Discrete Geometry *

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Summary

Definitions of affine subspace, affine closure, affine combination, affine dependent, hyper plane, convex set and convex combination. Separation theorem. Radon's Lemma.

1 Affine notions

Definition 1:A set $X \subseteq \mathbb{R}^d$ is called an *affine subspace* if there exists a linear subspace $L \subseteq \mathbb{R}^d$ and a vector $a \in \mathbb{R}^d$ such that $X = L + a$.

Definition 2: The *affine closure (affine hull)* of a set $X \subseteq \mathbb{R}^d$ is the intersection of all affine subspaces of $\mathbb{R}^d$ containig X.

Definition 3: An *affine combination* of $a_1, a_2, ..., a_n \in \mathbb{R}^d$ is a vector of the form $\sum_{i=1}^n \alpha_i a_i$ where $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$ and $\sum_{i=1}^n \alpha_i = 1$.

Claim: Let $X \subseteq \mathbb{R}^d$, Then the affine closure of X equals to the set of all affine combinations of X. **(Check!)**

Definition 4: The points $a_1, a_2, ..., a_n \in \mathbb{R}^d$ are *affine dependent* if there exist $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$ not all zeros such that both $\sum_{i=1}^n \alpha_i a_i = 0$ and $\sum_{i=1}^n \alpha_i = 0$.

*Lecture Notes for a course given by Dr. Shakhar Smorodinsky at the Ben-Gurion University, Israel.

Comment 1: If n vectors $a_1, a_2, ..., a_n$ are affinely independent in $\mathbb{R}^d$ then the vectors $a_1 - a_n, a_2 - a_n, ..., a_{n-1} - a_n$ are linearly independent in $\mathbb{R}^d$. The number of linearly independent vectors in $\mathbb{R}^d$ is at the most d, so the number of affinely independent vectors in $\mathbb{R}^d$ is at the most $d + 1$.

Comment 2: The points $a_1, a_2, ..., a_n \in \mathbb{R}^d$ are affinely independent if and only if the $d \times d$ matrix A with $a_i - a_{d+1}$ as it columns is invertible, that is $det(A) \neq 0$.

Definition 5: The *dimension* of the affine subspace $X = L + v$ is the dimension of the linear subspace L.

Defintion 6: An affine subspace of dimension $d - 1$ in $\mathbb{R}^d$ is called a *hyperplane*. Each hyperplane h can be expressed as the set $\{x \in \mathbb{R}^d | \langle a, x \rangle = b\}$ where $a \in \mathbb{R}^d - \{0\}$ and $b \in \mathbb{R}$. A closed *half-space* in $\mathbb{R}^d$ is a set of the form $\{x \in \mathbb{R}^d | \langle a, x \rangle \geq b\}$. Such a half-space bounded by the hyperplane $\{x \in \mathbb{R}^d | \langle a, x \rangle = b\}$.

Definition 7: An affine subspace of dimension $k = 0, 1, ..., d - 2$ is called a *k-flat*.

2 Convexity

Definition 1: A set $A \subseteq \mathbb{R}^d$ is called *convex* if for every two points $x, y \in A$ and for every $\lambda \in [0, 1]$ we have $\lambda x + (1 - \lambda)y \in A$.

Claim 1: The intersection of any family of convex sets is also convex. **(Check!)**.

Definition 2: Let $\{a_1, a_2, ..., a_n\} \in \mathbb{R}^d$, we say that a point $b \in \mathbb{R}$ is a *convex combination* of $\{a_1, a_2, ..., a_n\}$ if there exist nonnegative $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}^d$ such that $b = \sum_{i=1}^n \alpha_i a_i$ and $\sum_{i=1}^n \alpha_i = 1$.

Definition 3: Let $X \subseteq \mathbb{R}^d$. The *convex hull* of X is the intersection of all convex sets that contain X. We denote the convex hull of X by $CH(X)$.

Claim 2: If $X = \{a_1, a_2, ..., a_n\}$ then

$$CH(X) = \{b | b \text{ is a convex combination of } \{a_1, a_2, ..., a_n\}\}$$

(Check!).

Separation Theorem: If $C, D \subseteq \mathbb{R}^d$ are two disjoint convex sets, then there exists a hyperplane $h = \{x | \langle a, x\rangle = b\}$ such that $C \subseteq h^+ = \{x | \langle a, x\rangle \geq b\}$ and $D \subseteq h^- = \{x | \langle a, x\rangle \leq b\}$.
Sketch of proof: We will assume that C and D are compact (i.e., closed and bounded). The cartesian product $C \times D \subseteq \mathbb{R}^{2d}$ is a compact set too. Let us consider the function $f\colon (x, y) \rightarrow \| x - y \|$, when $(x, y) \in C \times D$. f attains its minimum, so there exist two points $a \in C$ and $b \in D$ such that $\| a - b \|$ is the minimum possible. The hyperplane h perpendicular to the segment $\overline{ab}$ and passing through its midpoint will be the one that we are searching for. From elemntary geomtric reasoning, it is easily seen that h indeed separates the sets C and D.

3 Radon's Lemma

The Theorem: If $A \subseteq \mathbb{R}^d$ and $|A| \geq d+2$ then there exist $A_1, A_2 \subseteq A$ such that $A_1 \cap A_2 = \emptyset$ and $CH(A_1) \cap CH(A_2) \neq \emptyset$.

Proof: Let $P = \{p_1, p_2, ..., p_{d+2}\} \subseteq \mathbb{R}^d$, this points are necessarily affine dependent, so, there exist $\alpha_1, \alpha_2, ..., \alpha_{d+2} \in \mathbb{R}$ not all zeros such that $\sum_{i=0}^{d+2} \alpha_i p_i = 0$ and $\sum_{i=0}^{d+2} \alpha_i = 0$. Let us put $I_1 = \{i | \alpha_i > 0\}$ and $I_2 = \{j | \alpha_j < 0\}$, $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$, put $P_1 = \{p_i | i \in I_1\}$ and $P_2 = \{p_j | j \in I_2\}$.
Put $S = \sum_{i \in I_1} \alpha_i$ (therefore also $\sum_{j \in I_2} \alpha_j = -S$).
Put $x = \sum_{i \in I_1} \frac{\alpha_i}{S} p_i$. We will show that $x \in CH(P_1) \cap CH(P_2)$:
$x \in CH(P_1)$ because x is a convex combination of the points in A_1. Indeed $\frac{\alpha_i}{S} > 0 \ \forall i \in I_1$ and $\sum_{i \in I_1} \frac{\alpha_i}{S} = \frac{1}{S} \sum_{i \in I_1} \alpha_i = \frac{1}{S} S = 1$.
Next we show that $x \in CH(P_2)$: since $\sum_{i=1}^{d+2} \alpha_i p_i = 0$, we have $\sum_{i \in I_1} \alpha_i p_i + \sum_{j \in I_2} \alpha_j p_j = 0$, or $\sum_{i \in I_1} \alpha_i p_i = \sum_{j \in I_2} -\alpha_j p_j$, and also $\frac{1}{s} \sum_{i \in I_1} \alpha_i p_i = \frac{1}{s} \sum_{j \in I_2} -\alpha_j p_j$, finally $x = \sum_{i \in I_1} \frac{\alpha_i}{S} p_i = \sum_{j \in I_2} \frac{-\alpha_j}{S} p_j$ when for all $j \in I_2$ $-\frac{\alpha_j}{S} > 0$ and $\sum_{j \in I_2} -\frac{\alpha_j}{S} = -\frac{1}{S} \sum_{j \in I_2} \alpha_j = -\frac{1}{S} \cdot (-S) = 1$. Hence $x \in CH(P_2)$. This completes the proof.

Summary

The finite and infinite versions of Helly's theorem.

4 Helly's theorem

The finite version of Helly's Theorem: Let $C_1, C_2, ..., C_n \in \mathbb{R}^d$ ($n \geq d+1$) be n convex sets such that every $d+1$ of these sets have a nonempty intersection, then all of the n sets have a nonempty intersection.

Proof to the finite version: We will prove the theorem for a fixed d by induction on n. The base of the induction ($n = d+1$) is clear.

