebraic curve (5) has a factor of genus zero or one, it follows from Theorem 5.1 that for every $i \geq 1$ there exist $d_i \geq 1$ and $S_i \in \mathbb{C}(z)$ such that the equality

$$A^{\circ d_i} \circ X_A = B^{\circ i} \circ S_i \tag{30}$$

holds. Therefore, if

$$\operatorname{ord}_p(\deg B) > 0 \tag{31}$$

for some prime number p, then for every $i \geq 1$ there exists $d_i \geq 1$ such that

$$d_i \operatorname{ord}_p(\deg A) + \operatorname{ord}_p(\deg X_A) \geq i \operatorname{ord}_p(\deg B),$$

implying that

$$\operatorname{ord}_p(\deg A) > 0. \tag{32}$$

By symmetry, inequality (32) implies in turn inequality (31). Therefore,

$$\mathcal{P}(\deg A) = \mathcal{P}(\deg B).$$

This proves Theorem 1.5 in the case where A and B are non-special. On the other hand, if at least one of the functions A and B is special, then Theorem 1.5 obviously follows from Theorem 1.6 proved below.

5.2. **Proof of Theorem 1.6 for A conjugate to $z^{\pm n}$ or $\pm T_n$.** For $s \geq 1$, we set

$$D_s = \frac{1}{2}\left(z^s + \frac{1}{z^s}\right).$$

To prove Theorem 1.6 in the case where A is conjugate to $z^{\pm n}$ or $\pm T_n$, we use the following result (see [20], Theorem 3.6).

Theorem 5.2. *Let A and U be rational functions of degree at least two.*

(1) *If $A = z^n$, then the algebraic curve $C_{A^{\circ d},U}$ has a factor of genus zero or one for every $d \geq 1$ if and only if $U = z^s \circ \mu$, $s \geq 2$, where μ is a Möbius transformation,*
(2) *If $A = T_n$, then the algebraic curve $C_{A^{\circ d},U}$ has a factor of genus zero or one for every $d \geq 1$ if and only if either $U = \pm T_s \circ \mu$, $s \geq 2$, or $U = D_s \circ \mu$, $s \geq 1$, where μ is a Möbius transformation.* □

Let us prove Theorem 1.6 in the case where A is conjugate to $\pm T_n$. Clearly, without loss of generality we may assume that $A = T_n$, if n is even, or $A = \pm T_n$, if n is odd. Since by Lemma 4.4 for every pair $(d,i) \in \mathbb{N} \times \mathbb{N}$ algebraic curve (5) has a factor of genus zero or one, it follows from the second part of Theorem 5.2 that if $A = T_n$, then for any $i \geq 1$ either

$$B^{\circ i} = \pm T_s \circ \mu, \qquad s \geq 2, \qquad \mu \in Aut(\mathbb{CP}^1), \tag{33}$$

or

$$B^{\circ i} = D_s \circ \mu, \qquad s \geq 1, \qquad \mu \in Aut(\mathbb{CP}^1). \tag{34}$$

The same is true if $A = -T_n$, since we can apply Theorem 5.2 to iterates of $A^{\circ 2}$. Setting $m = \deg B$, we show first that conditions (33), (34) imply the equality $B = \pm T_m$. Since an iterate of a rational function f of degree at least two equals $\pm T_s$ if and only if f equals $\pm T_{s'}$ (see e.g. [23], Lemma 6.3), it is enough to show that $B^{\circ 2} = \pm T_{m^2}$. Therefore, considering only even iterates of B, without loss of generality we may assume that the degree of B in (33), (34) is greater than two, implying that $\deg T_s > 2$ and $\deg D_s > 2$.

