INVARIANT FUNCTIONALS ON THE SPEH REPRESENTATION

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Abstract. We study \( \text{Sp}_{2n}(\mathbb{R}) \)-invariant functionals on the spaces of smooth vectors in Speh representations of \( \text{GL}_{2n}(\mathbb{R}) \).

For even \( n \) we give an expression for such a functional using an explicit realization of the space of smooth vectors in the Speh representation. Our construction, combined with the argument in [GOS12], gives a purely local and explicit construction of Klyachko models for all unitary representations of \( \text{GL}_{2n}(\mathbb{R}) \). Furthermore, we show that this functional is, up to a constant, the unique functional on the Speh representation which is invariant under the Siegel parabolic subgroup of \( \text{Sp}_{2n}(\mathbb{R}) \).

For odd \( n \) we show that the Speh representation does not admit an invariant functional with respect to the subgroup \( U(n) \) of \( \text{Sp}_{2n}(\mathbb{R}) \) consisting of unitary matrices.

1. Introduction

In recent years, there has been considerable interest in periods of automorphic forms to the Langlands program and equidistribution problems ([SV], [V10]). The study of periods admits a local counterpart, invariant linear functionals and with it the notion of distinction of a representation \( \pi \) of a reductive group \( G \) with respect to a subgroup \( H \subset G \). We recall that the representation \( \pi \) is called distinguished with respect to a subgroup \( H \subset G \) if \( \text{Hom}_H(\pi, \mathbb{C}) \neq 0 \). In many interesting cases the pair \( (G, H) \) is a Gelfand pair and this allows one to connect the global period integral to local linear functionals. Motivated by the work of Jacquet-Rallis [JR92] and Heumos-Rallis [HR90], the third author together with O. Offen classified in [OS07, OS08a, OS08b, OS09] those unitary representations of \( \text{GL}_{2n}(F) \) that are distinguished with respect to the subgroup \( \text{Sp}_{2n}(F) \), in the case that \( F \) is a non-archimedean local field. The case of Archimedean fields was treated subsequently in [GOS12] and [AOS12]. We remark that the pair \( \text{Sp}_{2n}(F) \subset \text{GL}_{2n}(F) \) is a Gelfand pair (see [OS08b] and [OS12a for the classification). The classification of distinguished unitary representations involves the family of unitary representations of \( \text{GL}_{2n}(\mathbb{R}) \) discovered by B. Speh. We remind that these unitary representations, and their generalizations to \( \text{GL}_{2n}(F) \), where \( F \) is a local field, play a central role in the classification scheme of the unitary dual of the general linear group over the local field \( F \). Indeed any irreducible representation of \( \text{GL}_{n}(F) \) is a Bernstein-Zelevinski product, in a unique way, of generalized Speh representations and their complementary series counterparts (see [Tad86, Vog86], [GOS12, AOS12]).

For a discrete series representation \( \sigma \) of \( \text{GL}_{r}(F) \) we denote by \( U(\sigma, n) \) the corresponding generalized Speh representation of \( \text{GL}_{nm}(F) \). For \( |\alpha| < \frac{1}{2} \) we denote by \( \pi(\sigma, n, \alpha) = U(\sigma, n) \cdot \sigma^{\alpha} \cdot U(\sigma, n) \cdot \sigma^{-\alpha} \) the complementary series, which is a unitary representation of \( \text{GL}_{2nm}(F) \). Recall that for archimedean \( F \) we have \( r \leq 2 \), and if \( F = \mathbb{C} \) then \( r = 1 \). If \( r = 1 \) then \( U(\sigma, n) \) is a character of \( \text{GL}_{n}(F) \), and \( \pi(\sigma, n, \alpha) \) is a Stein complementary series representation of \( \text{GL}_{2nm}(F) \). We denote by \( D_{m} \) the discrete series representations of \( \text{GL}_{2}(\mathbb{R}) \) and by \( \delta_{m} \) the corresponding Speh representations of \( \text{GL}_{2n}(\mathbb{R}) \).

The answer to the distinction is summarized in the next theorem, which in the archimedean case is a combination of [GOS12, Theorem A] and [AOS12, Theorem 1.1].

Theorem. Let \( \pi \) be an irreducible unitary representation of \( \text{GL}_{2n}(F) \). Write \( \pi = \times_{i=1}^{k} U(\sigma_{i}, n_{i}) \times \times_{j=1}^{l} \pi(\sigma_{j}', m_{j}, \alpha_{j}) \) with

- \( \sigma_{1}, \ldots, \sigma_{k} \) discrete series representations of \( \text{GL}_{p_{i}}(F) \), ..., \( \text{GL}_{p_{k}}(F) \) respectively
- \( \sigma_{1}', \ldots, \sigma_{l}' \) discrete series representations of \( \text{GL}_{q_{1}}(F) \), ..., \( \text{GL}_{q_{l}}(F) \) respectively
- \( \alpha_{1}, \ldots, \alpha_{l} \) real numbers in \( (-\frac{1}{2}, \frac{1}{2}) \).

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Lemma 4.2 we determine which one does.

One of the key steps in the proof that the generalized Speh representations \( U(\sigma, n) \) with even \( n \) are distinguished by the symplectic group. The proof of this result in \cite{OS07} and \cite{GOSS12} is based on a global argument involving periods of residues of automorphic Eisenstein series.

In \cite{SaSt90} Speh representations \( \delta_m \) of \( GL_{2n}(\mathbb{R}) \) have been constructed explicitly as a natural Hilbert space of distributions on matrix space. The paper \cite{SaSt90} also describes and uses a construction of the Speh representation as a quotient of a degenerate principal series representation induced from a character of the \((n, n)\) standard parabolic subgroup (see \( \S 2 \) below).

In the present paper we use the explicit constructions of \cite{SaSt90} and give a direct proof that the spaces of \( Sp_{2n}(\mathbb{R}) \)-invariant functionals on the Speh representations of \( GL_{2n}(\mathbb{R}) \) are zero if \( n \) is odd and one-dimensional if \( n \) is even. We also analyze functionals invariant with respect to subgroups of \( Sp_{2n}(\mathbb{R}) \).

To describe our result we need some further notation. Let \( G := G_{2n} \) denote the group \( GL_{2n}(\mathbb{R}) \).