We assume that the theorem holds for all $m < n$ and we will prove for $n \geq d+2$. Put $S_i = \cap_{j \neq i} C_j$. Each S_i is convex and by induction for every $i \in \{1, ..., n\}$ each $S_i \neq \emptyset$, so, for all $i \in \{1, ..., n\}$ we choose a point $p_i \in S_i$. Let $P = \{p_1, p_2, ..., p_n\}$ (remember $n \geq d+2$).By Radon's Lemma we can divide the point's indexes to two disjoint sets I_1 and I_2 such that $CH(\{p_i \mid i \in I_1\}) \cap CH(\{p_j \mid j \in I_2\}) \neq \emptyset$. Next, we pick a point x in this intersection: we want to show that $x \in C_i$ for all $i = 1, ..., n$. Indeed, if we choose some index i, there are two options: either $i \notin I_1$ or $i \notin I_2$, without loss of generality assume that $i \notin I_1$. Then for all $j \in I_1$, $p_j \in C_i$, and then $CH(\{p_i \mid i \in I_1\}) \subseteq C_i$, so, $x \in C_i$. Hence, $x \in \cap_{i=1}^{n} C_i$, i.e. all of the sets $C_1, C_2, ..., C_n$ have a nonempty interesection. This completes the induction step and hence the proof to the theorem.

Another proof for the finite version: By induction on the dimension d. The base of the induction, when $d = 1$, is clear. **Check.**

We will assume that the theorem is true for dimension $d-1$, and we will show that it is true for dimension d. Let $\{C_1, C_2, ..., C_n\}$ be the smallest collection of sets such that every $n-1 \geq d+1$ of them have a nonempty intersection, but all of them don't intersect, i.e., $C_n \cap (\cap_{i=1}^{n-1} C_i) = \emptyset$. We can assume that $\cap_{i=1}^{n-1} C_i$ and C_n are compact, therefore, there exists a hyperplane h that strictly separats them. Let $f := \{D_i = h \cap C_i\}_{i=1}^{n-1}$, we will show that every d sets from f have a nonempty intersection. Let $D_{i_1}, ..., D_{i_d}$ be d members of f, $\cap_{j=1}^{d} D_{i_j} = \cap_{j=1}^{d} (h \cap C_{i_j}) = h \cap (\cap_{j=1}^{d} C_{i_j}) \neq \emptyset$: $\cap_{i=j}^{n-1} C_{i_j} \subseteq \cap_{j=1}^{d} C_{i_j}$, so, we can find a point $p_1 \in \cap_{j=1}^{n-1} C_{i_j} \subseteq \cap_{j=1}^{d} C_{i_j}$, i.e. p_1 is on one side of h, also $\cap_{j=1}^{d} C_{i_j} \cap C_n \neq \emptyset$, so there is a point $p_2 \in \cap_{j=1}^{d} C_{i_j} \cap C_n$, i.e. p_2 is on the other side of h. Since $p_1, p_2 \in \cap_{i=1}^{d} C_i$ and $\cap_{j=1}^{d} C_{i_j}$ is convex, $\overline{p_1 p_2} \subseteq \cap_{j=1}^{d} C_{i_j}$. Also, $h \cap \overline{p_1 p_2} \neq \emptyset$, therefore there is a point $x \in h \cap (\cap_{j=1}^{d} C_{i_j})$. We get that every d sets from f have a nonempty intersection, and also because the sets $D_i = h \cap C_i$ are convex by induction all of the sets in f intersect, i.e. $\cap_{i=1}^{n-1} D_i = \cap_{i=1}^{n-1}(h \cap C_i) = h \cap (\cap_{i=1}^{n-1} C_i) \neq \emptyset$, a contradiction to how we chose h.

The infinite version of Helly's Theorem: Let $\{C_i\}_{i \in I}$ be an arbitrary family of convex compact sets in $\mathbb{R}^d$, not necessary finite, such that every $d+1$ of them have a nonempty intersection. Then all of the sets have a nonempty intersection.
Proof: Assume to the contrary that the intersection of all of the sets is empty. By a property of compactness there is a set $J \subseteq I$ such that $\mid J \mid < \infty$ and $\cap_{i \in J} C_i = \emptyset$. Then, by the finite version of Helly's theorem there are $d+1$ indexes in J such their intersection is empty, a contradiction.
Comment: The infinite version of the theorem isn't true if the sets aren't closed or bounded.
If we abandon the assumption that the sets are closed, we can chose the following convex sets $A_i = (0, \frac{1}{i})$ in $\mathbb{R}$. Every two sets have a nonempty intersection, but $\cap_{i=1}^{\infty} A_i = \emptyset$.
If we abandon the assumption that the sets are bounded, we can chose the following sets $B_i = [i, \infty)$, every two sets have a nonempty intersection, but $\cap_{i=1}^{\infty} B_i = \emptyset$

Summary

Definition of centerpoint, Centerpoint theorem, Carathéodory's theorem, Fractional Helly thereorem.

5 Centerpoint

Definition. *Let $X \subseteq \mathbb{R}^d$ a finite set. Point p (not necessrly from X) is called a centerpoint of X if every closed halfspace containing p contains at least $\frac{|X|}{d+1}$ points of X.*

Theorem 5.1 (Centerpoint Theorem)**.** *Let $X \subseteq \mathbb{R}^d$ finite set of points, X has a centerpoint.*

Proof for the Centerpoint Theorem. First we will notice that p is a centerpoint if and only if it lies in every open halfspace that contain more than

$\frac{d}{d+1} \mid X \mid$ points from X. **(Check).**
Put $F = \{CH(h \cap X) \mid \text{h is an open halfspace and } \mid h \cap X \mid > \frac{d}{d+1} \mid X \mid\}$. In F every $d+1$ sets have nonempty intersection, so if we apply Helly's theorem we get that all of the sets in F have nonempty intersection. Every point in the intersection is centerpoint. □

Comment: We can not find a better constant than $\frac{1}{d+1}$ in the definition of the centerpoint. One can see it by taking $d+1$ affinely independent points as the set X.

6 Carathéodory's Theorem

Theorem 6.1 (Carathéodory's Theorem)**.** *Let* $\{x_1, x_2, ..., x_n\} = X \subseteq \mathbb{R}^d$ *a finite set and* $p \in CH(x)$*, so* p *is a convex combination of most* $d+1$ *points from* X*.*

Proof for the Carathéodory's Theorem. By induction on n. The base ($n = d+1$) is clear. So, lets assume that $n \geq d+1$: $p \in CH(X)$, therefore $p = \sum_{i=1}^{n} \lambda_i x_i$ when $\sum_{i=1}^{n} \lambda_i = 1$ and each λ_i is positive (if one of the λ_i is zero than we finish). We will show that there exist a convex combination $p = \sum_{i=1}^{n} \delta_i x_i$ such that at least one of the δ_i is zero. Since $n = \mid X \mid \geq d+2$, X is affinely dependent, i.e. there are exist $\alpha_1, \alpha_2, ..., \alpha_n$ not all zeros such that $\sum_{i=1}^{n} \alpha_i x_i = 0$ and $\sum_{i=1}^{n} \alpha_i$. Notice that $p = p + t \cdot 0$, we will choose $t = min_{\alpha_j < 0} \frac{\lambda_j}{|\alpha_j|} = \frac{\lambda_k}{|\alpha_k|}$, so, $p = \sum_{i=1}^{n} (\lambda_i + t\alpha_i) x_i = \sum_{i \neq k} (\lambda_i + t\alpha_i) x_i + (\lambda_k + t\alpha_k) x_k$. Notice, that it is a convex combination. **(Check.)**
Also, notice that the coefficient of x_k is zero: $\lambda_k + t\alpha_k = \lambda_k - \frac{\lambda_k}{|\alpha_k|} \mid \alpha_k \mid$ and we will finish by induction. □

7 Fractional Helly Theorem

Theorem 7.1 (Carathéodory's Theorem)**.** *For every* $1 \geq \alpha > 0$ *there exists* $\beta = \beta(d, \alpha)$ *with the following property. Let* $C_1, C_2, ..., C_n$ *be convex sets in* $\mathbb{R}^d$*,* $n \geq d+1$*, and at least* $\alpha \binom{n}{d+1}$ *of the collection of sets of size* $d+1$ *have nonempty intersection, so there exists a point contained in at least* βn *sets.*

The best possible value of β is $\beta = 1 - (1-\alpha)^{\frac{1}{d+1}}$, but we will prove $\beta \geq \frac{\alpha}{d+1}$.

Comment: Notice that it is enough to prove the theorem for convex compact sets, because we can choose for each $I \subseteq \{1,...,n\}$, $\mid I \mid = d+1$ a point $p_I \in \cap_{i\in I} C_i$ and replace each C_j with $CH(\{p_I \mid j \in I)$. If the theorem holds for this convex sets then it also holds for our original sets $C_1, ..., C_n$.

