ON THE COHEN-MACAULAY PROPERTY OF SPHERICAL VARIETIES

AVRAHAM AIZENBUD AND EITAN SAYAG

Abstract. We study the space $\mathcal{S}(X)$ of Schwartz functions on a spherical variety $X = G/H$ as a smooth representation of a p-adic group G. Our main conjecture is that this is a Cohen-Macaulay object of the category of smooth representations of G. We show that this conjecture is related to several central questions of harmonic analysis on X:

(1) Behavior of the multiplicity $d(\pi) = \dim \mathrm{Hom}_H(\pi, \mathbb{C})$ in families of representation.
(2) Density of orbital integrals in the space of invariant distributions.
(3) A conjecture of P. Delorme regarding the disjointness of X-cuspidal representations and the unramified spectrum of X.
(4) Freeness properties of the module $\mathcal{S}(X)^K$ over subalgebras of the Hecke algebra $\mathcal{H}(G, K)$, for various compact subgroups $K \subset G$.

We verify the conjecture for the following cases:

(a) The diagonal case: $X = H \times H/\Delta(H)$. In this case, our conjecture is a corollary of an unpublished result of J. Bernstein.
(b) $X = GL(2)/U(2)$

Along the way, we provide a simple proof of a result of Hironaka ([Hir99]) which refers to the freeness of the module $\mathcal{S}(X)^{K_0}$ over $\mathcal{H}(GL(V), K_0)$, where X is the space of Hermitian forms on V and K_0 is the maximal compact subgroup of G.

Contents

Date: August 30, 2014.

1. Introduction

In this work we study the relationship between some well studied phenomena in harmonic analysis of a spherical variety X and certain properties of the module $\mathcal{S}(X)$ of Schwartz functions on X. We begin with an overview of these phenomena and then formulate a conjecture that provides an explanation for all of them.

1.1. **Phenomena.** Let F be a non-archimedean local field of characteristic zero and let G be a reductive group defined over F. Let X be a G-transitive spherical algebraic variety. It means that over the algebraic closure of F the Borel subgroup of G have a unique open orbit in X. Let $x \in X(F)$ and let $H = G_x$ be its stabilizer. We will sometimes fix a $G(F)$-equivariant sheaf $\mathcal{L}$ on $X(F)$ and we let $\chi = \mathcal{L}_x$ be the corresponding representation of $H(F)$. Notice that $\mathcal{L}$ is a locally constant sheaf of vector spaces.

1.1.1. *Continuity and Jumps in Multiplicities.* Let S be a complex algebraic variety and consider a family π_s of smooth admissible representations of G parameterized by $s \in S$. Let

$$m(s) = \dim\mathrm{Hom}_H(\pi_s, \chi)$$

be the multiplicity of H-invariant functionals on π_s. Notice that by Frobenius reciprocity we have

$$m(s) = \dim\mathrm{Hom}_G(\pi_s, C^\infty(X, \mathcal{L}))$$

where $C^\infty(X, \mathcal{L})$ is the space of sections of the sheaf $\mathcal{L}$ over X.

Assume for simplicity that there is a $G(F)$-invariant measure on $X(F)$. Let $\mathcal{S}(X(F),\mathcal{L}^*)$ denotes the space of compactly supported sections of the dual locally constant sheaf $\mathcal{L}^*$. By duality, we have

$$m(s) = \dim\mathrm{Hom}_G(\mathcal{S}(X(F),\mathcal{L}^*),\tilde{\pi}_s)$$

where $\tilde{\pi}_s$ is the smooth dual of π_s. In many cases the function $m : S \to \mathbb{N}$ is continuous. We will concentrate on the following setup. Let $P \subset G$ be a parabolic subgroup and M its Levi factor. Let $S \subset \mathfrak{X}^*(M)$ be a variety of unramified characters of M. Let ρ be an irreducible cuspidal representation of M. The family we consider is $\pi_s = \mathrm{Ind}_P^G(\rho \otimes s)$.

In this case it is proven in [FLO2012] (Lemma D.1) that $m(s)$ is upper semi-continuous.

The next conjecture capture some part of the continuity phenomenon mentioned above:

Conjecture I. *Let S be the smooth locus of the support of the function m on $\mathfrak{X}(M)$. Then the function m is continuous on S.*

Remark. *We note that the restriction to the subset S is essential. See Remark 4.3.4 below.*

Conjecture I is known to hold in some cases. For example the Whittaker case – G is split, H is the unipotent radical of the Borel subgroup of G and χ is non-degenerate (see [Rod73])

Conjecture I is inspired by the following folklore conjecture in the theory of Gelfand pairs:

Conjecture II. *Let $H \subset G$ be a spherical pair (i.e. G/H is a G-spherical variety). Suppose that "generic" H-distinguished irreducible unramified principal series representation π we have $\dim\mathrm{Hom}_H(\pi,\mathbb{C}) = 1$ then the pair (G,H) is a Gelfand pair (that is, $\dim\mathrm{Hom}_H(\pi,\mathbb{C}) \leq 1$ for any irreducible π.)*

This Conjecture is proven in many cases ([AG09b], [JR96], [BvD94]).

1.1.2. *Density Principles.* Let X_1 be another G-transitive spherical variety and let $\mathcal{L}_1$ be a G-equivariant sheaf on $X_1(F)$. Let x_1, H_1, χ_1 be the analogues to x, H, χ as before. Consider the space $\mathcal{I} = \mathcal{S}^*(X(F) \times X_1(F), \mathcal{L} \boxtimes \mathcal{L}_1)^{G(F)}$ of functionals on $\mathcal{S}(X(F) \times X_1(F), \mathcal{L} \boxtimes \mathcal{L}_1)$ that are invariant with respect to the diagonal action of $G(F)$. This space is canonically isomorphic to the space $\mathcal{S}^*(G)^{H \times H_1, \chi' \times \chi_1'} \cong \mathcal{S}^*(X,\mathcal{L})^{H_1,\chi_1'}$, where χ' and χ_1' are certain characters corresponding to χ and χ_1

In many cases, one can explicitly describe invariant distributions that span a dense subspace $\mathcal{J} \subset \mathcal{I}$. This is known as a Density principle. Let us recall few examples of known results in this direction:

(1) Density of regular semi-simple orbital integrals. Here $X = X_1 = H$ and $G = H \times H$ with the two sided action. In this case the space $\mathcal{I} \cong \mathcal{S}^*(H(F))^{Ad(H(F))}$ is the space of distributions on $H(F)$ that are $H(F)$-invariant with respect to conjugation. In this case let $\mathcal{J}$ be the space spanned by regular semi-simple orbital integrals, i.e. by $H(F)$-invariant distributions regular semi-simple orbits. (see [HC78]).

(2) Density of characters of irreducible representations.
Still in the group case, one can take $\mathcal{J}$ to be the space spanned by the characters of irreducible representations. (see [DKV84, §A.2] and [Kaz86, Appendix]).

(3) Density of twisted unipotent regular orbital integrals.
Here $G = GL(n)$, $H' = H = N_n \subset G$ the maximal unipotent subgroup, $\chi = \chi'$ a non-degenerate character of H, H'. Consider $\mathcal{J}$ be the space spanned by regular orbital integrals, i.e. orbits that lie in the open Bruhat cell. ([Jac98]).

(4) Density of twisted unipotent regular orbital integrals on the space of Hermitian matrices.
Here X is the space of Hermitian forms on a vector space V defined over a quadratic extension E/F. Let $G = GL(V)$, $H' = N_n$ the maximal unipotent subgroup and χ the trivial character of H'. Let $\mathcal{J}$ be the space spanned by regular orbital integrals, i.e. orbits that lie in the open Borel orbit on X. ([Jac98]).

(5) Rank one symmetric spaces.
In [RR96] there is a detailed analysis of the case $H = H' \subset G$ a symmetric subgroup such that $X = G/H$ is a rank one symmetric space. In particular, they study density of semi-simple regular orbital integrals in the space $\mathcal{S}^*(X(F))^{H(F)}$.

Nevertheless, at present there is no general conjecture concerning density. Even the symmetric case presents difficulties as noted in [RR96].

In §4.5 we analyze the case when X is an affine G-spherical space over the split group G and $H' = N(F) \rtimes T^0$ is a semi direct product of the maximal unipotent subgroup N with T^0 the maximal compact subgroup of the torus T. In this case we introduce a collection of H' orbits on $X(F)$ which we call *significant* (see Notation 4.2.8 in §4.2) and formulate the following conjecture.

Conjecture III. *Let $\mathcal{J} \subset \mathcal{S}^*(X(F))^{H'}$ be the space spanned by significant orbital integrals on X. Then $\mathcal{J}$ is dense.*

1.1.3. *Freeness of the Module $\mathcal{S}(X(F))^I$.* Let $K_0 \subset G(F)$ be a maximal open compact subgroup. The module $\mathcal{S}(X(F))^{K_0}$ was studied by many authors (see [Hir99], [Off], [MR09], [Sa08] to mention a few). In many cases this module turns out to be a free module over $\mathcal{H}(G(F), K_0)$. As a byproduct of our work we provide a rather simple proof of the following result.

Theorem A (Y. Hironaka)**.** *Let E/F be an unramified quadratic extension. Let V be a vector space over E, let $G = GL(V)$ and X be the space of Hermitian forms on V. Let $K_0 \subset G$ be a maximal compact subgroup. Then $\mathcal{S}(X(F))^{K_0}$ is a free $\mathcal{H}(G, K_0)$ module.*

We propose the following conjecture regarding the freeness of the module $M = \mathcal{S}(X(F))^I$ for the Iwahori subgroup $I \subset G(F)$.

Conjecture IV. *There exists a polynomial subalgebra $B \subset \mathcal{H}(G(F), I)$ such that M is finitely generated free B-module.*

1.1.4. *Disjointness of X-Cuspidal Representations and the Unramified Spectrum of X.* It is well known that unramified irreducible representations of a p-adic group are never cuspidal. The same is conjectured to hold in the relative case. Namely,

Conjecture V (Delorme)**.** *Let π be an X-cuspidal representation of $G(F)$ (see §§4.4 below) and assume that π is I-distinguished. Then $X(F)$ is compact modulo the center of $G(F)$.*

1.2. **The Main Conjecture.** We now formulate our main conjecture. We first recall the notion of Cohen-Macualy modules from commutative algebra (see e.g. [BBG97]). Let A be a commutative finitely generated algebra over a field k. Let M be an A-module. We say that M is a *Cohen-Macaulay* module if there exists a polynomial subalgebra $B \subset A$ such that M is a finitely generated and free B-module. More generally, if $\mathcal{C}$ is an abelian category we say that $M \in Ob(\mathcal{C})$ is a Cohen Macaulay object if for any projective object $P \in Ob(\mathcal{C})$, the module $\mathrm{Hom}(P, M)$ is a Cohen Macaulay object over the center $\mathcal{Z}(\mathrm{End}_{\mathcal{C}}(P))$ of the endomorphism ring of P

Recall that the abelian category $\mathcal{M}(G)$ of smooth representations of $G(F)$, admits a decomposition $\mathcal{M}(G) = \Pi_{\omega \in \Omega} \mathcal{M}_\omega(G)$ to its Bernstein blocks (see §§§ 2.3.3 below).

Conjecture VI. *Let X be a spherical G-space as above and consider $M = \mathcal{S}(X(F))$ with a Bernstein decomposition $M = \oplus_{\omega \in \Omega} M_\omega$. Then for any $\omega \in \Omega$ the module M_ω is Cohen Macaulay object in $\mathcal{M}(G)$.*

We prove the following theorem:

Theorem B.

(1) Part of conjecture VI, namely that the direct summands M_ω that corresponds to cuspidal representations are Cohen Macaulay object in $\mathcal{M}(G)$ holds for any affine spherical space.

(2) Conjecture VI holds in the following cases
- *(a) the diagonal case: $G = L \times L$ and $H = \Delta L$ is a diagonal copy of a reductive group L.*
- *(b) the case $G = \mathrm{GL}(V)$, where V is a 2-dimensional linear space over a quadratic extension of F and X is the space of Hermitian forms on V.*

1.3. **The Relation of the Main Conjecture with the Above Phenomena.** We view the main conjecture as formalization and refinement of the observed phenomena described above. The following theorem, which is the core of this paper, describe the relation between the main conjecture and the phenomena.

Theorem C (See §4)**.** *Assume that Conjecture VI holds for the spherical G-space X. Then*

(1) Conjecture IV holds for X.
(2) If in addition X is affine, Conjecture I (with trivial $\mathcal{L}$) and Conjecture III holds for X.
(3) If in addition X is a symmetric G-space, Conjecture V holds for X.

Let us now briefly elaborate on the relationship between the Cohen-Macaulay property of $\mathcal{S}(X(F))$ and each of these phenomena.

1.3.1. *Relation to Conjecture I – Continuity of Multiplicities.* In order to study the multiplicities $m(s) = \dim\mathrm{Hom}_G(\mathcal{S}(X(F)), \mathrm{Ind}_P^G(\rho\otimes s))$ we introduce a sheaf on $\mathfrak{X}(M)$ (see Notation 3.4.3) whose fibers describe the function m (see Lemma 3.4.4). We show that the support of this sheaf is a reduced scheme and provide a description of its structure (see Theorem 4.2.1 and Proposition 4.2.7). This gives a sufficient and necessary condition for a cuspidaly induced representation to be X-distinguished in terms of the cuspidal data. Note that a necessary condition was obtained by [Sa08] for unramified principal series.

In Corollary 3.4.7 we show that our main conjecture implies conjecture I by studying this sheaf. For this we use the following lemma from commutative algebra.

Lemma 1 (see 3.4.1 below)**.** *Suppose that M is a Cohen-Macaully module over a commutative algebra A. For a closed point $s \in Spec(A)$ let $m(s)$ be the dimension of the fiber M/sM of M at the point s. Then m is a continuous function on the smooth locus of the support of m.*

1.3.2. *Relation to Conjecture IV – Freeness of the Module $\mathcal{S}(X(F))^K$.* If $\mathcal{S}(X(F))$ is Cohen-Macaulay then it is easy to see that, for any open compact subgroup $K \subset G$ the module $\mathcal{S}(X(F))^K$ is Cohen-Macaulay over the Hecke algebra $\mathcal{H}(G,K)$. Thus $\mathcal{S}(X)^K$ is free over a certain sub-algebra.

1.3.3. *Relation to Conjecture III – Density Principle.* Recall the following

Lemma 2 (see Corollary 4.5.5 below)**.** *Let M be a Cohen-Macaulay with reduced support over a commutative algebra A. Let $m \in M$. Suppose for a generic point $x \in Supp(M)$ we have $m|_x = 0$. Then $m = 0$.*

We can rephrase the density property as follows: $\mathcal{J} \subset \mathcal{I}$ is dense if and only if for any $f \in \mathcal{S}(X, \mathcal{L}^*)_{H_1,\chi''}$ satisfying $\forall \xi \in \mathcal{J}, \langle \xi, f\rangle = 0$ we have $f = 0$. Here χ'' is an appropriate character of H_1.

The Cohen-Macaulay property combined with the lemma above can, in many cases, "explain" density phenomena. Indeed, assume that $M = \mathcal{S}(X, \mathcal{L}^*)_{H_1,\chi''}$ is a Cohen-Macaulay module over some algebra A. Then for each (closed) point $x \in Supp(M) \subset Spec(A)$ we can associate a one dimensional subspace $\mathcal{I}_x \subset \mathcal{I}$. The lemma implies that for any Zariski dense subset $S \subset Supp(M)$ the space $\mathcal{J} = \bigoplus_{x\in S} \mathcal{I}_x$ is dense in $\mathcal{I}$.

1.3.4. *Relation to Conjecture V – disjointness of X-Cuspidal Representations and the Unramified Spectrum of X.* By Borel-Matsumoto (see [Bo76], [Ma77]) the subcategory $\mathcal{M}(G,I)$, of representations generated by their Iwahori fixed vectors, consist of the Bernstein block ω_0 that corresponds to the principal series. It is easy to see that if X is not compact then the dimension of the support of the component $\mathcal{S}(X)_{\omega_0}$ is positive. When $\mathcal{S}(X)_{\omega_0}$ is Cohen-Macaulay it can not have direct summand with zero dimensional support. Thus an X-cuspidal representation can not be in $\mathcal{M}(G,I)=\mathcal{M}(G)_{\omega_0}$.

1.4. **The Group Case.** Let H be a group and $X=H$ the spherical space equipped with the two sided action of $G=H\times H$.

In §§4.1 we deduce our main conjecture for this class of spherical spaces (Theorem B)(2a)) from the following (unpublished) result of J. Bernstein.