Let \( \omega_{2n} \) be the standard symplectic form on \( \mathbb{R}^{2n} \). More explicitly \( \omega_{2n} \) is given by \( \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \) and let \( H := H_{2n} = Sp_{2n}(\mathbb{R}) < G_{2n} \) denote the stabilizer of this form. Let \( H := H_{2n} = Sp_{2n}(\mathbb{R}) < G_{2n} \) denote the stabilizer of this form. Let

\[
P := \left\{ \begin{pmatrix} g & X \\ 0 & g^{-1} \end{pmatrix} \mid g \in GL_n(\mathbb{R}), X \in \text{Mat}_{n \times n}(\mathbb{R}), X = X^t \right\} < H
\]
denote the Siegel parabolic subgroup. Let \( U(n) < H_{2n} < G_{2n} \) be the unitary group.

In this paper we prove the following result.

**Theorem A.**

(i) If \( n \) is even then

\[
\text{Hom}_H(\delta_m^\infty, \mathbb{C}) = \text{Hom}_P(\delta_m^\infty, \mathbb{C}) \simeq \mathbb{C}
\]

(ii) If \( n \) is odd then

\[
\text{Hom}_H(\delta_m^\infty, \mathbb{C}) = \text{Hom}_{U(n)}(\delta_m^\infty, \mathbb{C}) = 0.
\]

It is known that the restriction of \( \delta_m \) to \( SL_{2n}(\mathbb{R}) \) decomposes as a direct sum of two irreducible components. It follows from Theorem A that exactly one of them admits an \( H \)-invariant functional. In Lemma 4.2 we determine which one does.

It is easy to see that if \( n \) is odd and \( m \) is even then there are no functionals on \( \delta_m^\infty \) invariant with respect to \( -\mathrm{Id} \in H \), and thus neither \( P \)-invariant nor \( U_n \)-invariant functionals exist (see Remark 6.1).

**Remark.** Although the pair \((G_{2n}, P)\) is not a Gelfand pair for simple geometric reasons, we show that the Speh representation \( \delta_m \) still admits at most one \( P \)-invariant functional (at least for even \( n \)). The reason we suspected this result to hold is that, as shown in \cite{SaSt90}, Speh representations stay irreducible when restricted to a standard maximal parabolic subgroup \( Q \subset G \) satisfying \( Q \cap H = P \). It is possible that \((Q, P)\) is a generalized Gelfand pair, i.e. the space of \( P \)-invariant functionals on the space of smooth vectors of any irreducible unitary representation of \( Q \) is at most one dimensional. However, this statement does not imply our uniqueness result, since the space of \( G \)-smooth vectors of \( \delta_m \) could a priori have more functionals.

1.1. **Klyachko models.** For any \( n \), any even \( k \leq n \) and any field \( F \), \cite{Kly84} defines a subgroup \( Kl_k \) of \( GL_n(F) \) and a generic character \( \psi_k \) of \( Kl_k \). In particular, \( Kl_n = Sp_2(F) \) (if \( n \) is even) and \( Kl_0 \) is the group of upper unitriangular matrices. For local fields \( F \), it is shown in \cite{HR90, OS07, OS08a, OS08b, OS09, GOSS12, AOST12} that for any irreducible unitary representation \( \pi \) of \( GL_n(F) \) there exists a non-zero \((Kl_k, \psi_k)\)-equivariant functional on \( \pi^\infty \) for exactly one \( k \). The uniqueness of such functional is known only over non-archimedean fields (see \cite{OS08a}).

The proof of existence of \( k \) for \( F = \mathbb{R} \), given in \cite{GOSS12}, is done by reduction to the statement that certain representations are \( H \)-distinguished. This case is reduced, using the Vogan classification of the unitary dual, to the proof of existence of \( H \)-invariant functionals on Speh representations (for even \( n \)). This existence is proved using a global (adelic) argument. In this paper we give an explicit local construction of such a functional. Together with \cite{GOSS12} this gives a proof of existence of Klyachko models which uses only the representation theory of \( GL_n(\mathbb{R}) \) (and the theory of distributions).
1.2. Structure of the proof. We use the realization of $\delta_m^\infty$ as the image of a certain intertwining differential operator $\Box^m : \pi_{-m} \to \pi_m$, where $\pi_{-m}$ and $\pi_m$ are certain degenerate principal series induced from characters of a fixed $(n,n)$-parabolic subgroup $Q \subset G$ (see [AGS08]).

The study of the even case is divided into two parts. In § we first use the realization of $\delta_m$ as a quotient of the degenerate principal series $\pi_{-m}$ to lift a linear $P$-invariant functional on $\delta_m$ to an equivariant distribution on $G$. More precisely, we study $P \times \overline{Q}$-equivariant distributions on $G$. The technical heart is Corollary 3.3 which shows that such distributions do not vanish on the open cell $N\overline{Q}$.

This is based on the techniques of AGS08, classical invariant theory and a careful analysis of the double cosets $P \backslash G/\overline{Q}$, which is postponed to §. Then we analyze the distributions supported on the open cell by identifying them with the space of distributions on $N$ with a certain equivariance property. Identifying $N$ with its Lie algebra and using the Fourier transform we show that this space is at most one dimensional for even $n$. This finishes the proof of Proposition 3.1 which states that there exists at most one invariant $P$-invariant functional in the $n$ even case.

In the second part (§ we construct an $H$-invariant functional as an $H \times \overline{Q}$-equivariant distribution on $G$. For that we fix an explicit $H \times \overline{Q}$-equivariant polynomial $p$, consider the meromorphic family of distributions $|p|^\lambda$ (cf. [Ber72]) and take the principal part of this family at $\lambda = (n-m)/2$. This distribution defines an $H$-invariant functional on $\pi_{-m}^\infty$. To show that the restriction of this functional to $\delta_m^\infty$ is non-zero (Lemma 4.1) we use Corollary 3.3 along with another lemma from § on non-existence of equivariant distributions with certain support. The uniqueness of $P$-invariant functionals and the existence of $H$-invariant ones imply that the two spaces are equal. Our proof shows that the spaces of such functionals are equal and one-dimensional also for the (reducible) representations $\pi_m$ and $\pi_{-m}$.