Let us observe first that equality (34) is actually impossible for any $i \geq 1$. Indeed, otherwise considering the iterate $B^{\circ 2i}$ we conclude that there exists $\nu \in Aut(\mathbb{CP}^1)$ such that either

$$D_s \circ \mu \circ D_s \circ \mu = \pm T_{4s^2} \circ \nu, \tag{35}$$

or

$$D_s \circ \mu \circ D_s \circ \mu = D_{2s^2} \circ \nu. \tag{36}$$

Equality (35) is impossible since the function in its left part has more than one pole. Moreover, since any decomposition $D_l = U \circ V$ of D_l, up to the equivalency (23), reduces either to the decomposition

$$D_l = D_{l/d} \circ z^d,$$

or to the decomposition

$$D_l = \varepsilon^l T_{l/d} \circ D_d(\varepsilon z),$$

where $d|l$ and $\varepsilon^{2l} = 1$ (see e.g. [18], Section 4.2), it is easy to see comparing the ramification of the functions z^s, $\pm T_s$, and D_s that if $\deg D_s > 2$ equality (36) is impossible either.

Since (34) is impossible, $B = \pm T_m \circ \mu$ for some $\mu \in Aut(\mathbb{CP}^1)$ and

$$(\pm T_m \circ \mu) \circ (\pm T_m \circ \mu) = \pm T_{m^2} \circ \nu \tag{37}$$

for some $\nu \in Aut(\mathbb{CP}^1)$. Furthermore, since finite critical values of Chebyshev polynomials are ± 1, and the local multiplicity of $\pm T_s$ at each of the points in $T_s^{-1}\{-1, 1\}$ distinct from -1 and 1 is two, equality (37) implies by the chain rule that whenever $m > 2$ the equalities $\mu(\infty) = (\infty)$ and $\mu\{-1, 1\} = \{-1, 1\}$ hold. Thus, $\mu = \pm z$ and hence $B = \pm T_m$.

Let now $O_A(z_1)$ and $O_B(z_2)$ be orbits having an infinite intersection. Evidently, without loss of generality we may assume that $z_1 = z_2 = z_0$, and it is clear that $z_0 \neq \infty$. The equalities $A = \pm T_n$ and $B = \pm T_m$ imply that there exist a linear function α_A of the form nz or $nz+1/2$ and a linear function α_B of the form mz or $mz+1/2$ such that the diagrams

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\alpha_A} & \mathbb{C} \\ \downarrow{\scriptstyle \cos 2\pi z} & & \downarrow{\scriptstyle \cos 2\pi z} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}, \qquad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\alpha_B} & \mathbb{C} \\ \downarrow{\scriptstyle \cos 2\pi z} & & \downarrow{\scriptstyle \cos 2\pi z} \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \end{array}$$

commute. If z_0' is a point of $\mathbb{C}$ such that $\cos(2\pi z_0') = z_0$ and $k, l \geq 1$ are integers such that

$$A^{\circ k}(z_0) = B^{\circ \ell}(z_0), \tag{38}$$

then $(\alpha_A^{\circ k} \pm \alpha_B^{\circ \ell})(z_0')$ is an integer. Taking into account the form of α_A and α_B, this implies that either z_0' is a rational number, or $\alpha_A^{\circ k} = \pm\alpha_B^{\circ \ell}$. In the first case, however, z_0' is a preperiodic point both for α_A modulo 1 and for α_B modulo 1, implying that the orbits $O_A(z_1)$ and $O_B(z_2)$ are finite, and therefore cannot have an infinite intersection. Thus, $\alpha_A^{\circ k} = \pm\alpha_B^{\circ \ell}$, implying that $A^{\circ k} = B^{\circ \ell}$. This finishes the proof of Theorem 1.6 in the case where A is conjugate to $\pm T_n$.

In case A is conjugate to $z^{\pm n}$, the proof can be done in a similar way using the first part of Theorem 5.2 and the family of semiconjugacies

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\pm nz} & \mathbb{C} \\ \downarrow{\scriptstyle \exp z} & & \downarrow{\scriptstyle \exp z} \\ \mathbb{CP}^1 & \xrightarrow{z^{\pm n}} & \mathbb{CP}^1 \end{array},$$

where $n \in \mathbb{N}$. □

5.3. **Proof of Theorem 1.6 in the case where A is a Lattès map.** In this section, we need some further definitions and results concerning Riemann surface orbifolds. In particular, the definition of the orbifold $\mathcal{O}_0^A$ associated with a rational function A, and the description of Lattès maps as self-covering maps of orbifolds of zero Euler characteristic. All the necessary information can be found in the paper [20] (see Section 2.1 and Section 2.4).