Theorem 1.4.1 (Bernstein)**.** *The category $\mathcal{M}(H)$ is Cohen-Macaulay, i.e. $\mathcal{S}(H)$ is a (locally) Cohen-Macaulay object of $\mathcal{M}(H)$.*

For completeness we provide a proof of Bernstein's theorem (see theorem 2.4.2)

Note that an analogues of Bernstein's theorem was proven in the real case in [BBG97]

1.5. **Structure of the Paper.**

In §2 we overview the notion of Cohen-Macaulay modules in the context of representation theory of p-adic groups. In §§2.2 we review the notion of Cohen-Macaulay modules and introduce a notion of Cohen-Macaulay object in an abelian category. We use the theory of [BBG97] to provide few equivalent criteria for the Cohen-Macaulay property in the realm of abelian categories that are finite over their center. In §§2.3 we review the basic theory of the category $\mathcal{M}(G)$ of smooth representations of G, in particular Bernstein's decomposition theorem and the description of the center of this category. We also include a proof of Bernstein's theorem that $\mathcal{M}(G)$ is a locally Cohen-Macaulay category.

In §3 we use Bernstein's theory to study the behavior of Cohen-Macaulay objects of $\mathcal{M}(G)$, and prove part of Theorem C. In §§3.1 we introduce the notion of blockwise Cohen-Macaulay objects of $\mathcal{M}(G)$. In §§3.2-3.4 we study the relation of the blockwise Cohen-Macaulay property with the Hecke algebra and continuity of multiplicities. In §§3.5 we prove Theorem C(1).

In §4 we reformulate the main conjecture of the paper (Conjecture VI), we prove the rest of Theorem C and part of Theorem B. In §§4.1 we prove Theorem B(2a). In §§4.2 we study a collection of sheaves which encode the behavior of the representation $\mathcal{S}(X(F))$. As a result we prove Theorem B(1). In §§4.3 we prove that the main conjecture implies Conjecture I. In §§4.4 we prove Theorem C(3). Namely, we prove that the main conjecture implies conjecture V. In §§4.5 we prove that the main conjecture implies Conjecture III.

In §5 we consider the spherical space of non-degenerate hermitian forms. We rephrase the main conjecture in more explicit terms for this case.

In §6 we prove Theorem A.

In §7 we prove the rest of Theorem B.

1.6. **Acknowledgments.** The main conjecture of this paper came out during our discussions with Joseph Bernstein on multiplicity problems in representation theory. Bernstein also explained to us his works on the notion of Cohen-Macaulay in the context of abelian categories. We also thank Patrick Delorme, Dimitry Gourevitch and Erez Lapid for fruitful discussions, and Shachar Carmeli, Nadya Gourevitch, Friedrich Knop, Bernhard Krötz, Omer Offen, and Yiannis Sakellaridis for their useful remarks. A.A. was partially supported by NSF grant DMS-1100943 and ISF grant 687/13. E.S was partially supported by ISF grant 1138/10 and ERC grant 291612.

A large part of this paper was conceived and written during the Program *Multiplicity Problems in Harmonic analysis* held at the Hausdorff Institute (2012-2014).

2. Preliminaries

2.1. **General Conventions.**

Through this paper we will usually make the following notations and assumptions.

- F is a fixed non-archimedean local field of characteristic zero.
- Unless stated otherwise, algebraic groups and algebraic varieties are defined over F.
- Unless stated otherwise, vector spaces, algebras, modules and abelian categories are defined/enriched over $\mathbb{C}$.

2.2. Cohen-Macaulay Property.

2.2.1. *Algebras and Modules.* Let us now recall standard facts about the notion of Cohen-Macaulay Algebras and Modules. All algebras considered in this section are unital over $\mathbb{C}$. For a commutative algebra A we denote by $Spec_{\mathbb{C}}(A) = Mor_{\mathbb{C}}(A, \mathbb{C})$.

Remark 2.2.1. *Our usage of* $\mathrm{Spec}_{\mathbb{C}}$ *instead of the usual spectrum is due to the fact that for any product* $B = \prod A_\alpha$ *of commutative algebras we have* $\mathrm{Spec}_{\mathbb{C}}(B) = \coprod \mathrm{Spec}_{\mathbb{C}}(A_\alpha)$.

Theorem 2.2.2. *Let C be a unital commutative algebra. Let M be a finitely generated A-module. Then the following are equivalent*

1. *There exists a subalgebra $B \subset C$ isomorphic to the polynomial algebra $\mathbb{C}[x_1, ..., x_r]$ such that M is finitely generated free B-module.*
2. *For any subalgebra $B \subset C$ isomorphic to the polynomial algebra $\mathbb{C}[x_1, ..., x_r]$ such that M is finitely generated over B, the module M is a free B-module.*
3. *There exists a regular subalgebra $B \subset C$, of dimension r, such that M is finitely generated over B and M is projective B-module.*
4. *For any regular subalgebra $B \subset C$, of dimension r, such that M is finitely generated over B, the module M is projective.*

For the proof see §2 of [BBG97].

Definition 2.2.3.

- *A C-module M satisfying any of the equivalent conditions as above is called a* Cohen-Macaulay *module of dimension r.*
- *The algebra C is called Cohen-Macaulay if it is Cohen-Macaulay as a module over itself.*
- *Let A be an associative algebra which is finitely generated over its center $\mathcal{Z}(A)$. We say that an A-module M is Cohen-Macaulay of dimension r if it is Cohen-Macaulay of dimension r over $\mathcal{Z}(A)$.*
- *We say that A is Cohen-Macaulay of dimension r if A is a Cohen-Macaulay module of dimension r over itself.*
- *The notions above extends in a natural way to the case of coherent sheaves over schemes of finite types over $\mathbb{C}$.*

Remark 2.2.4.

- *The property of being Cohen-Macaulay of dimension r is a local property. This stops to be true if the dimension is not fixed.*
- *In some sources the notion Cohen-Macaulay is used for modules that satisfy the Cohen-Macaulay only locally (over A).*
- *In each of the situations above we will consider the corresponding $\mathbb{C}$-local notion. For example, the module M is said to be locally Cohen-Macaulay if locally on* $\mathrm{Spec}_{\mathbb{C}}(A)$ *this is true.*

We recall the following Proposition:

Proposition 2.2.5 (see §2 of [BBG97])**.** *Let C, D be commutative algebras.*

- *Let M be an $C \times D$-module which is finitely generated over C and finitely generated over D. Then M is r-dimensional Cohen-Macaulay module over-C if and only if it is r-dimensional Cohen-Macaulay module over D.*

- *Let M, N be two Cohen-Macaulay modules of dimension r. Then any extension L of M and N is Cohen-Macaulay of dimension r.*
- *A direct summand of Cohen-Macaulay module of dimension r is a Cohen-Macaulay module of dimension r.*

2.2.2. *Categories and Objects.* We will consider abelian categories enriched over $\mathbb{C}$ (i.e. Homomorphism spaces equipped with a structure of $\mathbb{C}$-vector spaces). Let us recall some notions regarding such categories.

Definition 2.2.6. *Let $\mathcal{C}$ be an abelian category.*

- *An object $M \in Ob(\mathcal{C})$ is called* finitely generated *iff the functor $H_M := \mathrm{Hom}(M, \cdot)$ commutes with direct sums.*
- *An object $M \in Ob(\mathcal{C})$ is called is* projective *iff H_M is exact*
- *An object $M \in Ob(\mathcal{C})$ is called a* projective generator *if H_M is exact and faithful.*
- *Denote by $\mathcal{Z}(\mathcal{C}) := \mathrm{End}_{\mathcal{C}}(Id)$ the algebra of endomorphism of the identity functor $Id : \mathcal{C} \to \mathcal{C}$. The algebra $\mathcal{Z}(\mathcal{C})$ is called the* Bernstein center *of $\mathcal{C}$.*

We record

Lemma 2.2.7 (see [P70, §4.11, Theorem 1])**.** *Let P be a finitely generated projective generator of an abelian category $\mathcal{C}$ that is closed under direct limits. Then $H_P : \mathcal{C} \to \mathcal{M}(\mathrm{End}(P))$ is an equivalence of categories.*

Definition 2.2.8. *A category $\mathcal{C}$ is called* **finite over the center** *if*

1. *$\mathcal{C}$ is Abelian and closed with respect to direct limits.*
2. *Any finitely generated object admits a surjection from a finitely generated projective object.*
3. *Any object in $\mathcal{C}$ is a direct limit of finitely generated objects.*
4. *For every finitely generated $M, N \in Ob(\mathcal{C})$ the $\mathcal{Z}(\mathcal{C})$-module $\mathrm{Hom}_{\mathcal{C}}(M, N)$ is finitely generated over a finitely generated subalgebra $B \subset \mathcal{Z}(\mathcal{C})$.*

The following simple lemma provide an example of such categories.

Lemma 2.2.9. *Let $\mathcal{C} = \mathcal{M}(A)$ be the category of modules over an associative algebra A. Suppose that A is finite over its center $\mathcal{Z}(A)$. Then, $\mathcal{C}$ is finite is over the center.*

The following proposition follows immediately from Proposition 2.2.5.

Proposition 2.2.10. *Let $\mathcal{C}$ be a category that is finite over its center and $X \in Ob(\mathcal{C})$ a finitely generated object. Let $P \in Ob(\mathcal{C})$ be a finitely generated projective object. The following are equivalent:*

1. *$Hom(P, X)$ is Cohen-Macaulay over $\mathrm{End}(P)$.*
2. *$Hom(P, X)$ is Cohen-Macaulay over $\mathcal{Z}(\mathcal{C})$.*
3. *P admits a surjection from a finitely generated projective object P' such that $Hom(P', X)$ is Cohen-Macaulay over $\mathcal{Z}(\mathcal{C})$.*

Definition 2.2.11. *Let $\mathcal{C}$ be a category that is finite over its center. A finitely generated object X is called* Cohen-Macaulay *if for any finitely generated projective object P the equivalent conditions of the proposition above (Proposition 2.2.10) are satisfied.*

Proposition 2.2.12. *Let $\mathcal{C}$ be a category that is finite over its center. The following are equivalent:*

1. *For any finitely generated projective $P, Q \in Ob(\mathcal{C})$, $Hom(P, Q)$ is Cohen-Macaulay over $\mathcal{Z}(\mathcal{C})$.*
2. *For any finitely generated projective $P, Q \in Ob(\mathcal{C})$, $Hom(P, Q)$ is Cohen-Macaulay over $\mathrm{End}(P)$.*
3. *For any finitely generated projective $P, Q \in Ob(\mathcal{C})$, $Hom(P, Q)$ is Cohen-Macaulay over $\mathrm{End}(Q)$.*
4. *Any finitely generated projective object is Cohen-Macaulay object.*
5. *For any finitely generated projective $P \in Ob(\mathcal{C})$, $\mathrm{End}(P)$ is Cohen-Macaulay.*
6. *Any finitely generated projective object in $\mathcal{C}$, admits a surjection from Cohen-Macaulay object.*

Proof. The equivalence of (1),(2),(3),(4),(6) and the implication (3)⇒(5) are clear. Item (5) implies (3) by considering $\mathrm{End}(P \oplus Q)$. □

Definition 2.2.13. *Let $\mathcal{C}$ be a category that is finite over the center. If each of the above conditions is satisfied we say that $\mathcal{C}$ is a* Cohen-Macaulay *category.*

Definition 2.2.14 (cf. [BBG97] §4.1)**.** *Let $\mathcal{C}$ be an abelian category and let $S \subset \mathcal{Z}(\mathcal{C})$ be a multiplicative closed subset. We say that a category $\mathcal{C}_S$ equipped with a functor $L_S : \mathcal{C} \to \mathcal{C}_S$ is a localization of $\mathcal{C}$ with respect to S if $L := L_S$ is universal with respect to the property that for any $X \in \mathcal{C}$ and $f \in S$ we have $L(f_X) : L(X) \to L(X)$ is invertible, where $f_X \in \mathrm{End}_{\mathcal{C}}(X)$ corresponds to the action of f on the object X.*

Lemma 2.2.15 (see [BBG97] §4.1)**.** *Let $\mathcal{C}$ be an abelian category that is closed under direct limits. Let S be a multiplicative closed set in $\mathcal{Z}(C)$. A category $\mathcal{C}_S$ exists and is unique up to a natural equivalence.*

We can now give a local counterpart of the above notions.

Definition 2.2.16. *Let $\mathcal{C}$ be an abelian category.*

(1) An object $M \in \mathcal{C}$ is called locally finitely generated *iff it is finitely generated locally on $\mathrm{Spec}_{\mathbb{C}}(\mathcal{Z}(\mathcal{C}))$. Namely for any $x \in \mathrm{Spec}_{\mathbb{C}}(\mathcal{Z}(\mathcal{C}))$ there exists $S \subset \mathcal{Z}(\mathcal{C})$ such that*
   - *for each $f \in S$ we have $f(x) \neq 0$ where $f(x) := x(f)$ is the value of the homomorphism x in the element $f \in \mathcal{Z}(\mathcal{C})$.*
   - *$S^{-1}M := L_S(M)$ is a finitely generated object in $\mathcal{C}_S$.*

(2) A locally finitely generated object $M \in \mathcal{C}$ is called locally Cohen-Macaulay *iff it is Cohen-Macaulay locally on $\mathrm{Spec}_{\mathbb{C}}(\mathcal{Z}(\mathcal{C}))$.*

(3) We call $\mathcal{C}$ locally Cohen-Macaulay *iff it is Cohen-Macaulay locally on $\mathrm{Spec}_{\mathbb{C}}(\mathcal{Z}(\mathcal{C}))$.*

The following straightforward proposition is the local conterpart of Proposition 2.2.12.

Proposition 2.2.17. *Let $\mathcal{C}$ be a category that is finite over its center. The following are equivalent:*

(1) The category $\mathcal{C}$ is locally Cohen-Macaulay.

(2) For any locally finitely generated projective $P, Q \in Ob(\mathcal{C})$, $Hom(P, Q)$ is locally Cohen-Macaulay over $\mathcal{Z}(\mathcal{C})$.

(3) For any locally finitely generated projective $P, Q \in Ob(\mathcal{C})$, $Hom(P, Q)$ is locally Cohen-Macaulay over $\mathrm{End}(P)$.

(4) For any locally finitely generated projective $P, Q \in Ob(\mathcal{C})$, $Hom(P, Q)$ is locally Cohen-Macaulay over $\mathrm{End}(Q)$.

(5) Any locally finitely generated projective object is locally Cohen-Macaulay object.

(6) For any locally finitely generated projective object $P \in Ob(\mathcal{C})$, $\mathrm{End}(P)$ is locally Cohen-Macaulay.

(7) Any locally finitely generated projective object in $\mathcal{C}$, admits a surjection from locally Cohen-Macaulay object.

2.3. **The Category $\mathcal{M}(G)$.**

2.3.1. *Representation Theory of l-Groups.* We will use the terminology of l-spaces and l-groups introduced in [BZ76]. An l-space is a locally compact second countable totally disconnected topological space. An l-group is a l-space with a continuous group structure. For further background on l-spaces, l-groups and their representations we refer the reader to [BZ76]. Let us recall the basic notions from there.

Notation 2.3.1. *Let X be an l-space and G be an l-group.*

(1) Denote by $\mathcal{S}(X)$ the space of locally constant compactly supported $\mathbb{C}$-valued functions on X.

(2) For an analytic variety X over a non-Archimedean local field, we denote by $\mu(X)$ the space of locally constant compactly supported measures on X.