Definition. *Lexicographic ordering is an ordering of points of* $\mathbb{R}^d$ *by their cordinate vectors.* $\leq_{lex}$ *will denote the lexicographic ordering.*

Lemma 7.2. *Let* $I \subseteq \{1,...,n\}$ *such that* $\mid I \mid = d+1$ *and* $\cap_{i\in I} C_i = F_i \neq \emptyset$, *so, there exists* $I' \subseteq I$, $\mid I' \mid \leq d$ *such that the lexicographic minimum of* $F_{I'}$ *equals to the lexicographical minimum of* F_I.

Proof for lemma. Let $p \in \mathbb{R}^d$ be the lexicographically minimum point of F_I and $C = \{q \in \mathbb{R}^d \mid q \leq_{lex} p\}$. Notice that C is convex (Check). Since p is the lexicographical minimum of F_I, we have $C \cap F_I = \emptyset$. By Helly's theorem we get that there are $d+1$ sets from $\{C_i\}_{i\in I} \cup \{C\}$ with empty intersection, the set C must be one of them, because all the others have nonempty intersection. So, there exists $\mid J \mid = d$ such that $J \subset I$ and $F_J \cap C = \emptyset$. $F_j \subset F_I$, so the lexicographical minimum of F_j must be smaller or equal than the lexicographical minimum of F_I, and because $F_J \cap C = \emptyset$ the lexicographical minimums are equal. □

Proof to the Fractional Helly Theorem. For each $I \subset \{1,...,n\}$ such that $\mid I \mid = d+1$ and $\cap_{i\in I} C_i = F_i \neq \emptyset$, we fix an index set of size d, $J = J(I) \subset I$ such that $min - lex(F_I) = min - lex(F_J)$. Each lexicographical minimum point of the $\alpha\binom{n}{d+1}$ $d+1$-tuples defined by one of the d-tuples J. Therefore there exists an index set J_0 such that the lexicographical minimum point that he defines appears in at least $\alpha\binom{n}{d+1}/\binom{n}{d} = \alpha\frac{n-d}{d+1}$ distinct I. Therefore the lexicographical minimum of F_{J_0} appears in at least $d+\alpha\frac{n-d}{d+1} > \alpha\frac{n}{d+1}$ sets among the F_i. Hence we may set $\beta = \frac{\alpha}{d+1}$. □

Summary

Fractional Helly for line Transversals, a weaker version of Lipton-Tarjan Theorem, Koebe Theorem.

8 Fractional Helly for line Transversals

Theorem 8.1. *Let $C_1, C_2, ..., C_n$ be n convex sets in the plane. For each α there exists a β such that if for $\alpha\binom{n}{3}$ triples from the sets above there exists a line that transects them, then there exists a line that intersects βn of the sets.*

Proof. We distinguish two cases:
a. There are at least $\frac{\alpha}{7}\binom{n}{2}$ pairs from the sets $C_1, C_2, ..., C_n$ that have a non-empty intersection. We project all the sets $C_1, C_2, ..., C_n$ vertically on the x-axis. In such a way we get $\frac{\alpha}{7}\binom{n}{2}$ intersecting pairs of intervals. By the fractional Helly theorem, at least $\beta' n$ intervals have a common point. The vertical line through this point intersects $\beta' n$ sets from $C_1, C_2, ..., C_n$.
b. There are at most $\frac{\alpha}{7}\binom{n}{2}$ intersecting pairs. We call a triple of sets C_i, C_j, C_k *good* if the sets are pairwise disjoint and have a line that intersects them. There are at most $\frac{\alpha}{7}\binom{n}{2}$ intersecting pairs, therefore at most $\frac{\alpha}{7}\binom{n}{2}n < \frac{\alpha}{2}\binom{n}{3}$ non good triples, and so at least $\frac{\alpha}{2}\binom{n}{3}$ good triples remain. Next, we claim that for every good triple C_i, C_j, C_k there is a line tangent to two of them and intersecting the third. To see that, start with a line that intersects the triple, translate it until it becomes tangent to one of the C_i, and then rotate it while keeping it tangent to C_i until it becomes tangent to a C_j, $i \neq j$. A pair of disjoint convex sets in the plane have exactly 4 common tangents, therefore we have at most $4\binom{n}{2}$ lines that are common tangent to at least two disjoint members from $C_1, C_2, ..., C_n$. Hence there is a line that intersect at least $\frac{\frac{\alpha}{2}\binom{n}{3}}{4\binom{n}{2}} = \beta n$ sets.

□

9 Planar Separators

We will prove the following weaker version of Lipton-Tarjan Theorem.

Theorem 9.1. *Let G be a planar graph with n vertices. There is a partition of the vertices into three disjoint sets $V = A \cup B \cup C$, such that $|A|, |B| \leq \frac{3}{4}n$, $|C| \leq 2\sqrt{n}$ and no vertex of A is adjacent to any vertex of B.*

In the original version of the theorem $|A|, |B| \leq \frac{2}{3}n$.
For the proof of the above theorem we will need the following theorem:

Theorem 9.2 (Koebe Theorem)**.** *Let $G = (V, E)$ a planar graph, $V(G) = \{v_1, v_2, ..., v_n\}$. There exist n discs $C_1, C_2, ..., C_n$ which form a packing (every two discs are either disjoint or tangent) such that C_i and C_j are tangent iff $(v_i, v_j) \in E(G)$.*

Proof for the weaker version of Lipton Tarjan Theorem. Let $G = (V, E)$ ($|V| = n$) be a planar graph. Consider a Koebe representation of the graph, we can apply a circle-preserving map on this representation and get a packing of circular caps on $\mathbb{S}^2$ that correspond to vertices of G, such that two caps touch iff the corresponding vertices of G are adjacent. For each i pick a point p_i on the cap C_i and let P be the set of the points p_i, $1 \le i \le n$. Next, we can find a map Φ that preserve the incidences and the circular caps such that the origin is a centerpoint of $\Phi(P)$. We show that there exists a plane h that contains the origin and intersects less than $2\sqrt{n}$ caps which implies the theorem. Let r_i denote the radius of tha cap $\Phi(C_i)$ The propability that the plane h intersects the cap C_i is $\le r_i$, so the expected value of the number of caps that h intersect is $\le \sum_{i=1}^{n} r_i$. By the Cauchy Schwarz inequality we have that $\sum_{i=1}^{n} r_i \le (\sum_{i=1}^{n} r_i^2)^{\frac{1}{2}}\sqrt{n}$. Since $\sum_{i=1}^{n} \pi r_i^2 \le 4\pi \Rightarrow \sum_{i=1}^{n} r_i^2 \le 4$, we get $\sum_{i=1}^{n} r_i \le 2\sqrt{n}$.

□

Summary

Colorful Carathéodory Theorem, Tverberg's Theorem.

10 Colorful Carathéodory Theorem

Theorem 10.1 (Colorful Carathéodory Theorem)**.** *Let $P_1, P_2, ..., P_{d+1} \subseteq \mathbb{R}^d$ be points sets such that the convex hull of each P_i contains the origin, i.e. $0 \in CH(P_i)$, $i = 1, ..., d+1$. Then there exists a point set S such that $|S| = d+1$, $|S \cap P_i| = 1$ for $i = 1, ..., d+1$ and $0 \in CH(S)$.*

Proof. Call a set $S \in \mathbb{R}^d$ with the property that $|S| = d+1$, $|S \cap P_i| = 1$ for $i = 1, ..., d+1$ a *rainbow simplex*. Assume to the contrary that there is no rainbow simplex that contains the origin. We choose a rainbow simplex R such that the distance of $CH(R)$ to 0 is the smallest possible. Let $x \in CH(R)$ be the closest point to 0. Let h be the hyperplane containing x and perpendicular to the segment $0x$ (R lies in the closed halfspace h^+). We have $x \in h \cap CH(R) = CH(h \cap R)$, hence by Carathéodory's Theorem there is a set $T \subseteq R$ such that $|T| \le d$ and $x \in CH(h \cap T)$. Let i be a color not occurring in T. There exists a point $y \in P_i$ such that $y \in h^-$ ($y \notin h$) because we assume that $0 \in CH(P_i)$. $R' = T \cup \{y\}$ is a rainbow simplex. It easy to see that $CH(R')$ is closer to the origin than $CH(R)$ (check) - a contradiction to our choice of R. This completes the proof of the Theorem. □

11 Tverberg's Theorem

Theorem 11.1. *For any finite set $A \subseteq \mathbb{R}^d$ of at least $(r-1)(d+1)+1$ points there exist $A_1, A_2, \ldots A_r \subseteq A$ pairwise disjoint subsets such that $\cap_{i=1}^{r} CH(A_i) \neq \emptyset$.*

Notice that in the case $r = 2$ we get Radon's Lemma. We will prove a weaker version of the Tverberg Theorem: If $|A| \ge (r-1)(d+1)^2+1$ then we can find $A_1, A_2, \ldots A_r \subseteq A$ pairwise disjoint subsets such that $\cap_{i=1}^{r} CH(A_i) \neq \emptyset$.