The first result we need is following (see [20], Theorem 3.5).

Theorem 5.3. *Let A and U be rational functions of degree at least two. If A is a Lattès map, then the algebraic curve $A^{\circ d}(x) - U(y) = 0$ has a factor of genus zero or one for every $d \geq 1$ if and only if U is a compositional left factor of $\theta_{\mathcal{O}_0^A}$.* □

In addition, we need the following two facts (see [20], Theorem 2.4, and [21], Lemma 3.5).

Theorem 5.4. *Let U be a rational function and $\mathcal{O} = (\mathbb{CP}^1, \nu)$ an orbifold. Then U is a compositional left factor of $\theta_{\mathcal{O}}$ if and only if $\mathcal{O}_2^U \preceq \mathcal{O}$.* □

Lemma 5.5. *Let A be a rational function such that $\chi(\mathcal{O}_2^A) = 0$, and U, V rational functions of degree at least two such that $A = U \circ V$ and*

$$\deg U, \deg V \notin \{2, 3, 4, 6, 8, 12\}.$$

Then $\mathcal{O}_2^V = \mathcal{O}_1^U$. □

Finally, we recall that if $\mathcal{O}=(\mathbb{CP}^1,\nu)$ is an orbifold distinct from the non-ramified sphere, then $\chi(\mathcal{O})=0$ if and only if the signature of $\mathcal{O}$ belongs to the list

$$\{2,2,2,2\}\quad \{3,3,3\},\quad \{2,4,4\},\quad \{2,3,6\}, \tag{39}$$

while $\chi(\mathcal{O})>0$ if and only if the signature of $\mathcal{O}$ belongs to the list

$$\{n,n\},\ \ n\geq 2,\quad \{2,2,n\},\ \ n\geq 2,\quad \{2,3,3\},\quad \{2,3,4\},\quad \{2,3,5\}.$$

To prove Theorem 1.6 in the case where A is a Lattès map we show first that if $\mathcal{O}=\mathcal{O}_0^A$ is the orbifold such that $A:\mathcal{O}\rightarrow\mathcal{O}$ is a covering map, then $B:\mathcal{O}\rightarrow\mathcal{O}$ is also a covering map. Assume, say, that $\nu(\mathcal{O})=\{2,3,6\}$. Since for every pair of integers $d\geq 1$, $i\geq 1$ algebraic curve (5) has a factor of genus zero or one, it follows from Theorem 5.3 and Theorem 5.4 that for every $d\geq 1$ the inequality $\chi(\mathcal{O}_2^{B^{\circ d}})\preceq\mathcal{O}$ holds, implying that the signature $\nu(\mathcal{O}_2^{B^{\circ d}})$ is either $\{2,3,6\}$, or one of the following signatures

$$\{2,2,3\},\quad \{2,3,3\},\quad \{2,2\},\quad \{3,3\}. \tag{40}$$

However, rational functions f such $\mathcal{O}_2^f$ belongs to the list (40) have bounded degrees (see e.g. [18]). Thus, for d big enough $\nu(\mathcal{O}_2^{B^{\circ d}})=\{2,3,6\}$. Furthermore, for d big enough $\deg B^{\circ d}>12$. Therefore, applying Lemma 5.5 to the decomposition

$$B^{\circ d^2}=B^{\circ d}\circ B^{\circ d},$$

we conclude that

$$\mathcal{O}_1^{B^{\circ d}}=\mathcal{O}_2^{B^{\circ d}}=\mathcal{O}.$$

Thus, $B^{\circ d}:\mathcal{O}\rightarrow\mathcal{O}$ is a covering map. Finally, the fact that $B^{\circ d}:\mathcal{O}\rightarrow\mathcal{O}$ is a covering map implies that $B:\mathcal{O}\rightarrow\mathcal{O}$ is a covering map (see [23], Corollary 4.6). The proof for other signatures from the list (39) is similar.