(3) *For an open compact subgroup $K \subset G$ we denote by $\mathcal{H}(G,K) = \mathcal{H}_K(G)$ the Hecke algebra of G w.r.t. K, i.e. the algebra of compactly supported measures on G that are invariant w.r.t. both left and right multiplication by K. The algebra $\mathcal{H}(G,K)$ is unital and we denote its unit by e_K.*
(4) *Denote $\mathcal{H}(G) = \bigcup_K \mathcal{H}(G,K)$.*
(5) *Denote by $\mathcal{M}(G)$ the category of smooth $\mathbb{C}$-representations of G, i.e representations of G s.t. each vector have an open stabilizer.*

Theorem 2.3.2 (See [BZ76])**.** *The category is canonically equivalent to the category of non-degenerate modules over $\mathcal{H}(G)$, i.e. modules M s.t. $M = \mathcal{H}(G)M$.*

Notation 2.3.3. *Let G be an l-group.*

(1) *For a closed subgroup $H < G$ we denote by $ind_H^G : \mathcal{M}(H) \to \mathcal{M}(G)$ the compactly supported induction functor and by $Ind_H^G : \mathcal{M}(H) \to \mathcal{M}(G)$ the full induction functor.*
(2) *Denote by Δ_G the modular character of G.*
(3) *Let $V \in \mathcal{M}(G)$ and χ be a character of $G(F)$. Denote $V[G,\chi] := \operatorname{Span}\{\chi(g)v - gv \mid v \in V,\, g \in G\}$ and the co-equivariants $V_{G,\chi} := V/V[G,\chi]$. If χ is trivial we omit it from the notation.*
(4) *For a representation V of G define its smooth part by $V^\infty := \{v \in V | stab_G(v) \text{ is open}\}$.*
(5) *For a smooth representation V of G define its smooth dual $\tilde{V}$ to be the smooth part of its dual V^*.*
(6) *For a smooth representation V, a vector $v \in V$ and a functional $\xi \in V^*$ define the matrix coefficient by $m_{v,\xi}(g) := \xi(gv)$.*
(7) *A smooth representation V of G is called* compact *if for any $v \in V$ and $\xi \in \widetilde{V}$ the matrix coefficient is a compactly supported function on G.*

Theorem 2.3.4 (Bernstein-Zelevinsky)**.** *Let G be an l-group. Then any compact representation of G is a projective object in the category $\mathcal{M}(G)$.*

2.3.2. *Representation Theory of Reductive p-adic Groups.*

Definition 2.3.5. *Let G be a reductive group.*

(1) *Denote $\mathcal{M}(G) := \mathcal{M}(G(F))$. We will use Hom_G instead of $\operatorname{Hom}_{G(F)}$ in order to denote the homomorphism space between two objects in this category.*
(2) *Let $P < G$ be a parabolic subgroup with unipotent radical U, and let $M := P/U$. Such M is called the* Levi factor *of P. Note that every representation of $M(F)$ can be considered as a representation of $P(F)$ using the quotient morphism $P \twoheadrightarrow M$.*
(3) *A subgroup $M \subset P$ which is mapped isomorphically to the Levi factor is called the* Levi subgroup *of P. By "a Levi subgroup of G" we will mean the Levi subgroup of some parabolic subgroup of G.*
(4) *We will fix a minimal parabolic B of G and mostly consider parabolic subgroups of G that contain B. These are called* standard parabolic *subgroups.*
(5) *A Levi subgroup or a Levi factor of standard parabolic is called* standard Levi. *All the Levi subgroups and factors that we consider are (unless stated otherwise) standard. Every standard Levi subgroup $M \subset G$ is a Levi subgroup of a unique standard parabolic.*
(6) *Given a parabolic subgroup P of G let M be it's Levi factor. Following [BZ77] we define the* **normalized** Jacquet functor *$r_{GM}^P : \mathcal{M}(G(F)) \to \mathcal{M}(M(F))$ by*

$$r_{MG}^P(\pi) := \Delta_{P(F)}^{\frac{1}{2}}|_{M(F)}(\pi|_{P(F)})_{U(F)}.$$

When the parabolic P is standard we omit the superscript P and write $r_{MG} := r_{GM}^P$. Note that the functor r_{MG} can be applied also to $P(F)$-modules.

(7) *Define the* **normalized** parabolic induction *functor* $i_{GM}^P : \mathcal{M}(M(F)) \to \mathcal{M}(G(F))$ *by*

$$i_{GM}^P(\tau) := ind_{P(F)}^{G(F)}(\tau \otimes \Delta_{P(F)}^{\frac{1}{2}}|_{M(F)}).$$

Again we omit the P *when it is standard.*

(8) *A representation* π *of* $G(F)$ *is called cuspidal if* $r_{MG}(\pi) = 0$ *for any Levi subgroup* M *of* G.

(9) *Denote by* G^0 *the preimage in* $G(F)$ *of the maximal compact subgroup of* $G(F)/[G,G](F)$.

(10) *Denote* $G^1 := G^0\mathcal{Z}(G(F))$. *Note that* G^1 *is of finite index in* $G(F)$.

Proposition 2.3.6 (Frobenius Reciprocity)**.** *The functor* i_{GM} *is right adjoint to* r_{MG}. *Namely we have a canonical isomorphism,*

$$\mathrm{Hom}_G(W, i_{GM}(V)) \cong \mathrm{Hom}_M(r_{MG}(W), V).$$

Theorem 2.3.7 (Harish-Chandra)**.** *Let* G *be a reductive group and* V *be a cuspidal representation of* $G(F)$. *Then* $V|_{G^0}$ *is a compact representation of* G^0.

2.3.3. *Bernstein Decomposition.*

Definition 2.3.8. *Let* G *be a reductive group.*

(1) *A complex character of* $G(F)$ *is called unramified if it is trivial on* G^0. *We denote the set of all unramified characters by* $\mathfrak{X}(G) := \mathfrak{X}(G(F))$.

(2) *Note that* $G(F)/G^0$ *is a lattice and therefore we can identify* $\mathfrak{X}(G)$ *with* $(\mathbb{C}^\times)^n$. *This defines a structure of algebraic variety on* $\mathfrak{X}(G)$.

(3) *For any smooth representation* ρ *of* $G(F)$ *we denote* $\Psi(\rho) := ind_{G^0}^G(\rho|_{G^0})$.

(4) *Note that* $\Psi(\rho) \simeq \rho \otimes \mathcal{O}(\mathfrak{X}(G))$. *Here* $G(F)$ *acts by* $g \cdot (v \otimes f) = \rho(g)v \otimes h_g f$, *where* $h_g \in \mathcal{O}(\mathfrak{X}(G))$ *is given by* $h_g(\chi) = \chi(g)$. *In other words, if we consider* $\rho \otimes \mathcal{O}(\mathfrak{X}(G))$ *as a space of functions on* $\mathfrak{X}(G)$ *with values in* ρ *then, the action of* G *is on the values, via* ρ *twisted by corresponding character in each point of* $\mathfrak{X}(G)$. *The above identification gives a structure of* $\mathcal{O}(\mathfrak{X}(G))$*-module on* $\Psi(\rho)$.

Theorems 2.3.4 and 2.3.7 give the following corollary:

Corollary 2.3.9. *Let* G *be a reductive group and* ρ *be a cuspidal representation of* $G(F)$. *Then*
(i) $\rho|_{G^0}$ *is a projective object in the category* $\mathcal{M}(G^0)$.
(ii) $\Psi(\rho)$ *is a projective object in the category* $\mathcal{M}(G(F))$.

Proof. (i) Clearly follows from Theorems 2.3.4 and 2.3.7.
(ii) Note that

$$\mathrm{Hom}_G(\Psi(\rho), \pi) \cong \mathrm{Hom}_{G/G^0}(\mathcal{O}(\mathfrak{X}(G)), \mathrm{Hom}_{G^0}(\rho, \pi)),$$

for any representation π. Therefore the functor $\pi \mapsto \mathrm{Hom}_G(\Psi(\rho), \pi)$ is a composition of two exact functors and hence is exact. □

To formulate Bernstein's theory of decomposition of the category $\mathcal{M}(G)$ we need some more notations.

Definition 2.3.10.

(1) *Recall that an abelian category* $\mathcal{A}$ *is a direct sum of two abelian subcategories* $\mathcal{B}$ *and* $\mathcal{C}$, *if every object of* $\mathcal{A}$ *is isomorphic to a direct sum of an object in* $\mathcal{B}$ *and an object in* $\mathcal{C}$, *and, furthermore, that there are no non-trivial morphisms between objects of* $\mathcal{B}$ *and* $\mathcal{C}$.

(2) *Let* G *be a reductive group and* $K < G(F)$ *be a compact open subgroup. Denote*

$$\mathcal{M}(G,K) := \{V \in \mathcal{M}(G(F)) \,|V \text{ is generated by } V^K\}$$

and

$$\mathcal{M}(G,K)^\perp := \{V \in \mathcal{M}(G(F) \,|V^K = 0\}.$$

(3) *We call K a* **splitting subgroup** *if the category $\mathcal{M}(G(F))$ is the direct sum of the categories $\mathcal{M}(G,K)$ and $\mathcal{M}(G,K)^{\perp}$, and $\mathcal{M}(G,K) \cong \mathcal{M}(\mathcal{H}_K(G))$.*
(4) *We denote $\mathfrak{z}_G := \mathcal{Z}(\mathcal{M}(G)) := \mathrm{End}(Id_{\mathcal{M}(G)})$ the center of the category $\mathcal{M}(G)$. We refer to $\mathfrak{z}_G$ as the Bernstein center of G.*
(5) *We fix an Iwahori subgroup $I := I_G \subset G(F)$. This group is uniquely defined up to conjugation (see [BT1, BT2], or for a short description, see [Cas]).*
(6) *A* cuspidal datum *of G is a pair (M, ρ) where $M \subset G$ is a Levi subgroup and ρ is a cuspidal representation of $M(F)$.*
(7) *On the set of cuspidal data we define an association relation as follows. Say that $(M, \rho) \simeq (M', \rho')$ if and only if there exists $g \in G$ such that*
   - $Ad(g)M = M'$
   - $Ad(g)\rho \otimes \chi \cong \rho$ *for some* $\chi \in \mathfrak{X}(M)$
(8) *We denote by $\Omega := \Omega_G$ the set of equivalence classes of pairs (M, ρ) under association. The set Ω is called the set of Bernstein Components.*
(9) *We denote by $[M, \rho]$ the class of the pair (M, ρ).*
(10) *Let ρ be a cuspidal representation of G. Denote by $\mathfrak{I}_\rho := \{\chi \in \mathfrak{X}(G) | \chi \cdot \rho \simeq \rho\}$. Note that $\mathfrak{I}_\rho$ is a finite subgroup of $\mathfrak{X}(G)$.*
(11) *Denote by $D_\rho := \mathfrak{X}(G)/\mathfrak{I}_\rho \subset Irr(G)$ the orbit of ρ under the natural action of $\mathfrak{X}(G)$ on the set $Irr(G)$ of irreducible representations of $G(F)$.*

We will use the following statements from Bernstein's theory on the center of the category $\mathcal{M}(G)$.

Theorem 2.3.11 (Bernstein)**.** *Let G be a reductive group, $P \subset G$ be its standard parabolic subgroup and M be its Levi subgroup.*

(1) *The set of splitting subgroups defines a basis at 1 for the topology of $G(F)$.*
(2) *Bernstein second adjointness: Let $\overline{P}$ denote the parabolic subgroup of G opposite to P, and let $\overline{r}_{GM} : \mathcal{M}(G(F)) \to \mathcal{M}((F))$ denote the Jacquet functor defined using $\overline{P}$. Then $\overline{r}_{GM}$ is right adjoint to i_{GM}. Explicitly, $\mathrm{Hom}_G(i_{GM}(V), W) \cong \mathrm{Hom}_M(V, \overline{r}_{MG}(W))$.*
(3) *i_{GM} and r_{MG} maps projective objects to projective objects and finitely generated objects to finitely generated objects.*
(4) *For any irreducible cuspidal representation ρ of $M(F)$, the representation $\Psi(M, \rho) := i_{GM}(\Psi(\rho))$ is a finitely generated projective object of $\mathcal{M}(G(F))$. Denote by $\mathcal{M}_{(M,\rho)} := \mathcal{M}_{(M,\rho)}(G)$ the full subcategory of $\mathcal{M}(G(F))$ generated by $\Psi(M, \rho)$.*
(5) *The categories $\mathcal{M}_{(M,\rho)}, \mathcal{M}_{(M',\rho')}$ are equal if and only if the pairs are associated. We denote henceforth the corresponding category by $\mathcal{M}_{[M,\rho]}$.*
(6) *Let $\mathcal{R}_{(M,\rho)} := \mathrm{End}(\Psi(M, \rho))$. The functor $H_\Psi : \mathcal{M}_{[M,\rho]} \to Mod(\mathcal{R}_{(M,\rho)})$ given by $H_{\Psi(M,\rho)}(V) = \mathrm{Hom}(\Psi(M, \rho), V)$ is an equivalence of categories.*
(7) *The ring $\mathcal{R}_{(M,\rho)}$ is finitely generated over $\mathcal{O}(\mathfrak{X}(M))$ which is finitely generated over the center $\mathcal{Z}(\mathcal{R}_{(M,\rho)})$ of the ring $\mathcal{R}_{(M,\rho)}$.*
(8) *We have $\mathcal{Z}(\mathcal{M}_{[M,\rho]}) \cong \mathcal{Z}(\mathcal{R}_{(M,\rho)}) \cong \mathcal{O}(D_\rho)^{W_M}$ where $W_M = N_G(M)/M$.*
(9) *Assume that K is a splitting subgroup. Then we have*
$$\mathcal{M}(G, K) = \bigoplus_{\omega \in \Omega_G(K)} \mathcal{M}_\omega,$$
for some finite subset $\Omega_G(K) \subset \Omega$.
(10) *We have the Bernstein Decomposition*
$$\mathcal{M}(G) = \Pi_{\omega \in \Omega} \mathcal{M}_\omega.$$
In particular, for any $V \in Ob(\mathcal{M}(G))$ we have a decomposition $V = \oplus_{\omega \in \Omega} V_\omega$ with V_ω an object of $\mathcal{M}_\omega$

(11) $\mathrm{Spec}_{\mathbb{C}}(\mathfrak{Z}_G)$ *is canonically isomorphic to the collection of* G*-orbits on the set of pairs* (M,ρ) *where* $M<G$ *is a standard Levi subgroup and* ρ *an isomorphism class of a cuspidal representation of* M*. In particular, the set of components of* $\mathrm{Spec}_{\mathbb{C}}(\mathfrak{Z}_G)$ *is* Ω

(12) *Let* ρ *be a cuspidal representation of* G*. Then* $\mathcal{R}_{(G,\rho)}$ *is an Azumaya algebra over* $\mathcal{O}(D_\rho)$ *of rank* $\#\mathfrak{I}_\rho$*. That means that* $\mathcal{R}_{(G,\rho)}$ *is a projective* $\mathcal{O}(D_\rho)$*-module and over any point* $z\in\mathrm{Spec}_{\mathbb{C}}(\mathcal{O}(D_\rho))$*, the algebra* $\mathcal{R}_{(G,\rho)}|_z$ *is a matrix algebra of rank* $\#\mathfrak{I}_\rho$*. Moreover, the subalgebra* $\mathcal{O}(\mathfrak{X}(G))|_z\subset\mathcal{R}_{(G,\rho)}|_z$ *is a Cartan subalgebra (i.e. the subalgebra of diagonal matrices if we choose an appropriate basis).*

For completeness we supply exact references for the most essential among these statements.

1. For statement 1 see e.g. [BD84, pp. 15-16] and [vD, §2].
2. statement 2 see [Ber87] or [Bus01, Theorem 3].
3. For statements 5,9 see [BD84, Proposition 2.10,2.11].
4. Item 12 follows from [BR, Proposition 28]

In this language one can reformulate a classical results of Borel, Casselman, Jacquet and Matsumoto as follows.

Theorem 2.3.12 (cf. [Bo76])**.** *The Iwahori subgroup is a splitting subgroup and* $\mathcal{M}(G,I)=\mathcal{M}_{[B,1]}$*. Here* B *is the minimal parabolic subgroup of* G *and* 1 *is the trivial representation of its Levi factor.*

2.4. **The Cohen-Macualy Property of** $\mathcal{M}(G)$**.** The theory of Bernstein center give the following corollary.

Corollary 2.4.1. $\mathcal{M}(G)$ *is a finite over its center.*

Proof. We verify the items of definition 2.2.8. Items (1)-(3) follows from Theorem 2.3.2. Item (4) follows from the Bernstein decomposition (Theorem 2.3.11) □

The next unpublished result of J. Bernstein was the starting point of this work and inspired our main conjecture. We will show in §§§4.1.1 that it implies our conjecture in the group case.

Theorem 2.4.2 (Bernstein)**.** $\mathcal{M}(G)$ *is a locally Cohen-Macaulay category.*

Proof. The proof is by induction on $\dim(G)$. Let us consider $\Pi=i_{GM}(\Psi(\rho))$. We have to show that for any projective object Q the module $\mathrm{Hom}_G(\Pi,Q)$ is locally Cohen-Macaulay over $\mathrm{End}(P)$. Since

$$\mathrm{Hom}_G(\Pi,Q)=\mathrm{Hom}_G(i_{GM}(\Psi(\rho)),Q)=\mathrm{Hom}_M(\Psi(\rho),\bar{r}_{MG}(Q))$$

and both $\bar{r}_{MG}(Q)$ and $\Psi(\rho)$ are projective objects in $\mathcal{M}(M)$ we need to verify that the module $\mathrm{Hom}_M(\Psi(\rho),\bar{r}_{MG}(Q))$ is locally Cohen-Macaulay over $\mathrm{End}_M(\Psi(\rho))$.

If $\dim(M)<\dim(G)$ we conclude by induction hypothesis that $\mathcal{M}(M)$ is a locally Cohen-Macaulay category and thus the module $\mathrm{Hom}_G(\Pi,Q)$ is locally Cohen-Macaulay module over $\mathrm{End}(\Psi(\rho))$ and hence over $\mathrm{End}(\Pi)$.

When $M=G$, we may consider the case $Q\in\mathcal{M}_{[M,\rho]}$. Using the equivalence H_Π (see Definition 2.2.6) we are reduce to the statement that $\mathrm{End}_G(\Psi(\rho))$ is Cohen-Macaulay. This follows from Theorem 2.3.11(12) □

3. Cohen-Macaulay Objects in $\mathcal{M}(G)$

In this section we fix a reductive F-group G and a spherical G-variety X.

3.1. **Cohen Macaulay and Bernstein Decomposition.**

Definition 3.1.1.