Proof. Set $n = (r-1)(d+1)^2+1$ and let $s = n-(r-1)(d+1)$. Every $d+1$ subsets of A of size s have a nonempty intersection. Indeed, if $A_1, A_2, \ldots, A_{d+1}$ are $d+1$ subsets of A such that they have an empty intersection, then $n = |A| = |\overline{\cap_{i=1}^{d+1} A_i}| = |\cup_{i=1}^{d+1} \overline{A_i}| \le \sum_{i=1}^{d+1} |\overline{A_i}| = \sum_{i=1}^{d+1}(n-s) = (r-1)(d+1)^2$, a contradiction to our assumption. Hence, by Helly's Theorem, there is a point x in the intersection of the convex hulls of all s-tuples. Of course $x \in A$, so by Carathéodory's Theorem there is a set $A_1 \subseteq A$ such that $|A_1| = d+1$ and $x \in CH(A_1)$. Since $|A \backslash A_1| \ge s$, $x \in CH(A \backslash A_1)$ and therefore again by Carathéodory's Theorem we can find $A_2 \subseteq A \backslash A_1$ such that $x \in CH(A_2)$ etc. We can continue in this manner r times and get r pairwise disjoint sets $A_1, A_2, \ldots A_r \subseteq A$ such that $x \in \cap_{i=1}^{r} CH(A_i)$ as claimed. □

Summary

First Selection Lemma.

12 First Selection Lemma

Definition. *Let $P \subset \mathbb{R}^d$ be a finite set. A $P-simplex$ is a convex hull of some $(d+1)$-tuple of points of P.*

Theorem 12.1. *There is a constant $c = c(d)$ (depending only on the dimension d) such for every $P \subset \mathbb{R}^d$ ($|P| = n$) there is a point (not necessarily in P) contained in at least $c\binom{n}{d+1}$ P-simplices.*

The best possible value of $c(d)$ is known only for $\mathbb{R}^2$ and it is $c(2) = \frac{2}{9}$. For $\mathbb{R}^d$, the first proof below shows that for n sufficiently large, we may take $c(d) \approx \frac{1}{(d+1)^{d+1}}$.

First proof: For the first proof we will use Colorful Carathéodory's and Tverberg's Theorems. Let $P \subset \mathbb{R}^d$ ($|P| = n$) (we may assume that n is sufficiently large). Put $r = \lceil \frac{n}{d+1} \rceil$, therefore $n \geq (r-1)(d+1)+1$. By Tverberg's Theorem there exist r pairwise disjoint sets $P_1, P_2, ..., P_r \subseteq P$ such that $0 \in \cap_{i=1}^{r} CH(P_i)$. So far we found a common point to r $P-simplicies$ (each P_i has at most $d+1$ points), we want to show that the point 0 is contained in many $P-simplicies$. Let $J = \{j_0, j_1, ..., j_d\} \subseteq \{1, 2, ..., r\}$ be a set of $d+1$ indices. $P_{j_0}, P_{j_1}, ..., P_{j_d}$ are points sets that the convex hull of each P_{j_i} contains the 0, hence by the Colorful Carathéodory's theorem, there is rainbow simplex that contains 0. For each such distinct $d+1$ tuple of indices there is a different rainbow simplex. Therefore, the number of P-simplicies that contain 0 is at least

$$\binom{r}{d+1} = \binom{\lceil \frac{n}{d+1} \rceil}{d+1} = c_d \binom{n}{d+1}.$$

□

Second proof: For the second proof we will use the Fractional Helly Theorem. Let $P \subset \mathbb{R}^d$ ($|P| = n$). Let F be the family of all P-simplicies. Put $N = |F| = \binom{n}{d+1}$ and set $t = (d+1)^2$. We will show that there are at least $\binom{n}{t}$ subsets of size $d+1$ from F that have a nonempty intersection, i.e. there are $\binom{n}{t} = \binom{n}{(d+1)^2} = \alpha\binom{\binom{n}{d+1}}{d+1} = \alpha\binom{N}{d+1}$ subsets of size $d+1$ that have nonempty intersection and by the Fractional Helly theorem we will get that there is a point common to βN P-simplicies.
Consider a t-point set $P' \subset P$. Using Tverberg's Theorem we find that P' can be partitioned into $d+1$ pairwise disjoint sets, of size $d+1$ each, whose

convex hulls have a common point. Every $d+1$ sets uniquely determine the set of t points and the theorem follows. □

Theorem 12.2. *Let $P \subset \mathbb{R}^2$ be a points set of size n, and let T be a set of t triangles from P, then there is a point in $\Omega(\frac{t^3}{n^6 \log^2 n})$ triangles.*

Theorem 12.3. *Let $P \subset \mathbb{R}^3$ be a point set of size n, and let T be a set of t triangles from P, then there is a line that intersects at least $\Omega(\frac{t^3}{n^6})$ triangles.*

Improving the above bounds is an open problem.

Summary

Szemerédi-Trotter Theorem; Crossing Number Theorem (Ajtai, Chvatal, Newbborn, Szemerdi; Leigthon); Example for using line-point incidences (Elekes Theorem).

13 Szemerédi-Trotter Theorem

P will denote a set of m distinct points, L will denote a set of n distinct lines, $I(P,L) = \mid \{(p,l) \mid p \in P, l \in L, p \in l\} \mid$, $I(m,n) = max_{|P|=m;|L|=n}\{I(P,L)\}$.

Theorem 13.1 (Szemerédi-Trotter Theorem)**.**

$$I(m,n) = O(m^{\frac{2}{3}} n^{\frac{2}{3}} + \ m + \ n).$$

We will prove the Szemerédi-Trotter Theorem by using the crossing number theorem, but first we need some definitions: A *drawing* of a graph $G = (V,E)$ is a mapping that assigns for each vertex $v \in V$ a point in $\mathbb{R}^2$ (different points for different vertices), and for each edge of G an arc connecting the corresponding two vertices of the edge. We call a point a *crossing point* if it is a point that belongs to at least two arcs but distinct from all vertices. The *crossing number* of a graph is the number of crossing points. If a point belongs to k arcs, we will count this point $\binom{k}{2}$ times. $Cr(G)$ will denote the minimal crossing number in the "best" drawing of G. For example, $Cr(G) = 0$ iff G is a planar graph, and $Cr(K_5) = 1$.

Theorem 13.2 (Ajtai, Chvatal, Newbborn, Szemérdi 82; Leigthon 83)**.** *Let $G = (V,E)$ be a simple graph such that $|E| \geq 4|V|$ then $Cr(G) \geq \frac{|E|^3}{64|V|^2}$.*

The lower bound is asymptotically tight, i.e. for every m, n there exists a graph $G = (V, E)$ such that $|V| = n$, $|E| = m$ and $Cr(G) = O(\frac{m^3}{n^2})$. (Check).