Let now $O_A(z_0)$ and $O_B(z_0)$ be orbits having an infinite intersection. Since $A:\mathcal{O}\rightarrow\mathcal{O}$ and $B:\mathcal{O}\rightarrow\mathcal{O}$ are both covering maps, there exist an elliptic curve $\mathcal{C}$ and holomorphic maps

$$\alpha_A:\mathcal{C}\rightarrow\mathcal{C},\quad \alpha_B:\mathcal{C}\rightarrow\mathcal{C},\quad \pi:\mathcal{C}\rightarrow\mathbb{CP}^1$$

such that the diagrams

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\alpha_A} & \mathcal{C} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}, \qquad \begin{array}{ccc} \mathcal{C} & \xrightarrow{\alpha_B} & \mathcal{C} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \end{array}$$

commute. Moreover,

$$\alpha_A=\psi_A+T_A,\qquad \alpha_B=\psi_B+T_B, \tag{41}$$

where $\psi_A,\psi_B\in\operatorname{End}(\mathcal{C})$ and T_A,T_B are points of finite order (see e.g. [13], Lemma 5.1).

If z_0' is a point of $\mathcal{C}$ such that $\pi(z_0')=z_0$ and $k,l\geq 1$ are integers such that (38) holds, then

$$(\alpha_A^{\circ k}-\alpha_B^{\circ\ell})(z_0')=0.$$

On the other hand, it follows from (41) that

$$\alpha_A^{\circ k}-\alpha_B^{\circ\ell}=\psi+T,$$

where $\psi \in \mathrm{End}(\mathcal{C})$ and T is a point of finite order d. Moreover, since the equality $(\psi + T)(z_0') = 0$ implies the equality $d(\psi + T)(z_0') = 0$, we see that $\psi(dz_0') = 0$. Therefore, either $\psi = 0$, or dz_0' belongs to the group of finite order $\mathrm{Ker}\,\psi$, implying that z_0' itself has finite order. Since points of finite order of $\mathcal{C}$ are mapped to preperiodic points of A and B (see e.g. [30], Proposition 6.44), in the second case the orbits $O_A(z_0)$ and $O_B(z_0)$ cannot have an infinite intersection. Therefore, $\psi = 0$, implying that $T = 0$. Thus, $\alpha_A^{\circ k} = \alpha_B^{\circ \ell}$, implying that $A^{\circ k} = B^{\circ \ell}$. □

6. The polynomial case

6.1. **Polynomial decompositions.** First of all, we recall that if A is a polynomial, and $A = U \circ V$ is a decomposition of A into a composition of rational functions, then there exists a Möbius transformation μ such that $U \circ \mu$ and $\mu^{-1} \circ V$ are polynomials. Thus, when studying decompositions of $A^{\circ d}$ we can restrict ourselves by the consideration of decompositions into compositions of polynomials. We also mention that since a polynomial cannot be a Lattès map, a polynomial is special if and only if it is conjugate to z^n or $\pm T_n$.

The following result follows easily from the fact that the monodromy group of a polynomial of degree n contains a cycle of length n.

Theorem 6.1 ([4]). *Let A, C, D, B be polynomials such that*

$$A \circ C = D \circ B. \tag{42}$$

Then there exist polynomials $U, V, \widetilde{A}, \widetilde{C}, \widetilde{D}, \widetilde{B}$, where

$$\deg U = \mathrm{GCD}(\deg A, \deg D), \quad \deg V = \mathrm{GCD}(\deg C, \deg B),$$

such that

$$A = U \circ \widetilde{A}, \quad D = U \circ \widetilde{D}, \quad C = \widetilde{C} \circ V, \quad B = \widetilde{B} \circ V,$$

and

$$\widetilde{A} \circ \widetilde{C} = \widetilde{D} \circ \widetilde{B}.$$ □

Notice that Theorem 6.1 implies that if $\deg D \mid \deg A$ in (42), then the equalities

$$A = D \circ R, \quad B = R \circ C$$

hold for some polynomial R. In particular, if (3) holds for *polynomials* A, f, g, then $f = \mu \circ g$ for some polynomial of degree one μ such that $A \circ \mu = A$. Moreover, Theorem 6.1 implies Theorem 4.1 in the case where A is a polynomial. Indeed, since (19) implies that $\mathcal{P}(X) \subseteq \mathcal{P}(A)$, we have:

$$\deg X = \prod_{p \in \mathcal{P}(A)} p^{\alpha_p},$$

where α_p obviously satisfies the inequality $\alpha_p \leq \log_2 \deg X$. Therefore,

$$\deg X \mid \deg (A^{\circ N})$$

for $N = \log_2 \deg X$, and applying Theorem 6.1 to the equality

$$A^{\circ d} = A^{\circ N} \circ A^{\circ (d-N)} = X \circ R,$$

where $d > N$, we conclude that

$$A^{\circ N} = X \circ R', \quad R = R' \circ A^{\circ (d-N)}$$

for some polynomial R'.

For a polynomial T we denote by $Aut(T)$ the set of polynomial Möbius transformations commuting with T. The following result classifies polynomials commuting with a given non-special polynomial (see [28], and [24], Section 6.2).

Theorem 6.2. *Let A be a polynomial of degree at least two, not conjugate to z^n or $\pm T_n$. Then there exists a polynomial T such that $A = \mu \circ T^{\circ k}$, where $\mu \in Aut(A)$ and $k \geq 1$, and any polynomial B commuting with A has the form $B = \nu \circ T^{\circ \ell}$, where $\nu \in Aut(A)$ and $l \geq 1$.* □

Corollary 6.3. *Let A be a polynomial of degree at least two, not conjugate to z^n or $\pm T_n$. Assume that B is a polynomial commuting with A such that $\deg B \geq \deg A$. Then $B = A \circ S$ for some polynomial S.*

Proof. Applying Theorem 6.2 and taking into account that $\nu, \mu \in Aut(A)$, we see that the equality $B = A \circ S$ holds for the polynomial

$$S = \mu^{-1} \circ \nu \circ T^{\circ(l-k)}.$$ □

6.2. **Equivalence relation.** Let A be a rational function. Following [19], we say that a rational function $\widehat{A}$ is an *elementary transformation* of A if there exist rational functions U and V such that $A = V \circ U$ and $\widehat{A} = U \circ V$. We say that A and B are *equivalent* and write $A \sim B$ if there exists a chain of elementary transformations between A and B. Notice that any pair A, $\widehat{A}$ as above gives rise to the semiconjugacies

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{\widehat{A}} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1\,, \end{array} \qquad \begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\ \mathbb{CP}^1 & \xrightarrow{\widehat{A}} & \mathbb{CP}^1\,, \end{array}$$

implying inductively that whenever $A \sim B$ the function A is semiconjugate to the function B, and the function B is semiconjugate to the function A.

Since for any Möbius transformation μ the equality

$$A = (A \circ \mu^{-1}) \circ \mu$$

holds, the equivalence class $[A]$ of a rational function A is a union of conjugacy classes. We denote the number of conjugacy classes in $[A]$ by $d(A)$. In this notation, the following statement holds.

Theorem 6.4. *Let A be a rational function of degree n. Then its equivalence class $[A]$ contains infinitely many conjugacy classes if and only if A is a flexible Lattès map. Furthermore, if A is not a flexible Lattès map, then $d(A)$ can be bounded in terms of n only.* □

The first part of Theorem 6.4 was proved in [19], using the McMullen theorem about isospectral rational functions [11]. This approach however provides no bound for $d(A)$. The fact that $d(A)$ can be bounded in terms of n was proved in the paper [21] (see Theorem 1.1 and Remark 5.2).

Lemma 6.5. *Let A be a special function, and $A' \sim A$. Then A' is special.* □

In the full generality Lemma 6.5 is proved in [21] (Lemma 2.11). Below we use this lemma only in the polynomial case, in which it follows from the well known description of decompositions of z^n and $\pm T_n$.