- *We say that $V \in \mathcal{M}(G)$ is* Blockwise Cohen-Macaulay *iff each of the modules V_ω in the decomposition $V = \oplus V_\omega$ is a Cohen-Macaulay object in the category $\mathcal{M}_\omega$. Note that any Blockwise Cohen-Macaulay is locally Cohen-Macaulay. Clearly any Cohen-Macaulay G-module is Blockwise Cohen-Macaulay G-module.*
- *Suppose A is an unital associate algebra which is finite over its Noetherian center $\mathcal{Z}(A)$. We will say that an A-module M is* blockwise Cohen-Macaulay *if for any connected component $S_0 \subset S = \mathrm{Spec}_{\mathbb{C}}(\mathcal{Z}(A))$ the module $M|_{S_0}$ is Cohen-Macaulay.*

In the present section we will study mainly the inter-relations between Blockwise Cohen-Macaulay properties and natural operations in the category $\mathcal{M}(G)$. One can easily adapt these properties to the case of locally Cohen-Macaulay modules.

3.2. **Behavior with Respect to Compact Subgroups.** Let $V \in \mathcal{M}(G)$.

Proposition 3.2.1. *The following are equivalent:*

1. *V is a Blockwise Cohen-Macaulay.*
2. *For any compact open subgroup $K \subset G$ the module V^K is blockwise Cohen-Macaulay over $H(G,K)$.*
3. *For a fundamental system of open compact subgroups K_i (i.e. a system that form a basis for the topology at $1 \in G$), the module V^{K_i} is blockwise Cohen-Macaulay over $H(G,K_i)$ for all i.*

For the proof we will need the following straightforward lemma:

Lemma 3.2.2. *Let A be a unital commutative algebra and M a blockwise Cohen-Macaulay module over A.*

1. *Let N be a direct summand of M. Then N is a blockwise Cohen-Macaulay module over A.*
2. *Let $A \to B$ be a morphism of unital algebras. Assume that the action of A on M is coming from an action of B on M. Then M is a blockwise Cohen-Macaulay module over B.*

Proof of Proposition 3.2.1.

(2) $\Rightarrow$ (1) For any splitting subgroup K the module V^K is blockwise Cohen-Macaulay over $\mathcal{H}(G,K)$. By Theorem 2.3.11 (1) and (9) we deduce that V is Blockwise Cohen-Macaulay

(1) $\Rightarrow$ (3) By Theorem 2.3.11 (1) we may choose K_i to be the system of spliting subgroups.

(3) $\Rightarrow$ (2) Let K be an open compact subgroup. Let $K' \subset K$ an open compact subgroups such that $V^{K'}$ is blockwise Cohen-Macaulay over $\mathcal{H}(G,K')$ (and hence over $\mathcal{Z}(\mathcal{H}(G,K'))$). Since K is compact we have canonical K-invariant decomposition $V^{K'} = V^K \oplus V^{K'}[K]$. It is easy to see that this decomposition commutes with the action of $\mathcal{Z}(\mathcal{H}(G,K'))$. Thus, by lemma 3.2.2 (1) V^K is blockwise Cohen-Macaulay over $\mathcal{Z}(\mathcal{H}(G,K'))$. We have a natural map $\mathcal{Z}(\mathcal{H}(G,K')) \to \mathcal{Z}(\mathcal{H}(G,K))$. Thus by 3.2.2 (2)) V^K is blockwise Cohen-Macaulay over $\mathcal{Z}(\mathcal{H}(G,K))$.

$\square$

3.3. **Behavior with Respect to Jacquet Functors and Parabolic Induction.**

Lemma 3.3.1.

1. *The functors i_{GM}, r_{MG} preserve the blockwise Cohen-Macaulay property.*
2. *If $i_{GM}(V)$ is Blockwise Cohen-Macaulay then V is Blockwise Cohen-Macaulay.*
3. *The following are equivalent:*
   - (a) *V is Blockwise Cohen-Macaulay object.*
   - (b) *The module $r_{MG}(V)$ is Blockwise Cohen-Macaulay for any Levi M.*

(c) *For any cuspidal datum* (M, ρ) *the module* $\mathrm{Hom}_{M^0}(\rho|_{M^0}, r_{MG}(V)|_{M^0})$ *is Cohen-Macaulay over* $\mathbb{C}[M/M^0] = \mathcal{O}(\mathfrak{X}(M))$.

Proof.

(1) Since the functors i_{GM} and r_{MG} commute with direct sums, it is enough to show that they preserve the Cohen-Macaulay property. Let Q be finitely generated projective object of $\mathcal{M}(G)$ and V be a Cohen-Macaulay object of $\mathcal{M}(M)$. In order to show that i_{GM} preserve the Cohen-Macaulay property we have to show that $\mathrm{Hom}_G(Q, i_{GM}(V))$ is Cohen-Macaulay over $\mathfrak{Z}_G$. By Frobenius reciprocity $\mathrm{Hom}_G(Q, i_{GM}(V)) = \mathrm{Hom}_M(r_{MG}(Q), V)$. By Theorem 2.3.11(3) The module $r_{MG}(Q)$ is projective and finitely generated. Thus $\mathrm{Hom}_M(r_{MG}(Q), V)$ is Cohen-Macaulay over $\mathfrak{Z}_M$ and hence also over $\mathfrak{Z}_G$. The proof for r_{MG} is similar using the second adjointness (Theorem 2.3.11(2)) in place of Frobenius reciprocity.

(2) Let $\omega = [(N, \rho)]$ where $N < M$ is a Levi subgroup and ρ a cuspidal representation of N. Let $V \in Ob(\mathcal{M}_\omega(M))$. Assume that $i_{GM}(V)$ is Blockwise Cohen-Macaulay. We have to prove that V is Cohen-Macaulay. Notice that $i_{GM}(V) \in Ob(\mathcal{M}_{\omega'}(G))$ where ω' is the image of ω under the map $\Omega_M \to \Omega_G$. Thus $i_{GM}(V)$ is Cohen-Macaulay.

Let Q be a finitely generated projective of $\mathcal{M}(M)$. Assume first that $Q = r_{MG}(R)$ for some finitely generated projective R of $\mathcal{M}(G)$. Then it is easy to see that $\mathrm{Hom}_M(Q, V) = \mathrm{Hom}_M(r_{MG}(R), V)$ is Cohen-Macaulay. By Proposition 2.2.10 it is enough to show that any finitely generated projective Q' of $\mathcal{M}(M)$ admits a surjection from an object Q as above. We can take $Q' = r_{MG}(i_{GM}(Q))$.

(3)

(a) $\Rightarrow$ (b) Follows from the first item.

(b) $\Rightarrow$ (c) Recall that $\Psi(\rho) := ind_{M^0}^M(\rho|_{M^0})$ is finitely generated projective object (see Definition 2.3.8). We have, $\mathrm{Hom}_{M^0}(\rho|_{M^0}, r_{MG}(V)|_{M^0}) = \mathrm{Hom}_M(\Psi(\rho), r_{MG}(V))$ is Cohen-Macaulay over $\mathfrak{Z}_M$, and hance also over $\mathbb{C}[M/M^0]$.

(c) $\Rightarrow$ (a) Recall that $\Psi(M, \rho) := i_{GM}(\Psi(\rho))$ is a (finitely generated) projective generator of $\mathcal{M}_{[M,\rho]}$. So it is enough to show that $\mathrm{Hom}_G(\Psi(M, \rho), V)$ is Cohen-Macaulay over $\mathrm{End}(\Psi(M, \rho))$. By the assumption, the module $\mathrm{Hom}_G(\Psi(M, \rho), V) = \mathrm{Hom}_G(\Psi(\rho), \bar{r}_{MG}(V)) = \mathrm{Hom}_{M^0}(\rho|_{M^0}, \bar{r}_{MG}(V)|_{M^0})$ is Cohen-Macaulay over $\mathbb{C}[M/M^0]$. Thus it is Cohen-Macaulay over $\mathrm{End}(\Psi(M, \rho))$.

□

3.4. **Multiplicities of Cohen-Macaulay Modules.**

We begin with a standard lemma from commutative algebra.

Lemma 3.4.1. *Let A be a finitely generated commutative ring and M locally Cohen-Macaulay module over A module. Let $Z \subset \mathrm{Spec}(A)$ be the support of M considered as a scheme and let Z_{reg} be the smooth locus of the scheme Z. In particular Z_{reg} is a reduced scheme. Then the dimension of the fiber $d(z) := \dim(M_z)$ is a locally constant function on $Z_{reg}(\mathbb{C})$.*

Proof.

Step 1. Reduction to the case that M is Cohen-Macaulay of dimension d.
clear.

Step 2. Reduction to the case that $Z_{reg} = Z$.
Set $S := Z - Z_{reg}$ and $U := \mathrm{Spec}(A) - S$. Clearly $M|_U$ is Cohen-Macaulay (i.e. there exist an affine open cover $U = \bigcup U_i$ s.t. $M|_{U_i}$ is Cohen-Macaulay) So we can replace A and M by $\mathcal{O}(U_i)$ and $M|_{U_i}$.

Step 3. Reduction to the case that $Z_{reg} = Z = \mathrm{Spec}(A)$.
Let $I \subset A$ be the annihilator of M. The action of A on M factors through $A/I = \mathcal{O}(Z)$. It is easy to see that M is Cohen-Macaulay module over $\mathcal{O}(Z)$.

Step 4. The case $Z_{reg} = Z = \mathrm{Spec}(A)$.
clear.

□

We now apply these considerations to the study of multiplicities.

Notation 3.4.2. *Given a cuspidal datum (M, ρ) and $V \in \mathcal{M}(G)$ we define*

(1) the multiplicity function on $\mathfrak{X}(M)$ by

$$d_{(V,(M,\rho))}(\chi) := \dim(\mathrm{Hom}_G(V, i_{GM}(\chi\rho)))$$

(2) The support

$$\mathrm{Supp}_{(M,\rho)}(V) := \{\chi \in \mathfrak{X}(M) | d_{V,(M,\rho)}(\chi) \neq 0\}$$

We can study those multiplicities using a sheaf:

Notation 3.4.3. *Given a cuspidal pair (M, ρ) and $V \in \mathcal{M}(G)$. We define*

$$\mathcal{F}(V, (M, \rho)) := \mathrm{Hom}_G(\bar{i}_{GM}\Psi(\rho), V) = \mathrm{Hom}_M(\Psi(\rho), r_{MG}(V)) = \mathrm{Hom}_{M^0}(\rho|_{M^0}, r_{MG}(V)|_{M^0}).$$

We consider $\mathcal{F}(V, (M, \rho))$ as a module over $\mathbb{C}[M/M^0] = \mathcal{O}(\mathfrak{X}(M))$ and view it as a sheaf over $\mathfrak{X}(M)$

The following lemma gives the interpretation of multiplicities in terms of this sheaf.

Lemma 3.4.4.

$$d_{V,(M,\rho)}(\chi) = \dim \mathcal{F}(V, (M, \rho))|_\chi$$

For the proof we need the following standard lemma:

Lemma 3.4.5. *Let C be a commutative finitely generated algebra over $\mathbb{C}$ without nilpotent elements. Let A be an Azumaya algebra over C and let M be a finitely generated A-module. Let $T \subset A$ be a commutative subalgebra such that for any $x \in spec_{\mathbb{C}}(C)$, $T|_x$ is a Cartan subalgebra of the matrix algebra $A|_x$. Let δ be an irreducible module over A. Let $\{x\} := Supp_C(\delta) \subset Spec(C)$ be the support of δ as a C-module. Let $y \in Spec_{\mathbb{C}}(T)$ be a point in the fiber of x. Then*

$$\dim Hom_A(M, \delta) = \dim M|_y.$$

Proof of Lemma 3.4.4.

$$d_{V,(M,\rho)}(\chi) = \dim(\mathrm{Hom}_G(V, i_{GM}(\chi\rho))) = \dim(\mathrm{Hom}_M(r_{MG}(V), \chi\rho))$$

By Lemma 2.2.7 and Theorem 2.3.11 (4)

$$\mathrm{Hom}_M(r_{MG}(V), \chi\rho) = \mathrm{Hom}_{End(\Psi(\rho))}(\mathrm{Hom}_M(\Psi(\rho), r_{MG}(V)), \mathrm{Hom}_M(\Psi(\rho), \chi\rho)),$$

and $\mathrm{Hom}_M(\Psi(\rho), \chi\rho)$ is irreducible module over $End(\Psi(\rho))$.

Now Lemma 3.4.5 and Theorem 2.3.11(12) implies the assersion. □

Remark 3.4.6. *In view of the last lemma (Lemma3.4.4) we will consider the set $\mathrm{Supp}_{(M,\rho)}(V)$ as a scheme by identifying it with the support of $\mathcal{F}(V, (M, \rho))$.*

Combining Lemma 3.4.4 with Lemma 3.4.1 we obtain the following corollary

Corollary 3.4.7. *Let (M, ρ) be a cuspidal pair and $V \in \mathcal{M}(G)$. Suppose that $V_{[M,\rho]}$ is Cohen-Macaulay. Let $Z := \mathrm{Supp}_{(M,\rho)}(V)$. Then $d_{[V,(M,\rho)]}$ is locally constant on $Z_{reg}(\mathbb{C})$.*

3.5. **The Iwahori Block.** Let us now restrict our considerations to Iwahori block of the category $\mathcal{M}(G)$. To describe the Cohen-Macaulay property for this block we will fix the following data and notations.

- Let B a minimal parabolic of G.
- Let $B = TU$ a Levi decomposition.
- We will write $T = T^0 A$ with A a maximal split torus in G.
- We fix $K_0 \subset G(F)$ maximal compact subgroup containing I.
- We fix $\pi \in \mathcal{O}_F$ a uniformizer.
- Set $\Lambda = \mathcal{X}_*(A)$ be the group of co-weights, i.e. co-characters of A.
- Set $\Lambda^{++} \subset \Lambda$ be the cone of dominant co-weights, i.e. co-weights which are paired positively with the positive roots in $X^*(A)$.
- For $\lambda \in \Lambda$ we set $\pi^\lambda = \lambda(\pi)$.
- We denote by $\mathcal{B}$ the Bernstein subalgebra of $\mathcal{H}(G, I)$, i.e. the commutative subalgebra generated by $\{e_I \delta_{\pi^\lambda} e_I | \lambda \in \Lambda^{++}\}$ and their inverses (see [Lus83]).
- Let $\omega_0 = [T, 1] \in \Omega$ be the principal Bernstein block.

Definition 3.5.1. *We call a representation $V \in \mathcal{M}(G)$ an* I-Cohen-Macaulay *if V^I is Cohen-Macaulay module over $\mathcal{H}(G, I)$.*

Repeating the argument from §3.2 and §3.3 we obtain the following proposition.

Proposition 3.5.2. *Let $V \in \mathcal{M}(G)$.*

(1) The following are equivalent
  - *(a) V is I-Cohen-Macaulay.*
  - *(b) V_{ω_0} is a Cohen-Macaulay object of $\mathcal{M}(G)$.*
  - *(c) $(r_{GT}(V))^{T^0}$ is Cohen-Macaulay module over $\mathbb{C}[\Lambda] = \mathbb{C}[T/T^0] = \mathcal{O}(\mathfrak{X}(M))$.*
  - *(d) V^I is a Cohen-Macaulay module over $\mathcal{B}$.*

(2) Assume that V is I-Cohen-Macaulay. Then
  - *(a) V^{K_0} is Cohen-Macaulay module over $\mathcal{H}(G, K_0)$.*
  - *(b) $d_{V,(T^0,1)}$ is a locally constant function on $\mathrm{Supp}_{(T^0,1)}(V)_{reg}$.*

4. The Main Conjecture and its Consequences

In this section we fix a reductive F-group G and a spherical G-variety X.

4.1. **Relative Analogue of Bernstein's Theorem.** The following is a standard conjecture in Harmonic analysis on spherical pairs.

Conjecture 4.1.1. *Let π be an irreducible smooth representation of $G(F)$. Then the multiplicity* $\dim\mathrm{Hom}_G(\mathcal{S}(X(F)), \pi)$ *is finite.*

This conjecture was proven in many cases (see [Del] for G general and X a symmetric spaces and in [SV] for G split and X a *wave front* spherical variety) and we will assume its validity for the G-space X. In [AAG11, AGS] it is shown that this conjecture is equivalent to the following one:

Conjecture 4.1.2. *Let G, X be as above. The G-module $\mathcal{S}(X(F))$ is locally finitely generated (i.e. for any open compact $K < G$, the module $\mathcal{S}(X(F))^K$ is finitely generated over the Hecke algebra $\mathcal{H}(G, K)$)*

Let us now formulate the main conjecture of this paper.