Lemma 13.3. *For every simple graph $G = (V, E)$, $Cr(G) \geq |E| - 3|V|$.*

Proof. Let $E' \subseteq E$ be a maximal set of edges such that $G' = (V, E')$ is a planar graph. Each edge in $E \backslash E'$ must cross at least one edge in E', so the number of crossings is at least $|E| - |E'| \geq |E| - 3|V|$ ($|E'| \leq 3|V|$ by the Euler formula). Therefore $Cr(G) \geq |E| - 3|V|$. □

Proof of the Ajtai, Chvatal, Newbborn, Szemerdi; Leigthon Theorem. We will present a probabilistic proof. Let $p \in (0, 1]$ be a parameter to be defined later. We choose a random subset $V' \subseteq V$ by choosing vertex $v \in V$ independently with probability p. Put $E' = \{(p, q) \in E \mid p, q \in V'\}$ and let X' be the set of all the crossings such that the four vertices that define the crossing are in V'. $|V'|$, $|E'|$ and $|X'|$ are random variables. By Lemma 1.3 we have $|X'| \geq |E'| - 3|V'|$. This inequality also holds for the expectations:

$$Exp[|X'|] \geq Exp[|E'|] - 3Exp[|V'|]$$

Hence,

$$|X| \cdot p^4 \geq |E| \cdot p^2 - 3|V| \cdot p$$

(The probability to choose a vertex is p, the probability to choose an edge is the probability to choose its two endpoints, the probability of a crossing point is the probability to choose the two edges that create the crossing point). Therefore,

$$|X| \geq \frac{m}{p^2} - \frac{3n}{p^3}$$

If we choose $p = \frac{4n}{m}$ (note that $p \leq 1$), we get that $|X| \geq \frac{m^3}{64n^2}$ as claimed. □

Proof of the Szemerédi-Trotter Theorem. Let P be a set of m points and let L be a set of n lines in the plane realizing the maximum number of incidences $I(m, n)$. We want to show that $I(m, n) = O(m^{\frac{2}{3}} n^{\frac{2}{3}} + m + n)$. We define a graph $G = (V, E)$ and it's drawing in the plane. The vertices V are the points P, and we draw an arc between any two consecutive points on each $l \in L$. Each line $l \in L$ with I_l points of P contribute $I_l - 1$ edges to E. So, $\mid E \mid = \sum_{l \in L}(I_l - 1) = \sum_{l \in L} I_l - n = I(m, n) - n$. Next, we show an upper bound on $\mid E \mid$:

If $\mid E \mid \leq 4m$, then $\mid E \mid = I(m, n) - n \leq 4m \Rightarrow I(m, n) \leq 4m + n$. Else, from the Crossing lemma and from the fact that every pair of lines can contribute at most one crossing point, we get $\binom{n}{2} \geq Cr(G) \geq \frac{|E|^3}{64m^2}$. Therefore $\mid E \mid= I(m,n) - n = O(m^{\frac{2}{3}}n^{\frac{2}{3}})$, so $I(m,n) = O(m^{\frac{2}{3}}n^{\frac{2}{3}} + n)$. Finally, we get that $I(m,n) = O(m^{\frac{2}{3}}n^{\frac{2}{3}} + n + m)$. □

14 Example for using line-point incidences (Elekes Theorem)

Let $A \subseteq \mathbb{R}$ be a finite set of n reals. Let $f(A)$ be the maximum of the sizes of $A + A$ and of $A \cdot A$ where $A + A = \{x + y \mid x \in A, y \in A\}$ and $A \cdot A = \{x \cdot y \mid x \in A, y \in A\}$. Put $f(n) = min_{A \subseteq \mathbb{R}; |A|=n}\{f(A)\}$.
It is an open problem to find the asymptotics of $f(n)$. Trivial bounds are $n \leq f(n) \leq n^2$. It is conjectured that $f(n) = \Omega(n^{n-\epsilon})$ for any constant ϵ.

Theorem 14.1 (Elekes Theorem)**.** *There exists a constant c such that* $f(n) \geq c \cdot n^{\frac{5}{4}}$.

Proof. Let $X =\mid A+A \mid$ and $Y =\mid A \cdot A \mid$, also let $P = (A+A) \times (A \cdot A)$, a set of $X \cdot Y$ points. For every $a_i, a_j \in A$ we define a line $l_{i,j} : y = a_i(x - a_j)$. Let L denote the set of all such lines. $\mid L \mid= \binom{n}{2}$. Each line contains at least n points from P (for some line $l_{i,j}$, for each $x = a_j + a_k$ when $1 \leq k \leq n$ we get that $y = a_i a_k$). Therefore $I(P, L) \geq \Omega(n^3)$, on the other hand, by Szemerédi-Trotter Theorem we get that $I(P, L) = O(\mid P \mid^{\frac{2}{3}} \mid L \mid^{\frac{2}{3}} + \mid P \mid + \mid L \mid) = O((XY)^{\frac{2}{3}} \cdot n^{\frac{4}{3}} + XY + n^2)$. If we combine the two bounds we get that either $n^3 \leq XY$ (and then we get even a better lower bound) or $n^3 \leq (XY)^{\frac{2}{3}} \cdot n^{\frac{4}{3}}$ and then $(XY)^{\frac{2}{3}} \geq n^{\frac{5}{3}}$ and then $XY \geq n^{\frac{5}{2}} \Rightarrow max\{X, Y\} \geq \Omega(n^{\frac{5}{4}})$. □

Summary

Unit distances and distinct distances.

15 Unit Distances

Let $P \subseteq \mathbb{R}^2$ be a set of n different points, let $U(P)$ denote the number of pairs of points with Euclidean distance 1 and put $U(n) = max_{|P|=n}\{U(P)\}$.

Erdős found a lower bound for $U(n)$: $U(n) = \Omega(n^{1+\frac{c}{\log\log n}})$ where c is some positive constant. The best known upper bound is $O(n^{\frac{4}{3}})$.
This problem is related to incidences between points and unit circles. The related problem is to find an upper bound on $I(P, C) = \mid \{(p, c) \mid p \in P, c \in C, p \in c\} \mid$ when P is a set of points and C is a set of unit circles. It is easy to see that $U(P) \leq \frac{1}{2} I(P, C)$ (if we draw around each point of P unit circle, we get a set C of $|P|$ unit circles, and each pair of points with Euclidean distance 1 will contribute 2 incidences).

Theorem 15.1. *Let P be a set of n points and C be a set of n unit circles. Then $I(P, C) = O(n^{\frac{4}{3}})$.*

Proof. We omit all the circles from C that contain less than three points from P. Let C' be the set of the remaining circles. In such a way we ignore less than $2n$ incidences from $I(P, C)$. We draw a graph $G = (P, E)$, where we draw an arc between every pair of points that are consecutive on some circle $c \in C'$. Notice that $\mid E \mid = I(P, C')$. Also notice that G isn't a simple graph, but the maximal multiplicity of each arc is 2. Next, we omit from any "double arc" one of its arcs. Let E' be the remaining arcs. We ignored less than $\frac{|E|}{2}$ arcs and $G = (P, E')$ is a simple graph. If $4n \geq \mid E' \mid \geq \frac{|E|}{2}$ then $\mid E \mid \leq 8n$, else $Cr(G') = \Omega(\frac{|E'|^3}{n^2})$ and also $Cr(G') \leq O(n^2)$ (every two circles intersect at most twice). Therefore, $\mid E' \mid = O(n^{\frac{4}{3}})$, so $\mid E \mid \leq 2 \mid E' \mid = O(n^{\frac{4}{3}})$, hence $I(P, C) = I(P, C') + 2n = O(n^{\frac{4}{3}})$. □

16 Distinct Distances

Let $g(n)$ be the minimum number of distinct distances determined by n points in the plane. It is easy to see that $g(n) = O(n)$ (for example, points of an arithmetic progression on the line). Erdős found a construction which shows that $g(n) = O(\frac{n}{\sqrt{\log n}})$. We will show that $g(n) \geq \Omega(n^{\frac{4}{5}})$.
It's easy to see that $g(n) = \Omega(n^{\frac{2}{3}})$. Indeed, let $r_1, r_2, .., r_t$ be the t distinct distances determined by given set of n points. Each distance appears in $O(n^{\frac{4}{3}})$ pairs of points. We partition the $\binom{n}{2}$ points to t sets $E_1, E_2, ..., E_t$ such that $E_i = \{(p, q) \mid dist(p, q) = r_i\}$, so $\binom{n}{2} = \sum_{i=1}^{t} \mid E_i \mid \leq O(tn^{\frac{4}{3}})$, therefore $t \geq \Omega(\frac{n^2}{n^{\frac{4}{3}}}) = \Omega(n^{\frac{2}{3}})$.

Theorem 16.1. *$g(n) \geq \Omega(n^{\frac{4}{5}})$.*

Proof. Let $r_1, r_2, .., r_t$ be the t distinct distances. We draw around each point in P t circles with radiuses $r_1, r_2, .., r_t$. We get a set C of nt circles, and $I(P,C) = 2\binom{n}{2} = n(n-1) = \Omega(n^2)$. Next, we delete all circles which contain at most 2 points. We delete at most $2nt$ incidences and $\Omega(n^2)$ incidences remain (we can assume that t is much smaller than n, because otherwise there is nothing to prove). Next, we draw a graph $G = (P, E)$, such that we draw an arc between every pair of consecutive points on some circle (the maximal multiplicity of each edge is $2t$) (check!).