For odd $n$ we prove that already a $U(n)$-invariant functional does not exist (Corollary 6.4). We do that by analyzing the $O(2n)$-types of $\delta_m$ described in [HL99, Sal95] and showing that none of these have a $U(n)$-invariant vector.

To summarize, Theorem A follows from Proposition 3.1 on uniqueness of $P$-invariant functionals for even $n$, Lemma 4.1 on existence of $H$-invariant functionals for even $n$ and Corollary 6.4 on non-existence of $U(n)$-invariant functionals for odd $n$.

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2. Preliminaries

2.1. Notation. Recall the notation $G = G_{2n} = GL_{2n}(\mathbb{R})$, and $H = H_{2n} = Sp_{2n}(\mathbb{R}) \subset G$. Let

\[ Q := \left\{ \begin{pmatrix} a & c \\ 0 & d \end{pmatrix} \in G \right\} \quad \overline{Q} := \left\{ \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} \in G \right\} \quad N := \left\{ \begin{pmatrix} \text{Id}_n & c \\ 0 & \text{Id}_n \end{pmatrix} \in G \right\} . \]

Recall that $P$ denotes $Q \cap H$ and let

\[ M := \left\{ \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \right\} \quad \text{and} \quad U := \left\{ \begin{pmatrix} \text{Id}_n & B \\ 0 & \text{Id}_n \end{pmatrix} \mid B = B^t \right\} \]

denote the Levi subgroup and the unipotent radical of $P$.

For $g \in \text{Mat}_{1 \times 1}(\mathbb{R})$ we denote $|g| := |\det(g)|$ and $\sgn(g) := \sgn(\det(g))$.

For $g = \begin{pmatrix} A & 0 \\ B & D \end{pmatrix} \in \overline{Q}$ we denote $\gamma(g) := |A||D|^{-1}$ and $\varepsilon(g) := \sgn(D)$.

For any integer $m$ let $L_m$ denote the character of $\overline{Q}$ given by $L_m := e^{m+1-n-(n+m)/2}$. Let $\pi_m$ denote the (unnormalized) induced representation $\text{Ind}_{\overline{Q}}^G(L_m)$. Considering $N$ as an open subset of $G/\overline{Q}$, one can restrict smooth vectors of $\pi_m$ to $N$. This restriction is an embedding since $N$ is an open subset of $G/\overline{Q}$. We sometimes identify $N$ and its Lie algebra $\mathfrak{n}$ with $\text{Mat}_{n \times n}(\mathbb{R})$ in the obvious way. This enables us to define the Fourier transform on $\mathfrak{n}$. Denote by $M^+_n$ (respectively $M^-_n$) the subset of $\text{Mat}_{n \times n}$ consisting

...
of matrices with nonnegative (resp. nonpositive) determinant. For \( f \in \pi_m^\infty \) we denote its restriction to \( n \) by \( f|_n \). We denote the space of all smooth functions obtained in this way by \( \pi_m^\infty|_n \).

2.2. Sahi-Stein realization of the Speh representations. For any \( m \in \mathbb{Z}_{\geq 0} \) define

\[
\hat{H}_m := \{ f \in S^*(n) \mid f \in L^2(n, |x|^{-m}dx) \} \quad \text{and} \quad \hat{H}_m^\pm := \{ f \in \hat{H}_m \mid \text{Supp} \ f \subset M_m^\pm \},
\]

where \( S^*(n) \) denotes the space of tempered distributions \( n \). The \( \hat{H}_m \) and \( \hat{H}_m^\pm \) are Hilbert spaces with the scalar product

\[
(f, g) = (\hat{f}, \hat{g})_{L^2(n, |x|^{-m}dx)}.
\]

Define an action of \( G \) on \( \hat{H}_m \) by

\[
\delta_m(g)f(x) := L_m(g)f(a^{-1}(c + xd)), \quad \text{for} \quad g = \begin{pmatrix} a & c \\ 0 & d \end{pmatrix},
\]

or equivalently on the Fourier transform side by

\[
\delta_m(g)\hat{f}(\xi) = \exp(2\pi i \text{Tr}(cq^{-1}\xi))L_m^{-1}(q)\hat{f}(d^{-1}\xi a).
\]

Summarizing the main results of [SaSt90] we obtain

**Theorem 2.1 (SaSt90).** Let \( m \in \mathbb{Z}_{\geq 0} \). Then

(i) The action of \( G \) extends to a unitary representation \( \delta_m \) of \( G \) on \( \hat{H}_m \).

(ii) \( (G, \delta_m, \hat{H}_m) \) is isomorphic to the Speh representation of \( G \).

(iii) There exists an epimorphism \( \pi-m \to \delta_m \) and an embedding \( \delta_m \subset \pi_m \). The latter is defined on the smooth vectors by the inclusion \( \delta_m^\infty \subset \pi_m^\infty|_n \).

(iv) The restriction of \( \delta_m \) to \( SL(2n, \mathbb{R}) \) is a direct sum of two irreducible representations \( \delta_m^\pm \), realized on the subspaces \( \hat{H}_m^\pm \).

Consider the determinant as a polynomial on \( n \) and let \( \Box_m \) denote the corresponding differential operator.

**Theorem 2.2.** The operator \( \Box_m \) defines a continuous \( G \)-equivariant map \( \pi_m^\infty \to \pi_m^\infty \) with image \( \delta_m^\infty \).

**Proof.** By [KV77] Proposition 2.3 (see also [Boe85]), the operator \( \Box_m \) defines a continuous \( G \)-equivariant map \( \pi_m^\infty \to \pi_m^\infty \), which is non-zero by [SaSt90]. By [HL99] Theorems 3.4.2-3.4.4 \( \pi_m \) has unique composition series in the strong sense, meaning that any quotient of \( \pi_m \) has a unique irreducible subrepresentation, and all these irreducible subquotients are pairwise non-isomorphic. It is easy to see that \( \pi_m \) is dual to \( \pi_m \), and thus their composition series are opposite. Hence the image of any nonzero intertwining operator from \( \pi_m \) to \( \pi_m \) is the unique irreducible subrepresentation of \( \pi_m \). Since \( \delta_m^\infty \) is an irreducible subrepresentation of \( \pi_m \), it is the image of \( \Box_m \).