Conjecture 4.1.3 (Main Conjecture)**.** *With the notations as above $\mathcal{S}(X(F))$ is a blockwise Cohen-Macaulay object in $\mathcal{M}(G)$.*

In view of the results of §3 (particularly Propositions 3.2.1 and 3.3.1) the main conjecture have several reformulations, listed in the next Corollary:

Corollary 4.1.4. *The following are equivalent:*

1. *The $G(F)$-module $\mathcal{S}(X(F))$ is a blockwise Cohen-Macaulay object in $\mathcal{M}(G)$.*
2. *For any $K \subset G(F)$ a splitting compact open subgroups the $\mathcal{H}(G,K)$-module $\mathcal{S}(X(F))^K$ is blockwise Cohen-Macaulay.*
3. *For any $K \subset G(F)$ a splitting compact open subgroups the $\mathcal{Z}(\mathcal{H}(G,K))$-module $\mathcal{S}(X(F))^K$ is blockwise Cohen-Macaulay.*
4. *For any $K \subset G(F)$ a compact open subgroups the $\mathcal{Z}(\mathcal{H}(G,K))$-module $\mathcal{S}(X(F))^K$ is blockwise Cohen-Macaulay.*
5. *For any $K \subset G(F)$ a compact open subgroups the $\mathcal{H}(G,K)$-module $\mathcal{S}(X(F))^K$ is blockwise Cohen-Macaulay.*
6. *For any Levi subgroup $M \subset G$ the module $r_{MG}(\mathcal{S}(X(F)))$ is a Blockwise Cohen-Macaulay object in $\mathcal{M}(M)$.*
7. *For any Levi subgroup $M \subset G$ and a cuspidal representation ρ of $M(F)$, the sheaf $\mathcal{F}(\mathcal{S}(X(F)),(M,\rho))$ is Cohen-Macaulay.*

We are especially interested in the following special case of Conjecture 4.1.3:

Conjecture 4.1.5. *The module $\mathcal{S}(X(F))$ is I-Cohen-Macaulay.*

In the sequel we will prove Conjectures 4.1.3 and 4.1.5 in some cases and show that they have a number of interesting consequences to representation theory and Harmonic analysis.

4.1.1. *The Group Case.* Bernstein's Theorem (Theorem 2.4.2) implies the following result.

Theorem 4.1.6. *Conjecture 4.1.3 is satisfied in the group case. Namely, $\mathcal{S}(G(F))$ is a Blockwise Cohen-Macaulay object of $\mathcal{M}(G \times G)$.*

Proof. Fix a open compact splitting subgroup $K \subset G$.

Step 1 $\mathcal{H}(G,K)$ is locally Cohen-Macaulay module over itself.
Consider the projective object $V := \mathcal{S}(G(F)/K) \in \mathcal{M}(G)$. By Theorem 2.4.2 it is locally Cohen-Macaulay. Thus, by Proposition 3.2.1 $\mathcal{H}(G,K) = V^K$ is locally Cohen-Macaulay module over $\mathcal{H}(G,K)$.

Step 2 $\mathcal{H}(G,K)$ is blockwise Cohen-Macaulay module over $\mathcal{Z}(\mathcal{H}(G,K))$.
Decompose $\mathcal{H}(G,K) = \bigoplus_{\omega\in\Omega_G(K)} \mathcal{H}(G,K)_\omega$. We know that $\mathcal{H}(G,K)_\omega$ is locally Cohen-Macaulay. We have to show that it is Cohen-Macaulay. So we have to show that its support is an equi-dimensional subvariety of $\operatorname{Spec}(\mathcal{Z}(\mathcal{H}(G,K)_\omega))$. Clearly its support is the entire $\operatorname{Spec}\mathcal{Z}(\mathcal{H}(G,K)_\omega)$. On the other hand Theorem 2.3.11 easily implies that $\operatorname{Spec}\mathcal{Z}(\mathcal{H}(G,K)_\omega)$ is irreducible.

Step 3 $\mathcal{H}(G,K)$ is blockwise Cohen-Macaulay module over $\mathcal{Z}(\mathcal{H}(G,K)) \times \mathcal{Z}(\mathcal{H}(G,K))$.
Follows from the previous step and Lemma 3.2.2

Step 4 $\mathcal{S}(G(F))$ is a Blockwise Cohen-Macaulay object of $\mathcal{M}(G \times G)$.
Follows from the previous step and Proposition 3.2.1.

□

4.2. **The Sheaf $\mathcal{F}(\mathcal{S}(X(F)),(M,\rho))$.** In this subsection we assume that X is affine and hence the stabilizer of a point in X is a reductive group. Our goal here is to describe the sheaf $\mathcal{F}(\mathcal{S}(X(F)),(M,\rho))$. In view of Corollary 4.1.4(7), this will give us a better understanding of the main conjecture. We will start with the case $M = G$.

Theorem 4.2.1. *Let ρ be a cuspidal representation of G , and let $\mathcal{L}$ be a $G(F)$-equivariant line bundle over $X(F)$. Then, the sheaf $\mathcal{F}(\mathcal{S}(X(F),\mathcal{L}),(G,\rho))$ is a direct image of a locally free sheaf on a smooth subvaraeity of $\mathfrak{X}(G)$. In particular it is Cohen-Macaulay.*

For the proof we will need the following lemmas

Lemma 4.2.2. *Let $\phi : S \to S'$ be finite étale map of algebraic varieties. Let $\mathcal{F}$ be a coherent sheaf over S. Suppose that $\phi_*(\mathcal{F})$ is a direct image of a locally free sheaf on a smooth subvariety of S'. Then, so $\mathcal{F}$ is a direct image of a locally free sheaf on a smooth subvariety of S.*

Proof. Without loss of generality we can assume that $\phi_*(\mathcal{F})$ is a locally free sheaf on S'. Now recall that a sheaf is locally free if and only if it is locally free in the étale topology. So we can assume that the map ϕ is a projection from a product of S' by a reduced zero dimensional scheme. In this case the assertion follows from the fact that a direct summand of a locally free sheaf is locally free. □

Lemma 4.2.3. *Let $\Lambda_{\mathcal{Z}} = \mathcal{X}_*(\mathcal{Z}(G))$. Consider the map $\Lambda_{\mathcal{Z}} \to G$ given by evaluation at the uniformizer π, and consider $\Lambda_{\mathcal{Z}}$ as a subset of G. Let $x \in X(F)$ Then $\Lambda_{\mathcal{Z}} \cap G(F)_x$ has finite index in $\Lambda_{\mathcal{Z}} \cap (G(F)_x \cdot G^0)$, where $G(F)_x$ is the stabilizer of x in $G(F)$.*

Proof. Let $H := G_x$ be the stabilizer and let H^0 and $H^1 := H^0 \mathcal{Z}(H(F))$ be defined analogously to G^0 and G^1. Let $\Lambda_{\mathcal{Z}(H)} = \mathcal{X}_*(\mathcal{Z}(H))$. We have the following commutative diagram:

$$\begin{tikzcd}
\Lambda_{\mathcal{Z}} \cap (G(F)_x \cdot G^0) \arrow[r] & \Lambda_{\mathcal{Z}} \arrow[r, "i_G"] & G(F) \arrow[r, "p_G"] & G(F)/G^0 \\
 & & H(F) \arrow[u, "i"] \arrow[r, "p_H"] & H(F)/H^0 \arrow[u, "i_0"] \\
\Lambda_{\mathcal{Z}} \cap G(F)_x \arrow[uu] \arrow[r] & \Lambda_{\mathcal{Z}(H)} \arrow[r, "i_H"] & H^1 \arrow[u, "j"] \arrow[r, "p_{H^1}"] & H^1/H^0 \arrow[u, "j_0"]
\end{tikzcd}$$

We have $\Lambda_{\mathcal{Z}} \cap (G(F)_x \cdot G^0) = (p_G \circ i_G)^{-1}(Im(i_0))$. Also, since $\Lambda_{\mathcal{Z}} \cap G^0$ is trivial, we have $\Lambda_{\mathcal{Z}} \cap G(F)_x = (p_G \circ i_G)^{-1}(Im(i_0 \circ j_0 \circ p_{H^1} \circ i_H))$. Since $p_{H^1} \circ i_H$ is onto, we have $\Lambda_{\mathcal{Z}} \cap G(F)_x = (p_G \circ i_G)^{-1}(Im(i_0 \circ j_0))$. The assertion follows now from the fact that $Im(j_0)$ has finite index in $H(F)/H^0$. □

Proof of Theorem 4.2.1.

Step 1. Proof for the case when $G(F) = G^1 := G^0 \mathcal{Z}(G(F))$

Let $\Lambda_{\mathcal{Z}} = \mathcal{X}_*(\mathcal{Z}(G))$ be as in the above lemma (Lemma 4.2.3). Decompose $G(F) = G^0 \times \Lambda_{\mathcal{Z}}$. Fix $x \in X(F)$. Let $\Lambda^0_{\mathcal{Z}} := \Lambda_{\mathcal{Z}} \cap G(F)_x$ and $\Lambda^1_{\mathcal{Z}} := \Lambda_{\mathcal{Z}} \cap (G(F)_x \cdot G^0)$. We can decompose $\Lambda_{\mathcal{Z}} := \Lambda^2_{\mathcal{Z}} \oplus \Lambda^3_{\mathcal{Z}}$, s.t. $\Lambda^1_{\mathcal{Z}}$ is a subgroup of finite index in $\Lambda^2_{\mathcal{Z}}$. We define $X^0 := \Lambda^2_{\mathcal{Z}} \cdot G^0 \cdot x$.

Using the fact that G^0 is normal in G, we get

$$X(F) \cong X^0 \times \Lambda^3_{\mathcal{Z}}$$

as $G^0 \cdot \Lambda^3_{\mathcal{Z}} \cdot \Lambda^0_{\mathcal{Z}} = G^0 \times \Lambda^3_{\mathcal{Z}} \times \Lambda^0_{\mathcal{Z}}$-spaces. Here the action of $\Lambda^0_{\mathcal{Z}}$ on $X^0 \times \Lambda^3_{\mathcal{Z}}$ is trivial, the action of $\Lambda^3_{\mathcal{Z}}$ is on the second component and the action of G^0 is on the first.

Now consider the fiber $\mathcal{L}|_x$ as a character of $G^0_x \times \Lambda^0_{\mathcal{Z}}$ and decompose it into a product $\chi_1 \otimes \chi_2$.

Thus we have isomorphisms of $G^0 \times \Lambda^3_{\mathcal{Z}} \times \Lambda^0_{\mathcal{Z}}$ representations:

$$\mathcal{F}(\mathcal{S}(X(F), \mathcal{L}), (G, \rho)) = \mathrm{Hom}_{G^0}(\rho, \mathcal{S}(X(F), \mathcal{L})) \cong \mathrm{Hom}_{G^0}(\rho, \mathcal{S}(X^0, \mathcal{L}|_{X^0})) \otimes \mathbb{C}[\Lambda^3_{\mathcal{Z}}] \otimes \chi_2$$

Let $L := \mathrm{Spec}(\mathbb{C}[\Lambda^3_{\mathcal{Z}} \times \Lambda^0_{\mathcal{Z}}])$. By Lemma 4.2.3 the map $\pi : \mathfrak{X}(G) = \mathrm{Spec}(\mathbb{C}[G/G^0]) \to L$ is finite étale morphism. We see that $\pi_*(\mathcal{F}(\mathcal{S}(X(F), \mathcal{L}), (G, \rho)))$ is a direct image of a free sheaf on $\mathrm{Spec}(\mathbb{C}[\Lambda^3_{\mathcal{Z}}])$ which is a smooth subvariety of L. Thus Lemma 4.2.2 implies the assertion.

Step 2. Proof for the general case.

as in the previous steps consider $\Lambda_{\mathcal{Z}} \subset G/G^0$. this give a finite étale map

$$\mathfrak{X}(G) = \mathrm{Spec}(\mathbb{C}[G/G^0]) \xrightarrow{p} \mathrm{Spec}(\mathbb{C}[\Lambda_{\mathcal{Z}}]) \cong \mathbb{C}[G^1/G^0].$$

Similarly to the previous step, the sheaf $p_*(\mathcal{F}(\mathcal{S}(X,\mathcal{L}),(G,\rho)))$ is a direct image of a free sheaf on a smooth subvariety of $\mathrm{Spec}(\mathbb{C}[\Lambda_{\mathcal{Z}}])$. Again Lemma 4.2.2 implies the assertion.

□

This theorem implies our main conjecture for the cuspidal blocks. Namely we have:

Corollary 4.2.4. *Let X be a G-spherical space and ρ be a cuspidal representation of $G(F)$. Then $\mathcal{S}(X)_{[G,\rho]}$ is Cohen-Macaulay.*

In order to study the structure of the sheaf $\mathcal{F}(\mathcal{S}(X(F)),(M,\rho))$ for general M, let us introduce the following notation.

Notation 4.2.5. *Let $P<G$ be a parabolic subgroup. Let M be its Levi factor. Let $x\in X(F)$. Let ρ be a cuspidal representation of $M(F)$. Define*

1. *P_x the stabilizer of x in P.*
2. *$M(x)$ the projection of P_x to M.*
3. *$X_{P,x} := U\backslash(P\cdot x) = M/M(x)$.*
4. *Let $\mathcal{L}_{P,x}$ be the natural M-equivariant line bundle on $X_{P,x}$ s.t. $r_{MG}(\mathcal{S}(P\cdot x)) = \mathcal{S}(X_{P,x},\mathcal{L}_{P,x})$*
5. *Let $\tilde{\mathcal{L}}_{P,x}$ be the natural M-equivariant line bundle on $X_{P,x}$ s.t. $C^\infty(X_{P,x},\tilde{\mathcal{L}}_{P,x})$ is the smooth contragredient of $r_{MG}(\mathcal{S}(P\cdot x))$*
6. *Consider the fiber $(\tilde{\mathcal{L}}_{P,x})|_x$ as a character of $M(x)$ and denote it by $\chi_{p,x}$*

The next lemma is a slight generalization of [AAG11, Lemma 2.2.4]

Lemma 4.2.6 ([AGS, Lemma B.3.3])**.** *Let $P<G$ be a parabolic subgroup and M be its Levi factor. Let $x\in X(F)$. Then*

1. *$X_{P,x}$ is M-spherical.*
2. *Suppose that Conjecture 4.1.2 holds for X. Then it holds for the pair $(X_{P,x},\mathcal{L}_{P,x})$, i.e. $\mathcal{S}(X_{P,x},\mathcal{L}_{P,x})^K$ is finitely generated over the Hecke algebra $\mathcal{H}_K(G)$ for any open compact subgroup $K\subset G$.*

Proposition 4.2.7. *Let (M,ρ) be a cuspidal datum of the group G. Choose a finite set $\mathfrak{R}\subset X(F)$ of representative for the P orbits in X (those that have F points). Then $\mathcal{F}(\mathcal{S}(X(F)),(M,\rho))$ admits a natural filtration, s.t. $\mathrm{gr}\mathcal{F}(\mathcal{S}(X(F)),(M,\rho)) = \bigoplus_{x\in\mathfrak{R}}\mathcal{F}(\mathcal{S}(X_{P,x}(F),\mathcal{L}_{P,x_i}),(M,\rho))$*

Proof. Stratify X by its P-orbits $\{O_1\dots O_n\}$. This gives a filtration of $\mathcal{S}(X(F))$ with $gr_i(\mathcal{S}(X(F)))=\mathcal{S}(O_i(F))$. Consider $r_{MG}(\mathcal{S}(X(F)))$ equipped with the induced filtration. Since the functor of U-coinvariants is exact, we obtain $\mathrm{gr}_i(r_{MG}(\mathcal{S}(X(F)))) = r_{MG}(\mathcal{S}(O_i(F))) = \mathcal{S}(X_{P,x}(F),\mathcal{L}_{P,x_i})$ where $\{x_i\} = O_i(F)\cap\mathfrak{R}$ (here we assume $O_i(F)$ is non-empty otherwise we can ignore this orbit). Thus

$$\mathrm{gr}_i(\mathcal{F}(\mathcal{S}(X(F)),(M,\rho))) = \mathrm{gr}_i(\mathrm{Hom}_{M^0}(\rho,r_{MG}(\mathcal{S}(X(F))))) = \mathrm{Hom}_{M^0}(\rho,\mathcal{S}(X_{P,x_i}(F),\mathcal{L}_{P,x_i})) = \\ = \mathcal{F}(\mathcal{S}(X_{P,x_i}(F),\mathcal{L}_{P,x_i}),(M,\rho))$$

□

Theorem 4.2.1 and Proposition 4.2.7 implies certain part of the main conjecture. In order to formulate it let us introduce the following notation:

Notation 4.2.8. *Let $P<G$ be a parabolic subgroup. Let M be its Levi factor. Let $x\in X(F)$. Let ρ be a cuspidal representation of $M(F)$. Define:*

1. *$\mathfrak{X}(M)_{x,\rho} = \mathrm{Supp}_{(M,\rho)}(\mathcal{S}(X_{P,x}(F),\mathcal{L}_{P,x})) = \{\chi\in\mathfrak{X}(M)|(\chi_{P,x}^{-1}\chi^{-1}\tilde{\rho})^{M(x)}\neq 0\}$,*
2. *The significance of $x\in X$ to be $Sig^{(M,\rho)}(x) := dim(\mathfrak{X}(M)_{x,\rho})$,*
3. *$Sig^{(M,\rho)}(X) := \max_{x\in X} Sig^{(M,\rho)}(x)$,*

(4) The set of significant points $X_{Sig}^{(M,\rho)} = \{x \in X : Sig^{(M,\rho)}(x) = Sig^{(M,\rho)}(X)\}$,
(5) $X_{Sig} = X_{Sig}^{(T,1)}$,
(6) $\mathfrak{X}(M)_{X,\rho}^{unsig} := \mathfrak{X}(M) - \bigcup_{x \in X - X_{Sig}^{(M,\rho)}} \mathfrak{X}(M)_{x,\rho}$.