Lemma 16.2. *Let $G = (V,E)$ be a multi graph, such that $\mid V \mid = n$, $\mid E \mid = m$ and the multiplicity of every edge is at most k. Then $Cr(G) = \Omega(\frac{m^3}{kn^2}) - O(k^2 n)$.*

Proof. Consider a fixed drawing of the graph $G = (V,E)$. We delete each edge $e \in E$ with probability $(1 - \frac{1}{k})$ (with probability $\frac{1}{k}$ we don't delete it). Next, if there remain edges with multiplicity we delete them. Let m' the number of the remainig edges and x' the number of crossing pairs of the remaining edges. The remaining graph G' is simple, so we can apply the crossing number theorem and get that $x' = Cr(G') \geq \frac{m'^3}{64n^2} - n$. Next, the probability that an edge $e \in E$ remains in G' while the edges connecting the same pair of vertices are deleted is $Prob(e \in E') = \frac{1}{k}(1 - \frac{1}{k})^{(k-1)} \geq \frac{1}{3k}$ (because $(1 - \frac{1}{k})^{(k-1)} \geq \frac{1}{3}$). We get $E[m'] \geq \frac{m}{3k}$ and $E[x'] = \frac{1}{k^2} \mid x \mid$ (for a crossing to survive we need that the two edges that create it will be chosen). From Jensen's inequality we have $E[m'^3] \geq (E[m'])^3$. Finally, since the crossing inequality holds for the expectations we have

$$\frac{1}{k^2} \mid x \mid = E[x'] \geq E[\frac{m'^3}{64n^2} - n] = \frac{E[m'^3]}{64n^2} - n \geq \frac{(E[m'])^3}{64n^2} - n \geq \frac{(\frac{m}{3k})^3}{64n^2} - n$$

and we get that

$$|x| \geq \Omega(\frac{m^3}{kn^2}) - O(k^2 n).$$

□

If we now use the lemma with $k = t$ (because the maximum edge multiplicity in the graph G defined above is at most $2t$) we get that $Cr(G) = \Omega(\frac{|E|^3}{tn^2}) - O(t^2 n)$, because $\mid E \mid = \Omega(n^2)$ and $t = O(n)$ we get that $Cr(G) = \Omega(\frac{n^6}{n^2 t}) - O(t^2 n) = \Omega(\frac{n^4}{t}) - O(t^2 n) = \Omega(\frac{n^4}{t})$. On the other hand, we have at most $O(n^2 t^2)$ intersecting circles, hence $c_1 \frac{n^4}{t} \leq c_2 n^2 t^2$ for some constants c_1 and c_2. Finally $t^3 \geq \frac{c_1}{c_2} n^2 \Rightarrow t = \Omega(n^{\frac{2}{3}})$.

The next idea is to deal with the edges of very high multiplicity separately. Let k be a parameter (the exact value will be revealed later). Let E' be the set of all pairs of vertices $\{u, v\}$ such that there are more than k edges between u and v. We will show that $| E' | < cn^2$ for a small c. Every edge in E' connecting the pair $\{u, v\}$ contributes one incidence of the perpendicular bisector line l_{uv} with a point $p \in P$. Each such incidence can be count at most $2t$ times, so if I denotes the number of such incidences, then $| E' | \leq 2It$. We want to show that I is relatively small.

Lemma 16.3. *Let P set of n points in the plane. Let L be a set of m lines such that every line contains at least k points from P. Then $I(P, L) = O(\frac{n^2}{k^2} + n)$.*

Proof. By Szemerédi-Trotter Theorem we get that $km \leq I(P, L) = O(n^{\frac{2}{3}} m^{\frac{2}{3}} + n + m)$. If $km \leq O(n^{\frac{2}{3}} m^{\frac{2}{3}})$ than $m = O(\frac{n^2}{k^3})$, hence in that case $I(P, L) = O(\frac{n^2}{k^2})$. Else $I(P, L) \leq n$, and if we combine the two results we get $I(P, L) = O(\frac{n^2}{k^2} + n)$. □

By the lemma we have that $I = O(\frac{n^2}{k^2} + n)$, so $| E' | \leq 2It = O(\frac{n^2 t}{k^2} + nt)$. We choose $k = C\sqrt{t}$. In the remaining graph $G' = (P, E \backslash E')$, the maximum multiplicity of each edge is $C\sqrt{t}$ and $| E \backslash E' | = \Omega(n^2)$, hence by the crossing lemma for multi graphs

$$Cr(G') = \Omega(\frac{|E \backslash E'|^3}{\sqrt{t} n^2}) - O(tn) = \Omega(\frac{n^4}{\sqrt{t}}) - O(tn) = \Omega(\frac{n^4}{\sqrt{t}})$$

on the other hand $Cr(G') = O(n^2 t^2)$, so $t = \Omega(n^{\frac{4}{5}})$ as claimed.

□

As mentioned above, the known upper bound for $g(n)$ is $O(\frac{n}{\sqrt{\log n}})$. The lower bound was improved by Solymosi and Tóth in 2001 to $\Omega(n^{\frac{6}{7}})$ and again by Tardos in 2001 to (approximately) $\Omega(n^{0.863})$.

Summary

Kövári-Sós-Turán Theorem, Point-Line Incidences via Cuttings.

17 Kövári-Sós-Turán Theorem

Theorem 17.1 (Kövári-Sós-Turán Theorem)**.** *Let $G = (V, E)$ be a graph that doesn't contain $K_{r,s}$ $(1 \leq r \leq s)$ as a subgraph. Then $|E| = O(|V|^{2-\frac{1}{r}})$.*

If $G = (V_1 \cup V_2, E)$ is a bipartite graph such that $|V_1| = m$ and $|V_2| = n$, that doesn't contain $K_{r,s}$ as a subgraph when the r vertices are from V_1 and the s vertices are from V_2, then $|E| = O(mn^{1-\frac{1}{r}}+n)$ and $|E| = O(m^{1-\frac{1}{s}}n+m)$.

Proof. We count the number of "star" configurations (a vertex with r neighbors). Let C be the number of such configurations, so $C = \sum_{i=1}^{|V|} \binom{d_i}{r} \leq (s-1)\binom{n}{r}$ (we can assume without loss of generality that the degree of each vertex is greater than r (check!)). Also, $\sum_{i=1}^{|V|} \binom{d_i}{r} = \Omega(\sum_{i=1}^{|V|} d_i^r)$ and $\sum_{i=1}^{|V|} d_i^r \geq \frac{(\sum_{i=1}^{|V|} d_i)^r}{n^{r-1}} = \frac{(2|E|)^r}{n^{r-1}}$ (this is easily implied by Hölder's inequality). Hence $(2|E|)^r \leq n^{r-1}(s-1)\binom{n}{r} = O(n^{2r-1}) \Rightarrow |E| \leq O(n^{2-\frac{1}{r}})$. □

18 Point-Line Incidences via Cuttings

To see the relevance of the above theorem for line-point incidences we define the follow bipartite graph $G = (V_1 \cup V_2, E)$. The vertices V_1 of the graph will represent the lines L and the vertices V_2 will represent the points P ($|L| = n$,$|P| = m$). G does not contain $K_{2,2}$ as a subgraph, because through two different points we can pass only one line. Therefore, by the Kövári-Sós-Turán Theorem we get that $|E| = I(m, n) = O(m\sqrt{n} + n) = O(n\sqrt{m} + m)$.

Let L be a set of n lines and let r be a parameter such that $n > r > 1$. A $\frac{1}{r}$ $-$ *cutting* is a subdivision of the plane into finitely many generalized triangles (a generalized triangle is an intersection of three halfplanes) such that the interior of each triangle is intersected by at most $\frac{n}{r}$ lines of L.

Lemma 18.1 (Cutting Lemma)**.** *For every $n > r > 1$ there is a $\frac{1}{r}$ $-$ cutting of size (number of the triangles) $O(r^2)$.*

The bound in the cutting lemma is asymptotically tight. Let L be a set of n lines such that every pair of them intersect in a distinct point. Let $\Delta_1, \Delta_2, ..., \Delta_t$ be the division of the plane to t generalized triangles and let n_i be the number of lines that intersect the triangle Δ_i. There are $\binom{n}{2}$ intersection points between the lines, so $\binom{n}{2} \leq \sum_{i=1}^{t} \binom{n_i}{2} \leq \sum_{i=1}^{t} n_i^2 \leq t(\frac{n}{r})^2$, hence $t \geq \frac{n^2}{(\frac{n}{r})^2} = \Omega(r^2)$. We will prove the cutting lemma later. First we show, with the help of this lemma, yet another proof for the Szemerédi-Trotter theorem.