**Remark 2.3.** One can deduce Theorem 2.2 also from [KS93], which computes the action of \( \Box_m \) on every \( K \)-type, where \( K = O(2n, \mathbb{R}) \). From the formula in [KS93] and the description of the \( K \)-types of the composition series of \( \pi_m \) in [HL99, Sah95] one can see that \( \Box_m \) does not vanish precisely on the \( K \)-types of \( \delta_m \).

2.3. Invariant distributions.

**Definition 2.4.** For an affine algebraic manifold \( M \) we denote by \( S(M) \) the space of Schwartz functions on \( M \), that is smooth functions \( f \) such that \( df \) is bounded for any differential operator \( d \) on \( M \) with polynomial coefficients. We endow this space with a Fréchet topology using the sequence of seminorms

\[
N_d(f) := \sup_{x \in M} |df(x)|, \quad \text{where} \quad d \quad \text{is a differential operator on} \quad M \quad \text{with polynomial coefficients}.
\]

Also, for an algebraic vector bundle \( E \) over \( M \) we denote by \( S(M, E) \) the space of Schwartz sections of \( E \). We denote by \( S^*(M, E) \) the space of continuous linear functionals on \( S(M, E) \) and call its elements tempered distributional sections. For a closed subvariety \( Z \subset M \) we denote by \( S^*_M(Z, E) \subset S^*(M, E) \) the subspace of tempered distributional sections supported in \( Z \). For the theory of Schwartz functions and distributions on general semi-algebraic manifolds we refer the reader to [AG08].
Proposition 3.1. For any integer \( n \) we have

\[
\dim((\pi_m^\infty)^*)^P \leq 1.
\]

Since \( \Delta_{\overline{Q}} = \gamma^{-n} \), we obtain from the definition of \( \pi_m \) and Lemma 2.8

\[
(\pi_m^\infty)^* \simeq S^*(G)_{\overline{Q}}^{\Delta^e n+1, \gamma(n-m)/2}
\]

and thus in order to prove Proposition 5.1 we have to show that for even \( n \)

\[
\dim S^*(G)_{\overline{Q}}^{P \times \overline{Q}, 1 \times e^{m+1, \gamma(n-m)/2}} \leq 1.
\]

We will need the following proposition, which we will prove in section 5.

Proposition 3.2. Denote \( K := P \times \overline{Q} \), and let \( x \notin N \overline{Q} \). Then

\[
\text{Sym}^*(X^G_{P \times \overline{Q}, x})|_{K_x, e^{m+1, \gamma(n-m)/2} \Delta_{K_x}^{-1}} = 0.
\]

From this proposition and Corollary 2.7 we obtain
Corollary 3.3.

$$S^*_{G}(G - NQ)^{P \times Q, 1 \times \epsilon^{m+1}, \gamma^{n-m}/2} = 0.$$  

By this corollary it is enough to analyze $$S^*(NQ)^{P \times Q, 1 \times \epsilon^{m+1}, \gamma^{n-m}/2}$$. Let $$S$$ denote the space of symmetric $$n \times n$$ matrices, and $$A$$ denote the space of anti-symmetric $$n \times n$$ matrices. Identify $$M \cong GL_n(\mathbb{R})$$ and let it act on $$S$$ and on $$A$$ by $$x \mapsto gxg^t$$.

**Lemma 3.4.**

$$S^*(NQ)^{P \times Q, 1 \times \epsilon^{m+1}, \gamma^{n-m}/2} \cong S^*(A)^{GL_n(\mathbb{R}), \det^{1-m} \cong S^*(A)^{GL_n(\mathbb{R}), \text{sgn}^{m+1} | \cdot |^{m-n}}$$

**Proof.** Identify $$U \cong S$$ and let it act on itself by translations. Then $$NQ$$ is isomorphic as a $$P \times Q$$-space to $$A \times S \times Q$$, where $$Q$$ acts on the third coordinate (by right translations), $$U$$ acts on the second coordinate and $$M$$ acts on the first and the second coordinates. Note that the action on $$S \times Q$$ is transitive and that $$\Delta_Q = \gamma^{-n}$$ and $$\Delta_P \left( \begin{array}{cc} g & 0 \\ 0 & g^t \end{array} \right) = |g|^{n+1}$$. The first isomorphism follows now from Frobenius descent.

The second isomorphism is given by Fourier transform on $$A$$ defined using the trace form. □

Let $$O \subset A$$ denote the open dense subset of non-degenerate matrices and $$Z$$ denote its complement. The following lemma is a straightforward computation.

**Lemma 3.5.**

(i) Every orbit in $$Z$$ includes an element of the form $$x = \left( \begin{array}{cc} 0_{k \times k} & 0 \\ 0 & \omega_{n-k} \end{array} \right)$$.

(ii) $$N^A_{GL_n(\mathbb{R})}x, x \cong \left\{ \left( \begin{array}{cc} 0_{k \times k} & b \\ 0 & 0 \end{array} \right) \right\}$$ and $$GL_n(\mathbb{R})_x = \left\{ \left( \begin{array}{cc} a_{k \times k} & 0 \\ c & d \end{array} \right) \text{ such that } d \text{ is symplectic} \right\}$$

(iii) $$\Delta_{GL_n(\mathbb{R})}_x = | \cdot |^{-(n-k)}$$

**Corollary 3.6.**

$$\text{Sym}^*(N^A_{GL_n(\mathbb{R})}x, x)^{GL_n(\mathbb{R}), \text{sgn}^{m+1} | \cdot |^{m-n} \cdot \Delta^{-1}_{GL_n(\mathbb{R})}_x} = 0$$

**Proof.** From the previous lemma $$\text{sgn}^{m+1} | \cdot |^{m-n} \cdot \Delta^{-1}_{GL_n(\mathbb{R})}_x = \text{sgn}^{k+1} \det^{n-k}$$. If $$n$$ is even then so is $$k$$ and thus this is not an algebraic character of $$GL_n(\mathbb{R})_x$$ and thus there are no tensors that change under this character. □

**Corollary 3.7.**

$$\dim S^*(A)^{GL_n(\mathbb{R}), \text{sgn}^{m+1} | \cdot |^{m-n}} \leq 1$$

**Proof.** By Corollary 3.6 and Corollary 2.7

$$(2) \quad S^*(Z)^{GL_n(\mathbb{R}), \text{sgn}^{m+1} | \cdot |^{m-n}} = 0.$$  

Therefore, the restriction of equivariant distributions to $$O$$ is an embedding. Now,  

$$\dim S^*(O)^{GL_n(\mathbb{R}), \text{sgn}^{m+1} | \cdot |^{m-n}} \leq 1$$

since $$O$$ is a single orbit.