6.3. **Polynomial orbits and iterates.** We start by reproving the main result of the paper [32], basing merely on the results of Sections 6.1-6.2.[2]

Theorem 6.6. *Let A be a polynomial of degree $n \geq 2$ not conjugate to z^n or $\pm T_n$. Then there exists an integer N, depending on n only, such that any decomposition of $A^{\circ d}$ with $d \geq N$ is induced by a decomposition of $A^{\circ N}$.*

Proof. It is enough to show that if a polynomial A is not conjugate to z^d or $\pm T_d$, then equality (16) for some polynomials X and Y with $\deg X$ big enough with respect to $\deg A$ implies that

$$X = A \circ R \tag{43}$$

for some polynomial R. Indeed, in this case the equality $A^{\circ(s-1)} = R \circ Y$ holds by Theorem 6.1, and applying this argument inductively, we obtain an analogue of Corollary 3.5, which holds for any non-special polynomials A. The rest of the proof is similar to the proof of Theorem 1.1.

Since (16) implies that $\mathcal{P}(X) \subseteq \mathcal{P}(A)$, the inequality $\gcd(\deg X, \deg A) > 1$ holds. Therefore, by Theorem 6.1, there exists a polynomial V_1 of degree at least two such that the equalities

$$A = V_1 \circ U_1, \qquad X = V_1 \circ X_1,$$

and

$$U_1 \circ A^{\circ(s-1)} = X_1 \circ Y \tag{44}$$

hold for some polynomials U_1 and X_1. In turn, equality (44) implies the equality

$$A_1^{\circ s} = X_1 \circ Y_1, \tag{45}$$

where

$$A_1 = U_1 \circ V_1, \qquad Y_1 = Y \circ V_1.$$

Applying now the same reasoning to (45) we can find polynomial U_2, V_2, X_2, $\deg V_2 \geq 2$, such that the equalities

$$A_1 = V_2 \circ U_2, \qquad X_1 = V_2 \circ X_2,$$

and

$$A_2^{\circ s} = X_2 \circ Y_2$$

hold for

$$A_2 = U_2 \circ V_2, \qquad Y_2 = Y_1 \circ V_2.$$

Continuing in the same way and taking into account that $\deg V_i \geq 2$, we see that there exist an integer $p \geq 1$ and a sequence of elementary transformations

$$L: \ A_0 = A \to A_1 \to A_2 \to \cdots \to A_p$$

such that the equalities

$$A_0 = V_1 \circ U_1, \quad A_i = U_i \circ V_i, \qquad 1 \leq i \leq p,$$
$$U_i \circ V_i = V_{i+1} \circ U_{i+1}, \quad 1 \leq i \leq p-1,$$

and

$$X = V_1 \circ V_2 \circ \cdots \circ V_p$$

hold.

[2] In distinction with [32], we do not provide an explicit bound for N. However, for applications similar to Theorem 6.7 the actual form of this bound is not really important.

Since a polynomial cannot be a Lattès map, the equivalence class $[A]$ contains at most finitely many conjugacy classes by Theorem 6.4. Setting

$$M = n^{d(A)K},$$

where K is a natural number to be defined later, assume that $\deg X > M$. Since $\deg V_i \leq n$, this implies that $p \geq d(A)K + 1$. Therefore, there exist indices

$$s_0, s_1, \dots, s_K, \qquad 0 \leq s_0 < s_1 < \cdots < s_K \leq p$$

such that A_{s_0}, $A_{s_1}, \dots, A_{s_K}$ are conjugate to each other. We consider now the commutative diagram

$$\begin{tikzcd}
\mathbb{CP}^1 \arrow[r, "A_p"] \arrow[d, "W_{K+1}"'] & \mathbb{CP}^1 \arrow[d, "W_{K+1}"] \\
\mathbb{CP}^1 \arrow[r, "A_{s_K}"] \arrow[d, dashed] & \mathbb{CP}^1 \arrow[d, dashed] \\
\mathbb{CP}^1 \arrow[r, "A_{s_1}"] \arrow[d, "W_1"'] & \mathbb{CP}^1 \arrow[d, "W_1"] \\
\mathbb{CP}^1 \arrow[r, "A_{s_0}"] \arrow[d, "W_0"'] & \mathbb{CP}^1 \arrow[d, "W_0"] \\
\mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1,
\end{tikzcd}$$

where

$$W_0 = V_1 \circ V_2 \circ \cdots \circ V_{s_0}, \quad W_{K+1} = V_{s_K+1} \circ V_{s_2+2} \circ \cdots \circ V_p,$$

and

$$W_i = V_{s_{i-1}+1} \circ V_{s_{i-1}+2} \circ \cdots \circ V_{s_i}, \quad 1 \leq i \leq K.$$