Proof. We want to show that $I(n,n) = O(n^{\frac{4}{3}})$. We choose a parameter $n > r > 1$, (to be revealed later), and take a $\frac{1}{r} - cutting$ of the plane. In such way we divided the plane into $t = O(r^2)$ triangles $\Delta_1, \Delta_2, ..., \Delta_t$, so that the interior of each triangle is intersected by at most $\frac{n}{r}$ lines of L. Let P_i denote the number of points of P lying inside Δ_i or on its boundry but not at the vertices of Δ_i, and let L_i denote the set of lines of L intersecting the interior of Δ_i.

$$I(P,L) = \sum_{i=1}^{i} I(P_i, L_i) + I(P', L) + I(P, L')$$

where $P' \subseteq P$ is the set of points that lie on some vertex of some triangle Δ_i, and $L' \subseteq L$ is the set of lines that contain some side of some triangle Δ_i. It easy to see that $|P'| \leq 3t$ and $|L'| \leq 3t$, hence,

$$I(P,L) \leq O(\sum_{i=1}^{t}(|p_i|\sqrt{L_i} + |L_i|) + |P'|\sqrt{|L|} + |L| + |P|\sqrt{L'} + |L'|) \leq$$

$$\leq O(\sum_{i=1}^{t}(|p_i|\sqrt{\frac{n}{r}} + \frac{n}{r}) + 3t\sqrt{n} + n + n\sqrt{3t} + 3t) \leq$$

$$\leq O(\sqrt{\frac{n}{r}}\sum_{i=1}^{t}|p_i| + nr + 3t\sqrt{n} + n + n\sqrt{3t} + 3t)$$

if we choose $r = n^{\frac{1}{3}}$, we get

$$I(P,L) = O(n^{\frac{4}{3}}).$$

□

For Circle-Point incidences we can show, using the weaker cutting lemma, in a similar way, that $I(n,n) = O(n^{1.4})$.

Now we present a proof to a weaker version of the cutting lemma. We are going to prove that for every set of n lines there is a $\frac{1}{r} - cutting$ of size $O(r^2 \log^2 n)$.

Proof for the weaker version of the Cutting Lemma. Let L be a set of n lines in the plane and $1 \leq r \leq n$, also put $S = 6r \ln n$. We select a random sample of $s \leq S$ lines from L. The selection of each line is independent and with replacements. We call a triangle Δ *dangerous* if its interior is intersected by at least $\frac{n}{r} = k$ lines of L and isn't intersected by any line from the sample of s lines we chose. The probability for it is the probability that out of k lines that intersect with Δ we didn't chose anyone, i.e. $Prob(\Delta \text{ is dangerous}) = (1 - \frac{k}{n})^S \leq (e^{-\frac{k}{n}})^S = e^{-\frac{kS}{n}} \leq e^{-6 \ln n} = \frac{1}{n^6}$ (since $1 + x \leq e^x$). We call a triangle *interesting* if its vertices are vertices of the arrangement of L, i.e. intersection points of two lines from L. There are less than n^6 interesting triangles. Hence the probability for a dangerous and interesting triangle to exists is $< n^6 \frac{1}{n^6} = 1$. Therefore, with positive probability, there is a set $L' \subseteq L$ of size less than S such that for this set no interesting triangle is dangerous, i.e. no triangle is intersected by more than $\frac{n}{r}$ lines of L. The number of triangles in a triangulation of L' is $O(\binom{S}{2}) = O(S^2) = O(r^2 \log^2 n)$. Hence, this triangulation forms a $\frac{1}{r} - \mathit{cutting}$. $\square$

Summary

Definition of levels in arrangements; Proof of the Cutting Lemma; Arrangements of hyperplanes.

19 Proof of the Cutting Lemma

We want to prove that there is a $\frac{1}{r} - \mathit{cutting}$ of size (number of triangles) $O(r^2)$. First we need some definitions and observations concerning levels. For simplicity, we will asuume that the lines L are in general position (no three lines intersect in a common point).

The *level* of a point $x \in \mathbb{R}^2$ is defined as the number of lines of L lying strictly below x.

Let $1 \leq k \leq n - 1$ be some parameter. The level k of the arrangement of L is a set of edges E_k such that each point on them are in level k. Put $t = |E_k|$. Let $e_0, e_1, ..., e_t$ be the the edges of E_k from left to right. We fix points p_i in the interior of each e_i. For an integer parameter $q \geq 2$, we define the q *-simplification* of E_k as the polygon $P = \overline{-\infty p_0}, \overline{p_0 p_q}, \overline{p_q p_{2q}}, ..., \overline{p_{\lfloor \frac{t-1}{q} \rfloor q} p_t}, \overline{p_t + \infty}$. Notice that the polygon P has at most $\frac{t}{q}+2$ edges.

Lemma 19.1. *a. The portion* Π *of the level* k *between the points* p_j *and* p_{j+q} *is intersected by at most* $q+1$ *lines.*
b. The segment $\overline{p_jp_{j+q}}$ *is intersected by at most* $q+1$ *lines.*
c. The q*-simplification of* E_k *is contained in the strip between the levels* $k-\lceil\frac{q}{2}\rceil$ *and* $k+\lceil\frac{q}{2}\rceil$*.*

Proof. a. Trivial: Each line of L intersecting Π contains one of the edges $e_j, e_{j+1}, ..., e_{j+q}$. b. By connectivity and convexity, each line intersecting $\overline{p_jp_{j+q}}$ must intersect Π as well.
c. Fix a segment $\overline{p_jp_{j+k}}$ of the q-simplification. If there is a point x on this segment at level $k+i$, then this segment must intersect more than $2i$ segments. By (b), $2i \leq q+1 \Rightarrow i \leq \lfloor\frac{q+1}{2}\rfloor = \lceil\frac{q}{2}\rceil$. □

Let r be a given parameter. We can assume that r is much smaller than n, for example $r \leq \frac{n}{10}$, because otherwise we can produce a 0-cutting of size $O(n^2)$ by simply triangulating the arrangement of L. Set $q = \lceil\frac{n}{10r}\rceil$ and let $E_0, E_1, ..., E_{n-1}$ be the different levels of the arrangement of L. We divide the levels into q families, where in the ith family $\mathcal{F}_i$ $(0 \leq i \leq q-1)$ we have $\mathcal{F}_i = \{E_j | j = i (\text{mod } q)\}$. Since $\sum_{i=0}^{n-1}|E_i| = O(n^2)$, there is a family $\mathcal{F}_i$ that contains $O(\frac{n^2}{q})$ edges. The family $\mathcal{F}_i$ consists of the levels $E_i, E_{i+q}, E_{i+2q}, ..., E_{i+\lfloor\frac{n}{q}\rfloor q}$. Let P_j denote the q-simplification of E_{i+jq} and let m_j denote the number of edges in E_{i+jq}. We have: $|P_j| \leq \frac{m_j}{q}+2$ and $\sum_{j=0}^{\lfloor\frac{n}{q}\rfloor}|P_j| = \sum_{j=0}^{\lfloor\frac{n}{q}\rfloor}(\frac{m_j}{q}+2) = \frac{1}{q}\sum_{j=0}^{\lfloor\frac{n}{q}\rfloor}m_j + 2\lfloor\frac{n}{q}\rfloor = O(\frac{n^2}{q^2}) = O(r^2)$. By the above lemma we know that no vertex of P_j is above P_{j+1}, hence the polygonal chains P_j never intersect properly. Next, we extend a vertical segment from each vertex of P_j upwards and downwards until they hit P_{j-1} and P_{j+1} or all the way to infinity. In such a way, we subdivide the palne into $O(r^2)$ trapezoids, and each trapeziod is intersected by at most $\frac{n}{r} = 10q$ lines of L. We look at a trapezoid in the strip between P_j and P_{j+1}, by the lemma above it lies between the levels $i+qj-\lceil\frac{q}{2}\rceil$ and $i+q(j+1)+\lceil\frac{q}{2}\rceil$, and therefore each of the vertical sides is intersected by at most $3q$ lines. Also, the bottom and the top sides are parts of edges of P_j and P_{j+1} respectively, therefore by the lemma they intersected by at most $q+1$ lines. Hence the total nimber of lines that intersect the trapeziod is smaller than $10q \leq \frac{n}{r}$. Finally, we can partition each trapeziod into triangles and obtain the wanted $\frac{1}{r}-$ *cutting.*

20 Arrangements of Hyperplanes

Let H be a set of n hyperplanes in $\mathbb{R}^d$ in general position (every d hyperplanes intersect only in one point and no $d+1$ have a common point). Let $\mathcal{A}(h)$ denote the arrangements of H. The *cells* of $\mathcal{A}(h)$ are the connected components of $\mathbb{R}^d \setminus \cup_{h \in H} h$.