□

Proposition 3.1 follows now from Corollary 3.7, Lemma 3.4, Corollary 3.3 and (1).

**Remark 3.8.** For odd $$n$$ Corollary 3.3 does not hold. For example, the smallest orbit does support an equivariant distribution.

Let $n$ be even. In this section we construct an $H$-invariant functional $\phi$ on $\pi_m^\infty$ for any $m \in \mathbb{Z}_{\geq 0}$ and show that its restriction to $\delta_m^\infty$ is non-zero. Define a polynomial $p$ on $Mat(2n \times 2n, \mathbb{R})$ by

$$p \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) := \det(D^TB - B^TD)$$

Note that $p$ is non-negative, $H$-invariant on the left and changes under the right multiplication by $\overline{Q}$ by the character $| \cdot |^{-1}$. Consider the meromorphic family of distributions on $Mat(2n \times 2n, \mathbb{R})$ given by

$$\xi^m_\lambda := p^\lambda | \cdot |^{-\lambda} \varepsilon^{m+1}.$$  

This family is defined by [Ber72]. For $\Re \lambda > 0$, the restriction of this distribution to $G = GL(2n, \mathbb{R})$ is a non-zero smooth function, and thus the restriction $\eta^m_\lambda$ of the family is not an identical zero. Note that $\eta^m_\lambda \in S^*(G(H \times \mathbb{G}, \mathbb{R}^{m+1}, \gamma))$.

Let $\alpha \in S^*(G)$ be the principal part of this family at $\lambda = \frac{n-m}{2}$. By (1) $\alpha$ defines a non-zero $H$-invariant functional $\phi$ on $\pi_m^\infty$.

Lemma 4.1. $\phi|_{\delta_m^\infty} \neq 0$.

Proof. By Theorem 2.3 it is enough to show that $\Box^m \phi \neq 0$. By Corollary 3.3 $\alpha|_{\mathbb{T} \delta_m^\infty} \neq 0$. It is enough to show that $(\Box^m \alpha)|_{\mathbb{T} \delta_m^\infty} \neq 0$. As in [3] let $A \subset N$ denote the subspace of anti-symmetric matrices and $O \subset A$ the open subset of non-degenerate matrices. Note that $\alpha|_{\mathbb{T} \delta_m^\infty}$ is $P \times \overline{Q}$-equivariant and let $\beta \in S^*(A)^{GL(n, \mathbb{R}), \det 1-m}$ be the distribution on $A$ corresponding to $\alpha$ by the Frobenius descent (see Lemma 3.3). Note that $F(\Box^m \beta) = F(\beta)$ multiplied by a polynomial. Thus it is enough to show that $F(\beta)$ has full support, i.e. $F(\beta)|_{\Omega} \neq 0$. This follows from the equivariance properties of $F(\beta)$ by [2]. \hfill $\Box$

This argument in fact proves slightly more.

Lemma 4.2. $\phi|_{(\delta_m^\infty)} \neq 0$.

Proof. If $g$ is a Schwartz function on $M^+_n \subset N$ then its Fourier transform $\hat{g}$ determines a vector in $(\delta^+_m)^\infty$ by Theorem 2.1. Thus it is enough to find such a $g$ for which $\zeta(\hat{g}) \neq 0$, where $\zeta$ denotes the $P$-invariant distribution on $N$ corresponding to $\alpha$.

Let $f$ be a compactly supported smooth function on $O$ such that $\beta(F(f)) \neq 0$. Since the determinant is positive on $O$, there exists a compact neighborhood $Z$ of zero in the space $S$ of symmetric $n$ by $n$ matrices such that $\supp(f) + Z \subset M^+_n$. Let $h$ be a smooth function on $S$ which is supported on $Z$ and s.t. $h(0) = 1$. Let $g := f \boxtimes h$ be the function on $N$ defined by $g(X + Y) := f(X)h(Y)$ where $X \in A$ and $Y \in S$. Let $F_S$ denote the Fourier transform on $S$. Then we have

$$\zeta(\hat{g}) = \zeta(F(f) \boxtimes F_S(h)) = \beta(F(f)) \neq 0.$$  

\hfill $\Box$

Remark 4.3. (i) For odd $n$, the polynomial $p$ is identically zero, since the matrix $D^TB - B^TD$ is an anti-symmetric matrix of size $n$.

(ii) The polynomial $p$ defines the open orbit of $H$ on $G/\overline{Q}$. In general, one can show that if a linear complex algebraic group $G$ acts on a complex affine algebraic manifold $M$, both defined over $\mathbb{R}$, $W$ is a basic open subset of $M$ defined by a $G$-equivariant polynomial $p$ with real coefficients, $\chi$ is a character of the group of real points $G(\mathbb{R})$ and there exists a non-zero $(G(\mathbb{R}), \chi)$-equivariant holonomic tempered distribution $\xi$ on $W(\mathbb{R})$ then there exists a non-zero $(G(\mathbb{R}), \chi)$-equivariant holonomic tempered distribution on $M(\mathbb{R})$.

To prove that consider the analytic family of distributions $|p|^\lambda \xi$ on $W$. For $\Re \lambda$ big enough, it can be extended to a family $\eta_\lambda$ on $M(\mathbb{R})$. By [Ber72] the family $\eta_\lambda$ has a meromorphic continuation to the entire complex plane. Note that the distributions in this family are equivariant with a character that depends analytically on $\lambda$. Thus taking the principal part at $\lambda = 0$ we obtain a non-zero $(G(\mathbb{R}), \chi)$-equivariant holonomic tempered distribution on $M(\mathbb{R})$. 