Since the equality

$$A_{s_K} = \nu^{-1} \circ A_{s_0} \circ \nu,$$

holds for some $\nu \in Aut(\mathbb{CP}^1)$, the polynomial

$$W = W_1 \circ W_2 \circ \dots W_K \circ \nu^{-1}$$

commutes with the polynomial A_{s_0}. Moreover, since A is non-special, the polynomial A_{s_0} is also non-special by Lemma 6.5.

Assume now that $K \geq \log_2 n$. Since $\deg V_i \geq 2$, in this case the inequality $\deg W \geq n$ holds, and hence $W = A_{s_0} \circ S$ for some polynomial S, by Corollary 6.3. Therefore,

$$X = W_0 \circ W \circ \nu \circ W_{K+1} = W_0 \circ A_{s_0} \circ S \circ \nu \circ W_{K+1} = A \circ W_0 \circ S \circ \nu \circ W_{K+1}.$$

Summarizing, we see that the condition

$$\deg X > n^{d(A) \log_2 n},$$

implies that equality (43) holds for some polynomial R. □

Now we reprove the main result of the papers [8], [9], basing on Theorem 5.1 and Theorem 6.6.

Theorem 6.7. *Let A and B be polynomials of degree at least two such that an orbit of A has an infinite intersection with an orbit of B. Then A and B have a common iterate.*

Proof. By Theorem 1.6, we may assume that the polynomials A and B are not special. Arguing as in Section 5.1, we see that there exist a Galois covering X_A and a rational function F such that diagram (29) commutes and for every $i \geq 1$ there exist $d_i \geq 1$ and $S_i \in \mathbb{C}(z)$ such that equality (30) holds. Moreover, $\mathcal{P}(B) \subseteq \mathcal{P}(A)$, implying that for every $i \geq 1$ there exist $s_i \geq d_i$ such that

$$\deg(B^{\circ i}) | \deg(A^{\circ s_i}). \tag{46}$$

Equality (30) implies the equality

$$A^{\circ d_i} \circ X_A \circ F^{\circ(s_i - d_i)} = B^{\circ i} \circ S_i \circ F^{\circ(s_i - d_i)},$$

which in turn implies the equality

$$A^{\circ s_i} \circ X_A = B^{\circ i} \circ S_i \circ F^{\circ(s_i - d_i)}. \tag{47}$$

Applying now Theorem 6.1 to (47) and taking into account (46), we conclude that for every $i \geq 1$ there exist $R_i \in \mathbb{C}[z]$ such that (24) holds. Finally, arguing as in the proof of Theorem 1.4, but using Theorem 6.6 instead of Theorem 1.1, we conclude that A and B have a common iterate. □

References

[1] R. Avanzi, U. Zannier, *The equation $f(X) = f(Y)$ in rational functions $X = X(t), Y = Y(t)$*, Compositio Math. 139 (2003), no. 3, 263-295.

[2] C. Cabrera, P. Makienko, *On decomposable rational maps*, Conform. Geom. Dyn. 15 (2011), 210–218.

[3] C. Cabrera, P. Makienko, *Amenability and measure of maximal entropy for semigroups of rational maps*, Groups Geom. Dyn. 15 (2021), no. 4, 1139-1174.

[4] H. Engstrom, *Polynomial substitutions,* Amer. J. Math. 63, 249-255 (1941).

[5] G. Faltings, *Complements to Mordell*, Rational points (Bonn, 1983/1984), 203-227, Aspects Math., E6, Friedr. Vieweg, Braunschweig, 1984.

[6] M. Fried, *Fields of definition of function fields and a problem in the reducibility of polynomials in two variables*, Ill. J. Math. 17, 128-146 (1973).