Theorem 4.2.1, Proposition 4.2.7 and Proposition 2.2.5, gives us the following corollary:

Corollary 4.2.9. *The sheaf* $\mathcal{F}(\mathcal{S}(X(F)),(M,\rho))|_{\mathfrak{X}(M)_{X,\rho}^{unsig}}$ *is Cohen-Macaulay of dimension* $Sig^{(M,\rho)}(X)$.

4.3. **Cohen-Macaulay and Multiplicities.** In this subsection we continue to assume that X is affine. Let us now describe the schematic support of the sheaves $\mathcal{F}(\mathcal{S}(X(F)),(M,\rho))$, where (M,ρ), a cuspidal datum. Theorem 4.2.1, Proposition 4.2.7, gives us the following corollary:

Corollary 4.3.1. *Let (M,ρ) be a cuspidal datum of the group G. Chose a finite set $\mathfrak{R} \subset X(F)$ of representative for the P-orbits in X (those that have F points). Then*
(1) $\mathrm{Supp}_{(M,\rho)}(\mathcal{S}(X(F)))$ *is reduced.*
(2) $\mathfrak{X}(M)_{x,\rho}$ *- is ($\mathbb{C}$-points of) a smooth subvariety of $\mathfrak{X}(M)$.*
(3) $\mathrm{Supp}_{(M,\rho)}(\mathcal{S}(X(F)))(\mathbb{C}) = \bigcup_{x\in\mathfrak{R}} \mathfrak{X}(M)_{(x,\rho)}$.

Remark 4.3.2. *In fact the variety $\mathfrak{X}(M)_{(x,\rho)}$ depend only on the stable orbit of x.*

This corollary implies:

Corollary 4.3.3. *Assume that the main conjecture (Conjecture 4.1.3) holds for the spherical space X. Let (M,ρ) be a cuspidal datum . Then we have:*
(1) $\mathrm{Supp}_{(M,\rho)}(\mathcal{S}(X(F)))(\mathbb{C}) = \bigcup_{x\in\mathfrak{R}\cap X_{Sig}^{(M,\rho)}} \mathfrak{X}(M)_{(x,\rho)}$.
(2) Conjecture I holds for the space X.

Remark 4.3.4. *We note that the restriction to the smooth locus in Conjecture I is essential. Indeed, consider the following example. Let $G = \mathrm{SL}_2 \times \mathrm{SL}_2$ and $X = \mathrm{SL}_2$ equipped with the two sided action. Identify the torus $T(F) \subset G(F)$ with $F^* \times F^*$. Let $\rho : F^* \times F^* \to \mathbb{C}^*$ given by*

$$\rho(z_1,z_2) = sgn(z_1)|z_1|^{\frac{1}{2}} sgn(z_1)|z_1|^{\frac{1}{2}}.$$

Identify $\mathfrak{X}(T)$ with $\mathbb{C}^2$.

$$\text{Then } m(s_1,s_2) = \begin{cases} 0, & (s_1+s_2)(s_1-s_2) \neq 0; \\ 2, & s_1 = s_2 = 0; \\ 1, & \textit{otherwise}. \end{cases}$$

4.4. **A Conjecture of Delorme.** Let (π, V) be a smooth representation of G. Given $\ell \in Hom_G(\mathcal{S}(X),\pi)$ and $v \in \tilde{V}$ denote by $m_{\ell,v} \in \mathcal{S}^*(X(F))$ the generalized matrix coefficient:

$$m_{\ell,v}(f) = v(\ell(f)).$$

It is easy to see that $m_{\ell,v}$ is a locally constant measure on X.

Recall the following definition:

Definition 4.4.1 (cf. [KT08])**.** *A pair (π,ℓ) as above is called X-cuspidal if the following equivalent conditions are satisfied:*
(1) For any $v \in V$ the generalized matrix coefficient $m_{\ell,v}$ is of compact support modulo center of G.
(2) The map $\ell : \mathcal{S}(X(F)) \to \pi$ admits a section that is G^0-invariant (if G is adjoint then there is a G-section to ℓ).

We will say that π is X-cuspidal if there exists a non-zero G-invariant functional $\ell : \mathcal{S}(X(F)) \to \pi$ such that (π,ℓ) is X-cuspidal.

Conjecture 4.4.2. *(Delorme) Let $X = G/H$ be a symmetric space. Let π be an irreducible representation of G with an I-fixed vector. Suppose that π is X-cuspidal. Then $X(F)$ is compact modulo center.*

Proposition 4.4.3. *Conjecture 3.5.1 implies Delorme's conjecture (Conjecture 4.4.2).*

For the proof we will need the following lemma.

Lemma 4.4.4. *Assume that $\tilde{U} := U(F)T^0$ have finitely many orbits on $X(F)$. Then $X(F)$ is compact.*

To prove this lemma we will need to recall some basic results from the structure theory of symmetric spaces.

Proposition 4.4.5 (see e.g. [HW93])**.**

(1) *Let (G, H, θ) be a symmetric pair. Then there exists a maximal θ-split torus A_0, and this torus is unique up to conjugation. In particular, $\theta|_{A_0}$ coincide with the inversion. Note that A_0 is not necessarily a maximal split torus.*
(2) *There exist a minimal parabolic subgroup P_0 s.t. $M_0 := \theta(P_0) \cap P_0$ is a Levi subgroup of P_0. Such a parabolic is unique up to conjuagtion and is called θ-parabolic. Note that P_0 in not necessarily a minimal parabolic of G. In what follows we fix our minimal parabolic B to be a subgroup of P_0 and its Levi subgroup T to be a subgroup of M_0.*
(3) *$X(F)$ is compact if and only if A_0 is trivial.*
(4) *If $P_0 = G$ then A_0 is in the center of G.*
(5) *Let P be a θ-parabolic and let $M := \theta(P) \cap P$ be its Levi. Consider the symmetric pair $(M, H_M, \theta|_M) := (M, M^\theta, \theta|_M)$. Then a maximal θ-split torus in M is a maximal θ-split torus in G.*

Proof of Lemma 4.4.3.

Step 1 Reduction to the case when $G = P_0$.
Follows from Theorem 4.4.5 (3) and (5).

Step 2 Proof for the case when $G = P_0$.
By Proposition 4.4.5(4) $A_0 \subset Z(G)$. There exists a θ-invariant subgroup $Z' \subset Z(G)$ such that $Z' \times A_0 \to Z(G)$ is an isogeny. Define $G' = [G, G]Z'$. We have finite map $A_0 \times G'/(H \cap G') \to X$. So, the fact that there is finite number of $\tilde{U}$ orbits on $X(F)$ implies that A_0 is trivial, which implies by Proposition 4.4.5(3) that $X(F)$ is compact.

∎

Remark 4.4.6. *In fact if a symmetric space $X(F)$ is compact than $G(F)$ act on $X(F)$ via compact quotient. Indeed let L be the quotient by which G act on X. If $L(F)$ is not compact, then there is a subgroup $L' \subset L$ which is isomorphic to the multiplicative group $\mathbb{G}_m$. This gives us a non-constant map ϕ (defined over F) from $\mathbb{G}_m$ to X. We can compactify X to a variety $\bar{X}$ and extend the map ϕ to a map $\bar{\phi} : \mathbb{P}^1 \to \bar{X}$. Since this map is not constant and X is affine, its image can not lie in X. This contradicts the fact that $X(F)$ is compact.*

Proof of Proposition 4.4.3.

Step 1. Proof in the case that G is semi-simple.
Let $V := \mathcal{S}(X(F))_{[T,1]}$ be the Iwahori component of the module $\mathcal{S}(X(F))$. We know that π is a direct summand of $\mathcal{S}(X(F))$ and hence of V. Thus π is a Cohen-Macaulay object of the same dimension as V. On the other hand, being an irreducible π is Cohen-Macaulay of dimension zero. Thus V is Cohen-Macaulay of dimension zero. Hence $r_{GT}(V)^{T^0}$ is Cohen-Macaulay $\mathbb{C}[T/T^0]$-module of dimension zero and hence finite dimensional. This implies that $\tilde{U}$ have finitely many orbits on $X(F)$. Thus by the previous lemma (Lemma 4.4.4) $X(F)$ is compact.

Step 2. The general case.
Follows easily from the previous case for the group $[G,G]$.

$\square$

4.5. **Density of Orbital Integrals.** In this subsection we continue to assume that X is affine. The question of density of orbital integrals is central in harmonic analysis. A basic result of Harish-Chandra is that regular semi-simple orbital integrals are dense in the space of $Ad(G)$-invariant distributions (see [HC78]). The generalization to the symmertic case was studied extensively in [RR96] where it is shown that the naive extension is false. Namely, regular H-orbits in $X := G/H$ are not dense in the space of H-invariant distributions on X. This density question is related to the study of H-distinguished representation in $\mathcal{M}(G)$. When we restrict our interest to the Iwahori block $\mathcal{M}_{[T,1]}(G)$, the natural density question becomes that of density of unipotent orbital integrals on X (more precisely, density of $\widetilde{U} = U \times T^0$ orbital integrals).

We will show that the main conjecture (conjecture 4.1.3) implies density of certain $\widetilde{U}$ orbital integrals.

First recall that since X is affine a result of Kostant and Rosenlicht (see [Ros61]) implas that the orbits of a unipotent group on X are closed, and thus the orbits of $\widetilde{U}$ are closed as well.

We conjecture that the following density property holds:

Conjecture 4.5.1. *The space* $\sum_{x \in X_{Sig}} \mathcal{S}^*(\widetilde{U}x)^{\widetilde{U}}$ *is dense in* $\mathcal{S}^*(X(F))^{\widetilde{U}}$

Theorem 4.5.2. *Our main conjecture for the Iwahori block (Conjecture 4.1.5) implies Conjecture 4.5.1.*

In order to prove this theorem we will need some auxiliary notions and lemmas.

Definition 4.5.3. *Let A be a commutative algebra. We say that an A-module M is relatively torsion free if for any $m \in M$, we have* $\dim(\mathrm{Supp}(m)) = \dim(M)$.

The following lemma is obvious

Lemma 4.5.4. *A Cohen-Macaulay module is of relatively torsion free.*

Proof. Let $B \subset A$ be such that M is finitely generated free B-module. Let $\pi : Spec(A) \to Spec(B)$ be the projection. Then $\pi|_{Supp(M)}$ is a finite map. The support of m in $\mathrm{Spec}(B)$ is the projecting $\pi(\mathrm{Supp}(m))$ and the support of M in $\mathrm{Spec}(B)$ is the projecting $\pi(\mathrm{Supp}(M))$. Now

$$\dim(\mathrm{Supp}(m)) = \dim(\mathrm{Supp}_B(m)) = \dim(\mathrm{Supp}_B(M)) = \dim(M).$$

$\square$

Corollary 4.5.5. *Let M a Cohen-Macaulay module whose support is reduced. Let $f \in M$ and assume that there exist an open dense set $U \subset \mathrm{Supp}(M)$ s.t. for each $x \in U$ we have $f|_x = 0$. Then $f = 0$.*

Proof. Consider M as a sheaf over $\mathrm{Supp}(M)$. There exist an open dense subset $U' \subset U$ s.t. $M|_{U'}$ is locally free. Since U' is reduced the assumption implies that $f|_{U'} = 0$. Thus the previous lemma (Lemma 4.5.4) implies that $f = 0$. $\square$

Notation 4.5.6. *For a character χ of T, let $\tilde{\chi}$ be the character of T s.t. for any representation $V \in \mathcal{M}(B(F))$ we have* $\mathrm{Hom}_T(r_{MG}(V), \chi) \cong (V^*)^{B(F),\tilde{\chi}}$. *Explicitly* $\tilde{\chi} = \chi^{-1}\Delta_B^{1/2}$.

Lemma 4.5.7. *Let* $V = \overline{\sum_{x \in X_{Sig}} \mathcal{S}^*(\tilde{U}x)^{\widetilde{U}}} \subset \mathcal{S}^*(X(F))^{\widetilde{U}}$. *Then for any*

$$\chi \in \mathfrak{X}(T) - \bigcup_{x \in X - X_{Sig}} \mathfrak{X}(T)_{x,1}$$

we have

$$\mathcal{S}^*(X(F))^{B(F),\tilde{\chi}} \subset V$$

Proof. If $\chi \notin \mathrm{Supp}_{(B,1)}(\mathcal{S}(X(F)))$ then $\mathcal{S}^*(X(F))^{B(F),\tilde{\chi}} = \{0\}$ and the statement is clear.

We will prove that $V^\perp \subset \mathcal{S}(X(F))[(B(F),\tilde{\chi})] := Span\{\pi(b)f - \tilde{\chi}(b)f : f \in \mathcal{S}(X(F)) \ , b \in B(F)\}$. Stratify $X(F)$ by its $B(F)$-orbits such that $X(F) = \cup_{i=1}^k O_i$ such that for any $j \geq 0$ the union $Z_j = \cup_{i \leq j} O_i$ is a closed subset of $X(F)$. Let $f \in V^\perp$. We will prove by induction on j that $f|_{Z_j} \in \mathcal{S}(Z_j)[(B(F),\tilde{\chi})]$. For $j = 0$ the claim is clear. Assuming the statement for any $i < j$ we will show it for j. We know $f|_{Z_{j-1}} \in \mathcal{S}(Z_{j-1})[(B(F),\tilde{\chi})]$ thus for some $b_1, ..., b_r \in B(F)$ and $\bar{\phi}_1, ..., \bar{\phi}_r \in \mathcal{S}(Z_{j-1})$ we have

$$f|_{Z_{j-1}} = \sum_{j=1}^{r} (\pi(b_j)\bar{\phi}_j - \tilde{\chi}(b_j)\bar{\phi}_j)$$

Now we extend $\bar{\phi}_i$ to Z_j and denote the extension by ϕ_i. Define $h = f|_{Z_j} - \sum_{j=1}^r (\pi(b_j)\phi_j - \tilde{\chi}(b_j)\phi_j)$ Then $h \in \mathcal{S}(O_j)$ and we will show that $h \in \mathcal{S}(O_j)[(B(F),\tilde{\chi})]$.

Case 1 $O_j \subset X_{Sig}$

In this case, for any $x \in O_i$ and any $\xi \in \mathcal{S}^*(\tilde{U}x)^{\tilde{U}}$ we have $\langle \xi, h \rangle = 0$. Thus, by Bernstein Localisation principal [1], $h \in \mathcal{S}(O_j)[\tilde{U}] \subset \mathcal{S}(O_j)[(B(F),\tilde{\chi})]$.

Case 2 $O_j \nsubseteq X_{Sig}$

In this case $O_j \cap X_{Sig} \neq \emptyset$ hence $\tilde{\chi} \notin \mathfrak{X}(T)_{x,1}$ for any $x \in O_j$. Thus $\mathcal{S}(O_j)[B(F),\tilde{\chi}] = \mathcal{S}(O_j)$ and we are done.

□

Proof of Theorem 4.5.2. Let $V := \sum_{x \in X_{Sig}} \mathcal{S}^*(\tilde{U}x)^{\tilde{U}}$ and let $m \in V_\perp \subset \mathcal{S}(X(F))_{\tilde{U}}$. Identify $\mathcal{S}(X(F))_{\tilde{U}}$ with

$$r_{GM}(\mathcal{S}(X(F)))^{T_0} \cong \mathcal{F}(\mathcal{S}(X(F)),(T,1)).$$

Let $\chi \in \mathrm{Supp}(\mathcal{F}(\mathcal{S}(X(F)),(T,1))) - \bigcup_{x \in X - X_{Sig}} \mathfrak{X}(T)_x$. By Corollaries 4.5.5 and 4.3.1, it is enough to show that $m|_\chi = 0$. We can consider χ as a character of $B(F)$. Thus it is enough to show that any $\xi \in \mathcal{S}^*(X(F))^{B(F),\tilde{\chi}}$ vanishes on m. This follows from the above lemma (Lemma 4.5.7) □

5. The Unitary Case

In this section we study some implications of our conjecture for the space of Hermitian matrices.

- Fix a natural number n. Let $H := H_n := GL_n$.
- Let E/F be an unramified quadratic extension.
- We let $\tau : E \to E$ be the Galois involution.
- In this section we concentrate on the case $G = G_n := Res_F^E(H_n)$ is the restriction of scalars of H to E (in particular $G(F) = H(E)$).
- We also fix $X := X_n$ the natural algebraic variety s.t. $X(F) = \{x \in G(E) | \tau(x^t) = x\}$.
- Let G act on X by
$$g \cdot x = gx\tau(g^t).$$
- Let $D \subset X$ be the subset of diagonal matrices.
- Note that in this case $T \subset G$ is a torus. We fix it to be the standard torus.