Note that since this construction involves taking principal part, the obtained distribution is not necessary an extension of the original $\xi$. This can already be seen in the case when $M$ is the affine line, $W$ is the complement to 0 and $G$ is the multiplicative group.

If $G$ has finitely many orbits on $M$ then any $G(\mathbb{R})$-equivariant distribution on $M(\mathbb{R})$ is holonomic.

5. Proof of Proposition 3.2

We start from the description of the double cosets $P \setminus G/Q$. Let $r_1, r_2, s, t$ be non-negative integers such that $r_1 + r_2 + 2s + 2t = n$. We will view $2n \times 2n$ matrices as $10 \times 10$ block matrices in the following way. First of all, we view them as $2 \times 2$ block matrices with each block of size $n \times n$. Now, we divide each block to $5 \times 5$ blocks of sizes $r_1, r_2, s, s, 2t$ in correspondence. Denote by $\sigma_{16}$ the permutation matrix that permutes blocks $1$ and $6$, by $\sigma_{39}$ the permutation matrix that permutes blocks $3$ and $9$, and by $\tau_{5,10}$ the matrix which has \( \begin{pmatrix} \text{Id}_{2t} & \omega_{2t} \\ \omega_{2t} & \text{Id}_{2t} \end{pmatrix} \) in blocks $5$ and $10$ and is equal to the identity matrix in other blocks.

Recall the notation $\omega_{2t} := \begin{pmatrix} 0 & \text{Id}_t \\ -\text{Id}_t & 0 \end{pmatrix}$. Denote

$$ x_{r_1, r_2, s, t} := \sigma_{16} \sigma_{39} \tau_{5,10}. $$

**Lemma 5.1.** Each double coset in $P \setminus \text{GL}_{2n}(\mathbb{R})/\overline{Q}$ includes a unique element of the form $x_{r_1, r_2, s, t}$. The orbits in $N\overline{Q}$ correspond to $r_2 = s = 0$.

**Proof.** Recall that $G/\overline{Q}$ is the Grassmannian of $n$-dimensional subspaces of $\mathbb{R}^{2n}$. Let $L := \text{Span}\{e_{n+1}, \ldots, e_{2n}\} \subset \mathbb{R}^{2n}$ be the standard Lagrangian subspace. To an $n$-dimensional subspace $W \subset \mathbb{R}^{2n}$ we associate the following invariants:

- $r_2 := \dim L \cap W \cap W^\perp$,
- $r_1 := \dim W^\perp \cap W - r_2$,
- $s := \dim L \cap W - r_2$,
- $t := (n - r_1 - r_2)/2 - s$.

Note that $n - r_1 - r_2$ is even since it is the rank of $\omega_{|W}$. Clearly, $W \in N\overline{Q}$ if and only if $r_2 = s = 0$.

The equality of vectors

$$ (v_1, 0, v_2, 0, \omega_{2t} u ) | 0, w_2, w_1, 0, u )^t = x_{r_1, r_2, s, t} (0, 0, 0, 0, 0 | v_1, w_2, w_1, v_2, u )^t. $$

It is enough to show that $W$ can be brought, using the action of $P$, to a space of vectors of the form

$$ (v_1, 0, v_2, 0, \omega_{2t} u ) | 0, w_2, w_1, 0, u )^t. $$

Clearly, $W$ can be brought to a space of vectors of the form $(v, Au + Bv | Cw, w, Dw)^t$, where $\text{size}(v) + \text{size}(w) = n$ and $A$ is a square matrix. Let us write this in more detailed form, with the same block sizes in the first $n$ coordinates and last $n$ coordinates:

$$ (v_1, v_2, A_{11} w_1 + A_{12} w_2 + B_{11} v_1 + B_{12} v_2, A_{21} w_1 + A_{22} w_2 + B_{21} v_1 + B_{22} v_2, C_{11} w_1 + C_{12} w_2, w_1, w_2, D_{11} w_1 + D_{12} w_2)^t. $$

Denote the first four blocks by $e_i$ and the last by $f_i$. For any $i$ and any $j \neq i$, $M$ allows us to do the following operations:

$$ (1)_{ij} \quad e_i \mapsto ge_i, \quad f_i \mapsto g^{-1} f_i, $$

$$ (2)_{ij} \quad e_i \mapsto e_i + ae_j, \quad f_i \mapsto f_j - A^t f_i. $$

Similarly, $U$ allows us to do two more operations:

$$ (3)_{ij} \quad e_i \mapsto e_i + bf_j, \quad e_j \mapsto e_j + b^t f_i $$

$$ (4)_{ij} \quad e_i \mapsto e_i + (c + c') f_i $$

Using $(2)_{11}$ and $(2)_{12}$, and redefining $C$ and $D$ we get $B = 0$. Using $(2)_{21}$ and $(2)_{22}$, and redefining $A$ we get $C = 0$ and $D = 0$.

Using $(3)_{32}$ and $(3)_{42}$ and $(3)_{43}$ we get $A_{11} = A_{21} = A_{22} = 0$. Using $(3)_{33}$ we make $A_{12}$ anti-symmetric.

Now, using $(1)_{11}$ we can replace $A_{12}$ by $gA_{12}g^t$ and thus we can bring it to the form $A_{12} = \begin{pmatrix} 0 & 0 \\ 0 & \omega_{2t} \end{pmatrix}$. \(\square\)
5.1. Proof of Lemma 5.2.

Note that since

(i) The stabilizer in

(ii) If \( s = 0 \) then

\[
\text{Sym}^*(N^G_{P \times Q})^K \epsilon_{m+1/2} (n-m)/2 \Delta_K \kappa \Delta_{K_s}^{-1} = 0.
\]

where \( \omega_d \) denotes the space of antisymmetric matrices and \( GL_i \) act by \( a \mapsto gag^t \).

For the proof of this lemma see 5.1.

Lemma 5.3. Let \( k, l \in \mathbb{Z}_{\geq 0}, r \in \mathbb{Z}_{>0} \).

(i) If \( k \neq l \pmod{2} \) then

\[
\text{Sym}^*(g_{r_i})^{GL_{r_i} \cdot |^l \text{sgn}^d} = 0.
\]

(ii) If \( k > 0 \) and \( r \) is odd then

\[
\text{Sym}^*(o_{r_i})^{GL_{r_i} \cdot \text{det}^k} = 0.
\]

Proof.