[7] E. Girondo, G. González-Diez, *Introduction to compact Riemann surfaces and dessins d'enfants,* London Mathematical Society Student Texts, 79. Cambridge University Press, Cambridge, 2012.

[8] D. Ghioca, T. Tucker, M. Zieve, *Intersections of polynomial orbits, and a dynamical Mordell-Lang conjecture,* Inventiones Math. 171 (2008), 463–483.

[9] D. Ghioca, T. Tucker, M. Zieve, *Linear relations between polynomial orbits,* Duke Math. J. 161 (2012), 1379-1410.

[10] M. Lyubich, Y. Minsky, *Laminations in holomorphic dynamics*, J. Differential Geom. 47 (1997), no. 1, 17-94.

[11] C. McMullen, *Families of rational maps and iterative root-finding algorithms,* Ann. of Math., 125, No. 3 (1987), 467-493.

[12] A. Medvedev, T. Scanlon, *Invariant varieties for polynomial dynamical systems,* Annals of Mathematics, 179 (2014), no. 1, 81 - 177.

[13] J. Milnor, *On Lattès maps,* Dynamics on the Riemann Sphere. Eds. P. Hjorth and C. L. Petersen. A Bodil Branner Festschrift, European Mathematical Society, 2006, pp. 9-43.

[14] F. Pakovich, *Prime and composite Laurent polynomials*, Bull. Sci. Math, 133 (2009) 693-732.

[15] F. Pakovich, *Algebraic curves $P(x) - Q(y) = 0$ and functional equations*, Complex Var. Elliptic Equ. 56 (2011), no. 1-4, 199-213.

[16] F. Pakovich, *Polynomial semiconjugacies, decompositions of iterations, and invariant curves*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), Vol. XVII (2017), 1417-1446.

[17] F. Pakovich, *On algebraic curves* $A(x) - B(y) = 0$ *of genus zero,* Math. Z., Vol. 288, Issue 1, (2018), 299-310.

[18] F. Pakovich, *On rational functions whose normalization has genus zero or one*, Acta Arith., 182 (2018), 73-100.

[19] F. Pakovich, *Recomposing rational functions,* Int. Math. Res. Not., 2019, no. 7, 1921-1935.

[20] F. Pakovich, *Algebraic curves* $A^{\circ \ell}(x) - U(y) = 0$ *and arithmetic of orbits of rational functions*, Mosc. Math. J., Vol. 20 (2020), no 1, 153-183.

[21] F. Pakovich, *Finiteness theorems for commuting and semiconjugate rational functions,* Conform. Geom. Dyn. 24 (2020), 202-229.

[22] F. Pakovich, *On rational functions sharing the measure of maximal entropy*, Arnold Math. J., 6, 387–396 (2020).

[23] F. Pakovich, *On generalized Lattès maps*, J. Anal. Math. 142 (2020), no. 1, 1-39.

[24] F. Pakovich, *Commuting rational functions revisited,* Ergodic Theory Dynam. Systems, 41 (2021), no. 1, 295 - 320

[25] F. Pakovich, *Invariant curves for endomorphisms of* $\mathbb{P}^1 \times \mathbb{P}^1$*,* Math. Ann., https://doi.org/10.1007/s00208-021-02304-5.

[26] J. Ritt, *Prime and composite polynomials*, American M. S. Trans. 23, 51-66 (1922).

[27] J. F. Ritt. *On the iteration of rational functions,* Trans. Amer. Math. Soc. 21 (1920), no. 3, 348-356.

[28] J. F. Ritt. *Permutable rational functions,* Trans. Amer. Math. Soc. 25 (1923), 399-448.

[29] N. Segol, *Injectivity of rational functions,* M.Sc. thesis, Technion, 2018.

[30] J. Silverman, *The arithmetic of dynamical systems,* Grad. Texts in Math., 241. Springer, New York, 2007.

[31] H. Ye, *Rational functions with identical measure of maximal entropy*, Adv. Math. 268 (2015), 373-395.

[32] M.Zieve, P. Müller, *On Ritt's polynomial decomposition theorem*, preprint, arXiv:0807.3578.

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