Conjecture 4.1.5, that $\mathcal{S}(X(F))^I$ is Cohen-Macaulay as $\mathcal{H}(G,I)$-module, implies the following:

Conjecture 5.0.1.

(1) $\mathcal{S}(X(F))^I$ is free $\mathcal{Z}(\mathcal{H}(G,I))$-module.
(2) $\mathcal{S}(X(F))^I$ is free $\mathcal{B}$-module.

[1]In fact in this case Bernstein Localisation principal is evident.

(3) *Multiplicities are constant:*

$$dim_{\mathbb{C}}\mathrm{Hom}_G(\mathcal{S}(X(F)), i_{TG}(\chi)) = 2^n$$

for any unramified χ character of T.

(4) *For every $x \in D(F)$ we define $\xi_x \in \mathcal{S}^*(X(F))$ by*

$$\xi_x(f) = \int_{u\in\widetilde{U}} f(ux)du.$$

Then $Span\{\xi_x | x \in D(F)\}$ is dense in $\mathcal{S}^*(X(F))^U$.

(5) *Define $\Omega : \mathcal{S}(X(F)) \to C^\infty(D(F))$ by*

$$\Omega(f)(x) = \xi_x(f).$$

Then $\Omega : \mathcal{S}(X(F))^I \to C^\infty(D(F))$ is injective. Equivalently, Ω gives an injection $\mathcal{S}(X(F))_{\widetilde{U}}$ to $C^\infty(D(F))$.

Observe that the next theorem of Y.Hironaka is a consequence of Conjecture 5.0.1

Theorem 5.0.2 (Hironaka [Hir99])**.** *The module $\mathcal{S}(X(F))^{K_0}$ is free of rank 2^n over $H(G, K_0)$ where $K_0 := GL(n, \mathcal{O}_E)$ is the standard maximal open subgroup of $G(F)$.*

We will provide an independent proof of this theorem in the next section (see §6.).

6. A Simple Proof of Hironaka's Theorem

In this section we prove Theorem 5.0.2

We continue with the notations of §5 and introduce a few more:

Notation 6.0.1.

- *π a uniformizer in $\mathcal{O}_E$.*
- *$q_F = |O_F/P_F|$, $q_E = |O_E/P_E|$.*
- *Λ the weight lattice of G. We identify it with the usual lattice $\mathbb{Z}^n$.*
- *$\Lambda^+ = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda | \sum_{i=1}^k \lambda_i \geq 0 \ \forall k = 1, \dots, n\}$.*
- *$\Lambda^{++} = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda | \lambda_k - \lambda_{k-1} \leq 0 \ \forall k = 2, \dots, n\}$.*
- *for $\lambda \in \Lambda$ we set $\pi^\lambda := \lambda(\pi) \in G(F)$.*
- *for $\lambda \in \Lambda$ we set x_λ to be $\lambda(\pi)$ considered as an element in $X(F)$.*
- *Let $a_\lambda = e_{K_0\delta_{\pi^\lambda}K_0} \in \mathcal{H}(G, K_0)$.*
- *Let $m_\lambda = e_{K_0\delta_{x_\lambda}} \in \mathcal{S}(X(F))^{K_0}$.*
- *We denote $\lambda \geq \lambda'$ iff $\lambda - \lambda' \in \Lambda^+$. In this case, if $\lambda \neq \lambda'$ we denote $\lambda > \lambda'$.*

Lemma 6.0.2.

(1) *The collection $\{\pi^\lambda | \lambda \in \Lambda^{++}\}$ is a complete set of representatives for the orbits of $K_0 \times K_0$ on G.*
(2) *The collection $\{x_\lambda | \lambda \in \Lambda^{++}\}$ is a complete set of representatives for the orbits of K_0 on X.*

Proof.

(1) This is the classical Cartan decomposition $G = K_0 A^{++} K_0$.
(2) This is proven in [Jac62].

□

Corollary 6.0.3.

(1) *The collection $\{a_\lambda | \lambda \in \Lambda^{++}\}$ is a basis for $\mathcal{H}(G, K_0)$.*
(2) *The collection $\{m_\lambda | \lambda \in \Lambda^{++}\}$ is a basis for $\mathcal{S}(X(F))^{K_0}$.*

This Corollary leads naturally to the following filtration on the module $M := \mathcal{S}(X(F))^{K_0}$ and the Hecke algebra $A := \mathcal{H}(G, K_0)$.

Definition 6.0.4. *For $\lambda \in \Lambda^{++}$ we now introduce the subspaces*

- $F_{\leq\lambda}(A) = Span_{\mathbb{C}}\{a_\mu | \mu \leq \lambda\}$, $\quad F_{<\lambda}(A) = Span_{\mathbb{C}}\{a_\mu | \mu < \lambda\}$
- $F_{\leq\lambda}(M) = Span_{\mathbb{C}}\{m_\mu | \mu \leq \lambda\}$, $\quad F_{<\lambda}(M) = Span_{\mathbb{C}}\{m_\mu | \mu < \lambda\}$

With this filtration we have the following Key Proposition:

Proposition 6.0.5.

(1) *For every $\lambda \in \Lambda^{++}$ and $\mu \in \Lambda^{++}$ there exists a non-zero $p(\lambda, \mu) \in \mathbb{C}$ such that*
$$a_\lambda a_\mu = p(\lambda, \mu) a_{\lambda+\mu} + r$$
with $r \in F_{<\lambda+\mu}(A)$.

(2) *For every $\lambda \in \Lambda^{++}$ and $\mu \in \Lambda^{++}$ there exists a non-zero $q(\lambda, \mu) \in \mathbb{C}$ and we have*
$$a_\lambda m_\mu = q(\lambda, \mu) m_{2\lambda+\mu} + \delta$$
where $\delta \in F_{<2\lambda+\mu}(M)$.

We postpone the proof of Proposition 6.0.5 to §6.1 and continue with the proof of Theroem 5.0.2 We first recall terminology regarding filtered modules and algebras.

Definition 6.0.6. *Let $(\mathcal{I}, \leq)$ a partially ordered set and let V be a vector space.*

- *An $\mathcal{I}$-filtration F on V is a family $\{F_i(V)\}_{i\in\mathcal{I}}$ of subspaces of V such that $i \leq j \Rightarrow F_i(V) \subset F_j(V)$. We say that the filtration is exhaustive if furthermore $V = \cup F_i(V)$ and $\cap F_i(V) = \{0\}$.*
- $Gr_F^i(V) := F_i(V)/\sum_{j<i} F_j(V)$.
- $Gr_F(V) := \bigoplus Gr_F^i(V)$.
- *For any $i \in \mathcal{I}$ we denote by $\sigma_i : F_i(V) \to Gr_F^i(V)$ the symbol map.*
- *A partially ordered commutative semi-group $(\mathcal{I}, \leq, +)$ is a partially ordered set such that for any $i \in \mathcal{I}$ and any $x, y \in \mathcal{I}$ satisfying $x \leq y$ we have $x + i \leq y + i$.*
- *Let $(\mathcal{I}, \leq, +)$ be a partially ordered commutative semi-group. An $\mathcal{I}$-filtration on an algebra A is an $\mathcal{I}$-filtration $F^i(A)$ on the underlying vector space such that $F_i(A)F_j(A) \subset F_{i+j}(A)$. Note that in such a case $Gr_F(A)$ is an $\mathcal{I}$-graded algebra.*
- *An action of a partially ordered semi-group $\mathcal{I}$ on a partially ordered semi-group $\mathcal{J}$ is a semi-group action $m : \mathcal{I} \to Aut(\mathcal{J})$ of $\mathcal{I}$ on $\mathcal{J}$ satisfying*
$$(i_1 \leq i_2) \wedge \ (j_1 \leq j_2) \Rightarrow m(i_1)(j_1) \leq m(i_2)(j_2)$$
- *Let A, $\mathcal{I}$, $\mathcal{J}$ be as above. Let M be an A-module we say that G is an $\mathcal{J}$-filtration on M if it is an $\mathcal{J}$-filtration on the underlying vector space of M and*
$$F_i(A)G_j(M) \subset G_{m(i)(j)}(M).$$
Note that in such a case $Gr_G(M)$ is a graded module over $Gr_F(A)$.

The following is an adaptation of a trick we learned from [BL2009] (see Lemma 4.2).

Lemma 6.0.7. *Let $\mathcal{I}$ be a partially ordered semi-group and $\mathcal{J}$ be a free $\mathcal{I}$- partially ordered set. Assume that any non-empty $S \subset \mathcal{I}$ have a (weak) minimal element, and similarly for $\mathcal{J}$.*

Let (M, G) be an $\mathcal{J}$-filtered module over an $\mathcal{I}$-filtered commutative algebra (A, F). Suppose that $Gr_G(M)$ is finitely generate free graded module over $Gr_F(A)$ (i.e. there exists finitely many homogenous elements that freely generate $Gr_G(M)$).

Then M is a finitely generated free A-module.

The proof is analoguous to Lemma 4.2 from [BL2009]. We are now ready to prove Hironaka's Theorem.

Proof of Theorem 5.0.2. Let Λ^{++} acts on itself by $a(\lambda)(\mu) = \mu + 2\lambda$. By proposition 6.0.5 the filtration F defined in 6.0.4 is a Λ^{++}-filtration on both A and M. Applying lemma 6.0.7 it is enough to show that $Gr_F(M)$ is finitely generated free $Gr_F(A)$-module. We now let $\bar{a}_\lambda, \bar{m}_\lambda$ be the reductions of a_λ, m_λ to the associated graded. By proposition 6.0.5 we get $\bar{a}_\lambda \bar{a}_\mu = p(\lambda,\mu)\bar{a}_{\lambda+\mu}$ and $\bar{a}_\lambda \bar{m}_\mu = q(\lambda,\mu)\bar{m}_{2\lambda+\mu}$. Let $L \subset \Lambda^{++}$ be a such that $\Lambda^{++} = \cup_{\ell \in L}(\ell + 2\Lambda^{++})$ is a disjoint covering. Clearly, the set $\{m_\ell | \ell \in L\}$ is a free basis of $Gr_F(M)$ over $Gr_F(A)$. This finishes the proof. □

6.1. **Proof of Proposition 6.0.5.** The proof of the proposition require an explicit version of Lemma 6.0.2. For this we require a definition.

Definition 6.1.1. *Let $V = E^n$ and $V_0 = F^n$*

(1) If L_1, L_2 are two O_E-lattices in V then we define

$$[L_1 : L_2] = \log_{q_E}(|L_1/(L_1 \cap L_2)||L_2/(L_1 \cap L_1)|^{-1})$$

(2) Let Q be a Hermitian form on V. Let $L \subset V_0$ be a lattice. Take an O_F basis $B = \{v_1, \dots, v_n\}$ to L. We define

$$\nu_L(Q) = \nu(det(Gram(B))) := \nu(det(Q(v_i, v_j))),$$

where ν is the valuation of E. This is independent of the choice of the basis.

Lemma 6.1.2. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^{++}$ and denote by $p_k = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ and let $q_k = \lambda_n + \lambda_{n-1} + \cdots + \lambda_{n-k+1}$.*

(1) Let $g \in K_0 \pi^\lambda K_0$. Then $p_k = \min_{W \in Grass(k,V)} [W \cap O_E^n : W \cap g O_E^n]$.

(2) Let $x = (x_{i,j}) \in K_0 x_\lambda$. Then $q_k = \min_{W \in Grass(k,V)} \nu_{O_E^n \cap W}(x|_W)$.

Proof. (1) We first note

$$\min_{W \in Grass(k,V)} [W \cap O_E^n : W \cap g O_E^n] = \min_{W \in Grass(k,V)} [W \cap O_E^n : W \cap \pi^\lambda O_E^n]$$

It remains to verify the statement of the lemma for $g = \pi^\lambda$. Clearly,

$$p_k \geq \min_{W \in Grass(k,V)} [W \cap O_E^n : W \cap \pi^\lambda O_E^n]$$

Thus it is enough to show that for any $W \in Grass(k,V)$ we have

$$p_k \leq [W \cap O_E^n : W \cap \pi^\lambda O_E^n]$$

For this we let $e_1, .., e_k$ be an O_E basis for $W \cap O_E^n$. Let $A \in Mat_{n \times k}(O_E)$ be the matrix whose i-the column is e_i, $i = 1, .., k$.

Denote by $r(A)$ the matrix obtained from A by reducing its elements to O/π. Since $e_1, ..., e_k$ is a basis we have $rank(r(A)) \geq k$ and we can find a $k \times k$ minor which is invertible in O_E. Explicitly, we have $\mathcal{I} = (i_1, i_2, .., i_k)$ such that the minor $M_{\mathcal{I},[1,k]}(A) \in O^\times$.

Notice that

$$\begin{aligned}[W \cap O_E^n : W \cap \pi^\lambda O_E^n] &= [Span_{O_E}(e_1, .., e_k) : \pi^\lambda(\pi^{-\lambda} W \cap O_E^n)] = \\ &= [Span_{O_E}(\pi^{-\lambda} e_1, .., \pi^{-\lambda} e_k) : \pi^{-\lambda} W \cap O_E^n] = \\ &= [Span_{O_E}(\pi^{-\lambda} e_1, .., \pi^{-\lambda} e_k) : Span_E(\pi^{-\lambda} e_1, .., \pi^{-\lambda} e_k) \cap O_E^n]\end{aligned}$$

Let $f_1, ..., f_k$ be an O_E-basis for $Span_E(\pi^{-\lambda} e_1, .., \pi^{-\lambda} e_k) \cap O_E^n$. Let $B \in Mat_{n \times k}(O_E)$ be the corresponding matrix as before.

Let $C \in Mat_{k\times k}(E)$ be such that $B = \pi^{-\lambda}AC$. Passing to the submatrix $B_{\mathcal{I},[1,..,k]}$ we have $B_{\mathcal{I},[1,..,k]} = diag(\pi^{-\lambda_{i_1}},...,\pi^{-\lambda_{i_k}})A_{\mathcal{I},[1,..,k]}C$. Thus $M_{\mathcal{I},[1,k]}(B) = \pi^{-\sum_{j=1}^{k}\lambda_{i_j}}M_{\mathcal{I},[1,k]}(A)det(C)$. Thus

$$0 \leq \nu(M_{\mathcal{I},[1,k]}(B)) = -\sum_{j=1}^{k}\lambda_{i_j} + \nu(M_{\mathcal{I},[1,k]}(A)) + \nu(det(C)) = -\sum_{j=1}^{k}\lambda_{i_j} + \nu(det(C))$$

Finally,

$$[W\cap O_E^n : W\cap \pi^{\lambda}O_E^n] = [Span_{O_E}(\pi^{-\lambda}e_1,...,\pi^{-\lambda}e_k) : Span_{O_E}(f_1,...,f_k)] = \nu(det(C)) \geq$$
$$\geq \sum_{j=1}^{k}\lambda_{i_j} \geq p_k$$

(2) as before, the only non-trivial part is to show that

$$\nu_{O_E^n\cap W}(x_\lambda|_W) \geq q_k.$$

If $x_\lambda|_W$ is degenerate this is obvious. So we will assume it is not. By Lemma 6.0.2 we can find a $x_\lambda|_W$-orthonormal basis $(e_1,\ldots,e_k)$ of $O_E^n\cap W$ and a $x_\lambda|_{W^\perp}$-orthonormal basis $(e_{k+1},\ldots,e_n)$ of $O_E^n\cap W^\perp$. Let $\mu_i = \tau(e_i^t)x_\lambda e_i$. By Lemma 6.0.2 the collection $(\mu_1,\ldots,\mu_n)$ coincides (up to reordering) with $(\lambda_1,\ldots,\lambda_n)$ thus

$$\nu_{O_E^n\cap W}(x_\lambda|_W) = \mu_1+\cdots+\mu_k \geq \lambda_n+\cdots+\lambda_{n-k+1} = q_k$$

$\square$

Proof of Proposition 6.0.5.