(i) The only algebraic characters of \( GL_r \) are powers of the determinant.

(ii) The stabilizer in \( GL_r \) of every matrix in \( o_r \) has an element with determinant bigger than 1. □

Proof of Proposition 5.2. By Lemmas 5.1 and 5.2 it is enough to show that

\[
\text{Sym}^*(g_{r_i})^{GL_{r_i} \cdot |^d \text{sgn}^1} = 0
\]

Note that since \( n \) is even, \( r_1 \) and \( r_2 \) are of the same parity. If they are even then (ii) follows from Lemma 5.3(i), and otherwise from Lemma 5.3(ii). □

5.1. Proof of Lemma 5.2. Let \( x = x_{r_1, r_2, s, t} \) be as in the lemma. We need to compute the space \( N^G_{x, P \times Q} \) the stabilizer \( K_x \) and its modular function. In order to do that we compute the conjugates of \( P \) and its Lie algebra \( p \) under \( x \).

Lemma 5.4. Let \( q = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in Q \). Then \( x^{-1} qx = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), where

\[
A = \begin{pmatrix}
d_{11} & 0 & d_{14} & 0 & 0 \\
0 & a_{22} & b_{24} & a_{24} & a_{25} \\
d_{41} & 0 & d_{44} & 0 & 0 \\
b_{41} & a_{42} & b_{44} & a_{44} & a_{45} \\
b_{51} - \omega d_{51} & a_{52} & b_{54} - \omega d_{54} & a_{54} & a_{55}
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
0 & d_{12} & d_{13} & 0 & d_{15} \\
0 & a_{21} & b_{23} & a_{23} & b_{25} + a_{25} \omega \\
0 & a_{41} & b_{42} & a_{43} & b_{45} + a_{45} \omega \\
a_{51} & b_{52} - \omega d_{52} & b_{53} - \omega d_{53} & a_{53} & b_{55} + a_{55} \omega - \omega d_{55}
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
b_{11} & a_{12} & b_{14} & a_{14} & a_{15} \\
d_{21} & 0 & d_{24} & 0 & 0 \\
d_{31} & 0 & d_{34} & 0 & 0 \\
b_{41} & a_{42} & b_{44} & a_{44} & a_{45} \\
d_{51} & 0 & d_{54} & 0 & 0
\end{pmatrix}
\]

\[
D = \begin{pmatrix}
a_{11} & b_{12} & b_{13} & a_{13} & b_{15} + a_{15} \omega \\
0 & d_{22} & d_{23} & 0 & d_{25} \\
0 & d_{32} & d_{33} & 0 & d_{35} \\
a_{31} & b_{32} & b_{33} & a_{33} & b_{35} + a_{35} \omega \\
0 & d_{52} & d_{53} & 0 & d_{55}
\end{pmatrix}
\]

This lemma is a straightforward computation, which can be done using a computer.

We can identify \( T_2 \mathbb{G} \cong \mathfrak{gl}_{2n} \). Under this identification \( T_{x} P \times Q \cong x^{-1} px + \overline{q} \) and

\[
N^G_{x, P \times Q} \cong \mathfrak{gl}_{2n} / (x^{-1} px + \overline{q}) \cong n / (n \cap (x^{-1} px + \overline{q})).
\]

From the previous lemma we obtain
Corollary 5.5. Let $V \subset \mathfrak{n}$ denote the subspace consisting of matrices of the form

$$\begin{pmatrix}
  n_{11} & n_{12} & 0 & n_{14} & n_{15} \\
  n_{12} & n_{22} & 0 & 0 & 0 \\
  n_{31} & 0 & 0 & n_{34} & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  n_{15} & 0 & 0 & 0 & 0
\end{pmatrix},$$

such that $n_{22} = -n_{12}^2$.

Then $V$ projects isomorphically onto $\mathfrak{n}/(\mathfrak{n} \cap (x^{-1}px + \mathcal{Q}))$.

Now let us analyze the stabilizer $K_x$. From Lemma 5.3 we obtain

Corollary 5.6.

(i) Using the projection on the first coordinate

$$K_x \cong P \cap \mathcal{Q}x^{-1} \cong \left\{ \begin{pmatrix} A & B \\ 0 & A^{-t} \end{pmatrix} \in P \text{ s.t.} \begin{pmatrix}
  A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\
  0 & A_{22} & 0 & 0 & 0 \\
  0 & A_{32} & A_{33} & 0 & 0 \\
  0 & A_{42} & 0 & A_{44} & 0 \\
  0 & A_{52} & 0 & 0 & A_{55},
\end{pmatrix} \right\},$$

where $A_{55}$ is symplectic and $B$ is a symmetric matrix of the form $B = \begin{pmatrix}
  B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\
  B_{12} & B_{13} & 0 & 0 & 0 \\
  B_{13} & 0 & B_{33} & 0 & 0 \\
  B_{14} & 0 & 0 & B_{44} & 0 \\
  B_{15} & 0 & B_{35} & 0 & B_{45}
\end{pmatrix}$.

(ii) The modular function of $K_x$ is given by

$$\Delta_{K_x}(\begin{pmatrix} A & B \\ 0 & A^{-t} \end{pmatrix}) = |A_{11}|^{2n-r_1+1}|A_{22}|^{-n+r_1+r_2}|A_{33}|^{n-r_1-s+1}|A_{44}|^{n-r_1-s+1}. \tag{33}$$

(iii) Let $q = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in \mathcal{Q} \cap x^{-1}P_x$. Let $k = (xq^{-1}, q) \in K_x$. Then $k$ acts on $V$ by

$$k \cdot n = \text{pr}_V(AnD^{-1}),$$

where $\text{pr}_V : \mathfrak{n} \to V$ denotes the projection.