(1) This is well known. See e.g. [Mac98, Chapter 5 (2.6)]

(2) Since $x_{2\lambda+\mu} \in \pi^\lambda K_0x_\mu$, it is enough to show that $\pi^\lambda K_0x_\mu \subset \bigcup_{\nu\leq 2\lambda+\mu}K_0x_\nu$. Let $x\in K_0x_\mu$.
By Lemma 6.1.2(2) we have to show

$$\min_{W\in Grass(i,V)}\nu_{W\cap O^n}(\pi^\lambda\cdot x|_W) \leq \sum_{j=n-i+1}^{n}(\mu_j+2\lambda_j).$$

By Lemma 6.1.2 we have,

$$\min_{W\in Grass(i,V)}\nu_{O^n\cap W}(\pi^\lambda\cdot x|_W) = \min_{W\in Grass(i,V)}\nu_{\pi^\lambda O^n\cap\pi^\lambda W}(x|_{\pi^\lambda W}) =$$
$$= \min_{W\in Grass(i,V)}\nu_{\pi^\lambda O^n\cap W}(x|_W) = \min_{W\in Grass(i,V)}(2[O^n\cap W:\pi^\lambda O^n\cap W] + \nu_{O^n\cap W}(x|_W)) \leq$$
$$\leq 2\min_{W\in Grass(i,V)}([O^n\cap W:\pi^\lambda O^n\cap W]) + \sum_{j=n-i+1}^{n}\mu_j = \sum_{j=n-i+1}^{n}(2\lambda_j+\mu_j).$$

$\square$

Remark 6.1.3. *It appears that the method of this section can be generalized to cases where Cartan decomposition of the spherical space X is nice. We hope to explore this in the future.*

7. Proof of the Main Conjecture in a Special Case

In this section we prove the main conjecture for the symmetric space of two by two Hermitian matrices. We first prove the conjecture for the Iwahori block (Conjecture 4.1.5) for this case and then deduce the general case.

7.1. **The Iwahori Block.** Most of the section is dedicated to the study of the $\mathcal{H}(G_n, I)$-module $\mathcal{S}(X_n)^I$ for general n. However in order to make the argument complete we need at the end to restrict to the case $n = 2$.

The proof is based on the following proposition (which is valid for any n).

Proposition 7.1.1. *Consider $M := \mathcal{F}(\mathcal{S}(X(F)), (T, 1)) = r_{G,T}(\mathcal{S}(X(F)))^{T^0}$ as a module over $\mathbb{C}[T/T^0] = \mathcal{O}(\mathfrak{X}(T))$. Identifying $T(F)$ with $(F^\times)^n$ we get an identification of $\mathfrak{X}(T)$ with $(\mathbb{C}^\times)^n$*

(1) Let $U \subset \mathfrak{X}(T)$ be the Zariski open set given by

$$U = \{(t_1, ..., t_n) \in \mathfrak{X}(T) | \forall i, j \text{ we have } t_i/t_j \neq q\}.$$

Then M_U is locally-free over $\mathcal{O}(U)$

(2) Let $V \subset \mathfrak{X}(T)$ be the Zariski open set given by

$$V = \{(t_1, ..., t_n) \in \mathfrak{X}(T) | \forall i \neq j \text{ we have } t_i \neq t_j\}.$$

Then M_V is locally-free over $\mathcal{O}(V)$

This proposition implies Conjecture 4.1.5 for the spherical space X_2. Indeed:

Theorem 7.1.2. *The sheaf $\mathcal{F}(\mathcal{S}(X(F)), (T, 1))$ is locally free sheaf over $\mathfrak{X}(T)$ in particular it is Cohen-Macaulay*

Proof. We have to prove that M is locally free. For this by Proposition 7.1.1 it is enough to prove that $U \cup V = \mathfrak{X}(T)$. This is clear. $\square$

Remark 7.1.3. *Note that for $n \geq 3$ this deduction fails.*

Now it is left to prove Proposition 7.1.1 (for general n). For this we will need the following standard lemmas.

Lemma 7.1.4. *If the dimension of the fiber of a coherent sheaf over an algebraic variety is a locally constant function then the sheaf is locally free.*

For brievity we will write π_χ for $i_{GT}(\chi)$.

Lemma 7.1.5 (appendix D in [FLO2012])**.** *for any $\chi \in \mathfrak{X}(T)$ we have*

$$d_{\mathcal{S}(X(F)),(T,1)}(\chi) := \dim \mathrm{Hom}(\mathcal{S}(X(F)), \pi_\chi) \geq 2^n$$

The following is well known and follows from the equality $G = K_0 B$

Lemma 7.1.6. *for any $\chi \in \mathfrak{X}(T)$ we have $\dim \pi_\chi^{K_0} = 1$.*

The irreducibility of unramified principal series is analyzed in [Cas], in particular we have:

Lemma 7.1.7 (see Prop. 3.5. in [Cas])**.** *For any $\chi \in U$ the representation π_χ is irreducible.*

Lemma 7.1.8. *The set of involutions of the Weyl group, considered as a subset of $X(F)$, gives a complete set of representatives for the B-orbits that have F points.*

Proof of Proposition 7.1.1. Recall that by Lemma 3.4.4 the dimension of the fiber of M at point χ is the multiplisity $d_{\mathcal{S}(X(F)),(T,1)}(\chi)$.

Proof of (1) By lemmas 7.1.5 and 7.1.4 it is enough to prove that for any $\chi \in U$ we have $d_{\mathcal{S}(X(F)),(T,1)}(\chi) \leq 2^n$. Consider the map $i : \mathrm{Hom}(\mathcal{S}(X(F)), i_{GT}(\chi)) \to \mathrm{Hom}(\mathcal{S}(X(F))^{K_0}, i_{GT}(\chi)^{K_0})$. By Lemma 7.1.7 it is injective. Indeed if $i(\phi) = 0$, then $Im\phi \cap i_{GT}(\chi)^{K_0} = 0$ and thus $\phi = 0$. By Lemma 7.1.6 and Theorem 5.0.2

$$\mathrm{Hom}(\mathcal{S}(X(F))^{K_0}, \pi_\chi^{K_0}) = 2^n.$$

Thus

$$d_{\mathcal{S}(X(F)),(T,1)}(\chi) = \dim \mathrm{Hom}(\mathcal{S}(X(F)), \pi_\chi) \leq \dim \mathrm{Hom}(\mathcal{S}(X(F))^{K_0}, \pi_\chi^{K_0}) = 2^n.$$

Proof of (2) This follows immediately from Corollary 4.2.9 and Lemma 7.1.8

□

7.2. **Deduction of Conjecture 4.1.3.** In this section we deduce conjecture 4.1.3 from conjecture 4.1.5 for the space $X = X_2$. By Corollary 4.2.4 the cuspidal blocks are always Cohen-Macaulay. So it is enough to prove the following theorem:

Theorem 7.2.1. *For any $\chi = (\chi_1, \chi_2)$ character of T the component $\mathcal{S}(X(F))_{[T,\chi]}$ is Cohen-Macaulay.*

Proof. By twisting with unramified character we may assume that $\chi_1(\pi) = \chi_2(\pi) = 1$ where $\pi \in O_E$ is the fixed uniformizer. Let $O'_E = \{t \in O^*_E | t\gamma(t) = 1\}$. Let $\gamma : E \to E$ be the Galois involution. Now we will prove the assertion by considering several cases:

case 1 $\chi_1 = \chi_2 = \gamma(\chi_1)$ (or equivalently $\chi_1 = \chi_2$ and $\chi_1|_{O'_E} = 1$).
$\mathcal{S}(X(F))_{[T,(\chi_1,\chi_1)]} = (\mathcal{S}(X(F)) \otimes \chi_1^{-1})_{[(T,1)]} = \mathcal{S}(X(F))_{[(T,1)]}$ which is Cohen Macaulay by Theorem 7.1.2.

case 2 $\chi_1 = \gamma(\chi_1) \neq \chi_2 = \gamma(\chi_2)$.
Let $\{1, w\}$ be a set of representatives as in Lemma 7.1.8. The assertion follows from Corollary 4.2.9, since $\mathfrak{X}(T)_{1,(\chi_1,\chi_2)} = \mathfrak{X}(T)$ and $\mathfrak{X}(T)_{w,(\chi_1,\chi_2)} = \emptyset$

case 3 $\chi_1 = \gamma(\chi_2) \neq \chi_2$.
The assertion follows from Corollary 4.2.9, since $\mathfrak{X}(T)_{1,(\chi_1,\chi_2)} = \emptyset$ and $\mathfrak{X}(T)_{w,(\chi_1,\chi_2)} = \{(t_1, t_2) \in \mathfrak{X}(T) | t_1 = t_2\}$

case 4 $\chi_1 \neq \gamma(\chi_1), \gamma(\chi_2)$ and $\chi_2 \neq \gamma(\chi_2)$. In this case $\mathcal{S}(X(F))_{[T,\chi]} = 0$, since $\mathfrak{X}(T)_{1,(\chi_1,\chi_2)} = \mathfrak{X}(T)_{w,(\chi_1,\chi_2)} = \emptyset$.

□

References

[AAG11] A. Aizenbud, N. Avni and D. Gourevitch, *Spherical pairs over close fields*, Commentarii Mathematici Helvetici. See also arXiv:0910.3199[math.RT].

[AG09a] A. Aizenbud, D. Gourevitch, *Multiplicity one theorem for* $(GL_{n+1}(\mathbb{R}), GL_n(\mathbb{R}))$, Selecta Mathematica, Vol. 15, No. 2., pp. 271-294 (2009). See also arXiv:0808.2729[math.RT].

[AG09b] Aizenbud, A.; Gourevitch, D.: *Generalized Harish-Chandra descent, Gelfand pairs and an Archimedean analog of Jacquet-Rallis' Theorem.* Duke Mathematical Journal, Volume 149, Number 3,509-567 (2009). See also arXiv: 0812.5063[math.RT].

[AGRS] A. Aizenbud, D. Gourevitch, S. Rallis, G. Schiffmann, *Multiplicity One Theorems*, arXiv:0709.4215v1 [math.RT], Annals of Mathematics Volume 172, Issue 2, (2010).

[AGS] A. Aizenbud, D. Gourevitch, E. Sayag : *Z-finite distributions on p-adic groups.* ArXiv: 1405.2540.

[AGS08] A. Aizenbud, D. Gourevitch, E. Sayag : $(GL_{n+1}(F), GL_n(F))$ *is a Gelfand pair for any local field* F. Compositio Mathematica, **144** , pp 1504-1524 (2008), doi:10.1112/S0010437X08003746. See also arXiv:0709.1273 [math.RT].

[BZ77] I. N. Bernstein, A. Zelevinsky, *Induced representations of reductive p-adic groups. I*, Ann. Sci. cole Norm. Sup. (4) 10 (1977), no. 4, 441472.

[Ber87] J.N. Bernstein, *Second adjointness for representations of reductive p-adic groups.* Unpublished, available at `http://www.math.uchicago.edu/~mitya/langlands.html`.

[BR] J.N. Bernstein and K. Rumelhart, *Lectures on p-adic groups.* Unpublished, available at `http://www.math.tau.ac.il/~bernstei/Publication_list/`.

[BD84] J.N. Bernstein, *Le centre de Bernstein (edited by P. Deligne)* In: Representations des groupes reductifs sur un corps local, Paris, 1984, pp. 1-32.

[BBG97] J.N. Bernstein, A. Braverman, D. Gaitsgory, *The Cohen-Macaulay property of the category of* $(\mathfrak{g}, K)$*-modules* Selecta Mathematica, 1997, 303-314.

[BL2009] J.N. Bernstein and V. Lunts, *A simple proof of Kostant's theorem that* $U(\mathfrak{g})$ *is free over its center*

[BZ76] I. N. Bernšteĭn and A. V. Zelevinskiĭ, *Representations of the group* $GL(n, F)$*, where* F *is a local non-Archimedean field*, Uspehi Mat. Nauk **31** (1976), no. 3(189), 5–70. MR MR0425030 (54 #12988)

[Bo76] A. Borel, *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Inventiones Math. 35 (1976), 233259.

[Bus01] Colin J. Bushnell, *Representations of Reductive p-Adic Groups: Localization of Hecke Algebras and Applications*. J. London Math. Soc. (2001) **63**: 364-386; doi:10.1017/S0024610700001885

[BvD94] E. P. H. Bosman and G. Van Dijk, *A new class of Gelfand pairs*, Geom. Dedicata 50 (1994), 261 – 282. MR 1286380

[BT1] F. Bruhat and J. Tits, Groupes reductifs sur un corps local, Inst. Hautes Etudes Sci. Publ. Math. (1972), 5251. (French)

[BT2] Groupes reductifs sur un corps local. II. Schemas en groupes. Existence d'une donnee radicielle valuee, Inst. Hautes Etudes Sci. Publ. Math. (1984), 197376. (French) Bruhat, Tits

[Cas] W. Casselman, *The unramified principal series of p-adic groups. I. The spherical function*, Compositio Math. 40 (1980), no. 3, 387406.

[Del] P. Delorme, *Constant term of smooth H_ψ-spherical functions on a reductive p-adic group*. Trans. Amer. Math. Soc. **362** (2010), 933-955. See also `http://iml.univ-mrs.fr/editions/publi2009/files/delorme_fTAMS.pdf`.

[vD] G. van Dijk: *Some recent results of Harish-Chandra for p-adic groups,* Colloque sur les Fonctions Spheriques et la Theorie des Groupes (Univ. Nancy, Nancy, 1971), Exp. No. 2, 7 pp. Inst. lie Cartan, Univ. Nancy, Nancy, 1971.

[DKV84] P. Deligne, D. Kazhdan, M.-F. Vigneras, *Representations des algebres centrales simples p-adiques* In: Representations des groupes reductifs sur un corps local, Paris, 1984, pp. 33-117.

[HC78] Harish-Chandra: *Admissible distributions on p-adic groups,* Queen's paper in pure and applied Math. 48, 1978, 281-346.

[FLO2012] B. Feigon, E. Lapid, O. Offen *On representations distigiushed by unitary groups*,Publ. Math. Inst. Hautes Études Sci., 115, N. 1, (2012), 185-323

[HW93] A.G. Helminck and S. P. Wang *On rationality properties of involutions of reductive groups* , Advances in Mathematics, vol. 99 (1993), 26-97.

[Hir99] Y. Hironaka *Spherical functions and local densities on Hermitian forms*, J. Math. Soc. Jpn., 51 (1999), 553-581. MR 1691493 (2000c:11064).

[Jac98] H. Jacquet *A theorem of density for Kloosterman integrals.* Asian J. Math. 2 (1998), no. 4, 759-778.

[JR96] H. Jacquet, S. Rallis, *Uniqueness of linear periods.*, Compositio Mathematica , tome **102**, n.o. 1 , p. 65-123 (1996).

[Jac62] R. Jacobowitz, *Hermitian forms over local fields*, Amer. J. Math. 84 pp. 441-465, (1962).

[Kaz86] D. Kazhdan: *Cuspidal geometry of p-adic groups*, J. Analyse Math. **47** (1986), 1-36.

[Lag08] N. Lagier , Terme constant de fonctions sur un espace symétrique réductif p-adique, J. of Funct. An., 254 (2008) 1088-1145.

[Lus83] G. Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities*, Analysis and topology on singular spaces, II, III (Luminy, 1981), 208-229, Astrisque, 101-102, Soc. Math. France, Paris, 1983.

[MR09] Z. Mao and S. Rallis *A Plancherel formula for Sp_{2n}/Sp_nxSp_n and its application*, Compos. Math. 145 (2009), no. 2, 501-527.

[Ma77] H. Matsumoto, *Analyse Harmonique dans les Syst'emes de Tits Bornologiques de Type Affine*, Springer Lecture Notes N. 590, Berlin 1977.

[Mac98] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Mathematical Monographs, Second Edition, 1998.

[Off] O. Offen *Relative spherical functions on p-adic symmetric spaces (three cases)*, Pacific J. Math. 215 (2004), no. 1, 97-149.

[P70] B. Pareigis. *Categories and functors.* Translated from the German. Pure and Applied Mathematics, Vol. 39. Academic Press, New York, 1970.

[Rod73] F. Rodier, *Whittaker models for admissible representations of reductive p-adic split groups.* Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI,Williams Coll., Williamstown, Mass., 1972), pp. 425-430. Amer. Math. Soc., Providence, R.I., 1973.

[Ros61] M. Rosenlicht, *On quotient varities and the embedding of certain homogeneous spaces*, Trans. Amer. Math. Soc. 101 (1961), 211-233.

[RR96] C. Rader and S. Rallis, *Spherical characters on p-adic symmetric spaces, Periods and harmonic analysis on spherical varieties*, Ame. J. of Math. 118, 1996, 91-178.

[Sa08] Y. Sakellaridis, *On the unramified spectrum of spherical varieties over p-adic fields*, Compositio Mathematica 144 (2008), no. 4, 978-1016.

[SV] Y. Sakellaridis and A. Venkatesh, *Periods and harmonic analysis on spherical varieties.* ArXiv:1203.0039.

[SZ] B. Sun and C.-B. Zhu, *Multiplicity one theorems: the archimedean case*, arXiv:0903.1413[math.RT].

[KT08] S.I. Kato, K. Takano, *Subrepresentation theorem for p-adic symmetric spaces*, Int. Math. Res. Not. IMRN 2008, no. 11, Art. ID rnn028, 40 pp.

Avraham Aizenbud, Massachusetts Institute of Technology, Department of Mathematics, Cambridge, MA 02139 USA.
E-mail address: aizner@gmail.com
URL: http://www.wisdom.weizmann.ac.il/~aizenr/

Eitan Sayag, Department of Mathematics, Ben-Gurion University of the Negev, ISRAEL
E-mail address: sayage@math.bgu.ac.il
URL: http://www.math.bgu.ac.il/~sayage/