Corollary 5.7. Denote

$$\chi := \varepsilon^{m+1}k^{(n-m)/2} \cdot \Delta_K|_{K_x} \cdot \Delta_{K_x}^{-1}. \tag{34}$$

Let

$$q = \text{diag}(a, b, c, b^{-t}, d, d^{-t}, d, d^{-t}, \text{Id}).$$

Let $k := (xq^{-1}, q) \in K_x$. Then

$$\chi(k) = (\text{sgn}(a) \text{ sgn}(b) \text{ sgn}(c) \text{ sgn}(d))^{m+1}|a|^{-m-r_1}|b|^{2s+2t-m+1}|c|^{-r_1-s}|d|^{-r_1-s}. \tag{35}$$

Proof.

$$\gamma(q) = |a|^2|b|^2 \quad \text{and} \quad \Delta_{\mathcal{Q}}(q) = |a|^{-2n}|b|^{-2n}$$

$$xq^{-1} = \text{diag}(a^{-t}, b, d^{-t}, c, \text{Id}, a, b^{-t}, d, c, \text{Id})$$

$$\Delta_K(k) = |a|^{-3n-1}|b|^{-n+1}|c|^{-n-1}|d|^{-n-1}$$

$$\Delta_{K_x}(k) = |a|^{-2n+r_1-1}|b|^{-n+r_1+r_2}|c|^{-n+r_1+s-1}|d|^{-n+r_1+s-1}.$$  \hfill \square

Now we are ready to prove Lemma 5.2.
Proof of Lemma 6.3. If $s > 0$ then $\text{Sym}^s(V)^{K\times} = 0$, since tensors cannot have negative homogeneity degrees. Otherwise, $V$ involves only 3 blocks - the ones numbered 1, 2 and 5.

Let $p \in \text{Sym}^s(V)^{K\times}$. Identify $K_2$ with a subgroup of $Q$ using the second coordinate.

Consider the action of the block $A_{21}$. It can map any non-zero vector in the block $n_{11}$ to any vector in the block $n_{12}$. This action does not change any element in any other block of $V$ (it does effect $n_{22}$, but not its anti-symmetric part). Also, the character $\chi$ does not depend on $A_{21}$. Therefore $p$ does not depend on the variables in the block $n_{12}$.

In the same way, using the action of $A_{52}$, we can show that $p$ does not depend on the variables in the block $n_{15}$. Therefore, $p$ depends only on $n_{11}$ and $n_{22}$. Hence

$$\text{Sym}^s(V)^{K\times} \cong \text{Sym}^s(\mathfrak{gl}_1)^{K_{11}} \cdot |\cdot|^{-m-r_1} \text{sgn}^{m+1} \otimes \text{Sym}^s(\mathfrak{gl}_2)^{K_{22}} |\cdot|^{n+r_2} \text{sgn}^{n+1}.$$ $\square$


In this section we prove that if $n$ is odd then there are no $U(n)$-invariant functionals on the Speh representation and therefore there are no $H$-invariant functionals. We do that using $K$-type analysis. The maximal compact subgroup of $GL_{2n}(\mathbb{R})$ is $K := O(2n, \mathbb{R})$, and $U(n) = K \cap H$ is a symmetric subgroup of $K$. We show that no $K$-type of $\delta_n$ has a $U(n)$-invariant vector.

The root system of $K$ is of type $D_n$, and we make the usual choice of positive roots

$$\{\varepsilon_i \pm \varepsilon_j : i < j\}$$

where $\varepsilon_i$ is the $i$-th unit vector in $\mathbb{R}^n$. With this choice, the highest weights of $K$-modules are given by integer sequences $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ such that

$$\mu_1 \geq \cdots \geq \mu_{n-1} \geq \mu_n \geq 0. \tag{6}$$

Remark 6.1. From the definition of $\pi_m$ we see that if $n$ is odd and $m$ is even then the central element $-\text{Id} \in G$ acts by scalar $-1$, and there are neither $P$-invariant nor $U(n)$-invariant functionals on $\delta_m^\infty$.

Since $\delta_m^\infty$ is the irreducible quotient of $\pi_{-m}$, the following theorem follows from [HL99] Theorems 3.4.2 - 3.4.4 (see also [Sah95]).

Theorem 6.2. The $K$-types of $\pi_{m \pm 1}$ are given by sequences as in (6) with $\mu_i \equiv m + 1 \mod 2$, while the $K$-types of the Speh representation $\delta_m$ satisfy the additional condition $\mu_n \geq m + 1$.

Lemma 6.3. If $n$ is odd then no $K$-type $(\mu_1, \ldots, \mu_n)$ with $\mu_n \neq 0$ has $U(n)$-invariant vectors.

Proof. Let $\rho$ be an irreducible representation of $K$ with $\mu_n \neq 0$. Suppose that $\rho$ has a non-zero $U(n)$-invariant vector. Then $\rho = \rho_1 \oplus \rho_2$, where $\rho_i$ are irreducible non-zero representations of $K^n = SO(2n, \mathbb{R})$. The pair $(K, U(n))$ is a symmetric pair of compact groups and therefore a Gelfand pair. Hence the $U(n)$-invariant vector is unique up to a scalar and belongs to one of the $\rho_i$. Denote it by $v$ and say $v \in \rho_1$.

Consider $g := \begin{pmatrix} 1 \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \in K$. Since $n$ is odd, $g \notin K^0$. Hence $\rho(g)v \notin \rho_1$, since otherwise $\rho$ would be reducible. However, $g$ normalizes $U(n)$ and hence $\rho(g)v$ is $U(n)$-invariant and therefore proportional to $v$. Contradiction. $\square$

Corollary 6.4. If $n$ is odd then there are no $U(n)$-invariant functionals on $\delta_m^\infty$.

Proof. By Remark 6.1 we can assume that $m$ is odd. Then by Lemma 6.3 and Theorem 6.2 no $K$-type of $\delta_m$ has a $U(n)$-invariant vector. Therefore, the space of $K$-finite vectors, which decomposes to a direct sum of $K$-types, does not have a $U(n)$-invariant functional. This space is dense in $\delta_m^\infty$, hence there are no $U(n)$-invariant functionals on $\delta_m^\infty$ either. $\square$

Remark 6.5. Using the Cartan-Helgason theorem and the table in [Kna85] Appendix C, §2, it can be shown that the $K$-types that have $U_n$-invariant vectors are of the form $\mu_{2i-1} = \mu_{2i}$ for $1 \leq i \leq n/2$ and if $n$ is odd then $\mu_n = 0$, which gives an alternative proof of Lemma 6.3.
References


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