Proposition 20.1. *The number of cells in an arrangment of n hyperplanrs in general position is* $\varphi_d(n) = \sum_{i=0}^{d} \binom{n}{i}$.

Proof. By induction on the dimension d and the number of hyperplanes n. For $d = 1$ the number of cells is $\binom{n}{0} + \binom{n}{1} = n+1$. Let H be a family of n hyperplanes in $\mathbb{R}^d$ in general position. Fix a hyperplane $h \in H$. Let $\mathcal{A}' = \mathcal{A}(H \setminus h)$. The number of cells in $\mathcal{A}'$ is the number of cells in $\mathcal{A}'$ plus the number of cells in $(d-1)$-dimensional arrangement of the family $\{h \cap \Pi | \Pi \in H \setminus \{h\}\}$. So $\varphi_d(n) = \varphi_d(n-1) + \varphi_{d-1}(n-1)$ (h is of dimension $d-1$).

$$\varphi_d(n) = \varphi_d(n-1) + \varphi_{d-1}(n-1) = \binom{n-1}{0} + \left(\binom{n-1}{0} + \binom{n-1}{1}\right)$$

$$+ \left(\binom{n-1}{1} + \binom{n-1}{2}\right) + \ldots + \left(\binom{n-1}{d-1} + \binom{n-1}{d}\right) =$$

$$= \binom{n-1}{0} + \sum_{i=0}^{d-1} \left(\binom{n-1}{i} + \binom{n-1}{i+1}\right) = \sum_{i=1}^{d} \binom{n}{i}.$$

□

As we can see from the proposition above the number of cells in an arrangement of n hyerplanes in $\mathbb{R}^d$ is $O(n^d)$.

Summary

Number of vertices of level at most k, Clarkson's Theorem.

21 Clarkson's Theorem

In the following proofs we assume that the given arrangements are simple, i.e. containing no three lines that meet at a single point. Let H be a set of n hyper-planes in $\mathbb{R}^d$. $\mathcal{A}(H)$ will denote the arrangements of H, $v_{\leq k}(H)$ will denote the number of vertices in $\mathcal{A}(H)$ at level at most k and $v_k(H)$ will denote the number of vertices in $\mathcal{A}(H)$ at level exactly k.

Theorem 21.1 (Clarkson's Theorem)**.** *Let H be a set of n hyper-planes in $\mathbb{R}^d$, then*

$$|v_{\leq k}(H)| = O(n^{\lfloor \frac{d}{2} \rfloor} k^{\lceil \frac{d}{2} \rceil}).$$

We will present a proof to the above theorem in the case that $d = 2$.

Proof. Let L be a set of n lines in the plane, we want to prove that $|v_{\leq k}(H)| = O(nk)$. We choose a random sample $L' \subseteq L$, by including each line $l \in L$ with probability $p \in (0, 1]$ (the exact value of p will be revealed later). We estimate the expectation of the random variable $X = |v_0(L')|$ in two ways:
On one hand, we have that $X = |v_0(L')| \leq |L'|$, so $E[X] \leq E[|L'|] = np$.
On the other hand, for each vertex $v \in \mathcal{A}(L)$ we define an indicator random variable:

$$X_v = \begin{cases} 1 & \text{if both lines that define } v \text{ are in } L' \text{ and no line below } v \text{ is in } L'. \\ 0 & \text{otherwise.} \end{cases}$$

We have $E[X_v] = p^2(1-p)^{l(v)}$, where $l(v)$ denotes the level of the vertex v. Notice that $X = \sum_{v \in A(L)} X_v \geq \sum_{v \in V_{\leq k}(L)} X_v$, so by linearity of the expectation we have

$$E[X] = \sum_{v \in A(L)} E[X_v] \geq \sum_{v \in V_{\leq k}(L)} E[X_v] = \sum_{v \in V_{\leq k}(L)} p^2(1-p)^{l(v)} \geq$$

$$\geq \sum_{v \in V_{\leq k}(L)} p^2(1-p)^k = |v_{\leq k}(L)| p^2 (1-p)^k$$

if we combine the two bounds,

$$|v_{\leq k}(L)| p^2 (1-p)^k \leq E[X] \leq np$$

so,

$$|v_{\leq k}(L)| \leq \frac{n}{p(1-p)^k}$$

if we choose $p = \frac{1}{k+1}$ and because $(1 - \frac{1}{k+1})^k \geq \frac{1}{e} > \frac{1}{3}$ we get

$$|v_{\leq k}(L)| \leq 3(k+1)n = O(nk)$$

as required. □

The proof for higher dimensions is very similar. The following two theorems are needed:
a. Let Y be a binomial random variable in $B(n,p)$. Then $E[Y^{\lfloor \frac{d}{2} \rfloor}] = O((np)^{\lfloor \frac{d}{2} \rfloor})$.
b. Let H be a set of n hyper-planes in $\mathbb{R}^d$. Then $|v_0(H)| = O(n^{\lfloor \frac{d}{2} \rfloor})$.

The bound that we saw is asymptoticly tight. For example, in the case that $d = 2$ and for some k $(0 \leq k \leq n)$ we can choose $\frac{n}{k}$ lines so that the lower unbounded cell in the arrangement is a convex polygon. We replace each such line with $\frac{k}{2}$ very close parallel lines, in such a way we get $\frac{n}{k} \cdot \frac{k}{2}^2 = \Omega(nk)$ vertices of level at most k.

For every set P of points or for every set L of lines we can define the dual problem. For example, for a point $p = (a,b) \in P$ we can define the line $p^* : y = -ax + b$ and for a non-vertical line $l : y = cx + d \in L$ we can define a point $l^* = (c,d)$.

Lemma 21.2. *A point p is above (on or under) a not non-vertical line l iff the line p^* is above (passing through or under respectively) the point l^*.*

Summary

Bounds on the number of halving edges.

22 Halving edges

Let P be a set of n points in the plane (n is even). We define a graph $G = (P,E)$, when E is the set of all pairs (p,q) such that $p,q \in P$ and the line $l_{p,q}$ that passes through p and q has below it exactly $\frac{n-2}{2}$ points. We call such edges *halving edges.*

Theorem 22.1. *The number of halving edges E is at most $O(n^{\frac{3}{2}})$.*

Proof. We find upper and lower bounds for the crossing number of G. On one hand, we know that $Cr(G) \geq \Omega(\frac{|E|^3}{n^2})$ from the crossing number theorem (Ajtai, Chvatal, Newbborn, Szemerdi; Leigthon). On the other hand, we will show that $Cr(G) \leq |E|n$. If we combine the two bounds we get that $|E| = O(n^{\frac{3}{2}})$, as required.

Lemma 22.2. *The graph G defined above is antipodal.*

Proof. Consider a point $p \in P$ and all the edges incident with p. We draw a vertical line s through p. Let $e = (p, p'), e' = (p, p'') \in E$ two angularly consecutive edges on the left side of s, we claim that the wedge antipodal to the wedge $p'pp''$ contains some point $q \in P$. Indeed, the line l_1 passing through p' and p has $k = \frac{n-2}{2}$ points below it, if we rotate the line l_1 clockwise, towards p'', just after the beginning of the rotation we have $k+1$ points under the line and when we meet the line l_2 passing through the points p and p'' we have again k points under the line. Therefore, we have to lose some point in the way, and this point will be some point in the angle opposite to the angle $p'pp''$. This means that for any two consecutive points of P there is exactly one point in the corresponding opposite wedge. Therefore the difference between the number of points on one side of s to the number of points on the other side of s is at most 1. ☐

Lemma 22.3 (Lovász Lemma)**.** *Let $P \subset \mathbb{R}^2$ and let l be a line that is not parallel to any of the halving edges of P. Then l intersects at most n halving edges.*

Proof. Start with a line l' parallel to l such that l' doesn't intersect any of the halving edges. Next, we begin moving l' towards l. The number of edges that l intersects can change only when we meet some point $p \in P$. By the above lemma, each time we meet such point the number of edges that l' intersects changes at most by one. We meet at most n points of P, so l can intersect at most n edges. ☐

From Lovász Lemma we get that any edge can intersect at most n other edges, therefore $Cr(G) \leq O(n|E|)$ and this completes the proof. ☐

23 A Better upper bound

The best known upper bound on the number of halving edges of an n-point set in the plane is $O(n^{\frac{4}{3}})$, this bound was first proved by Dey [Dey98]. The proof to this bound is similar to the proof in the previous section. Dey showed a better upper bound on the number of pairs of halving edges that cross. Dey showed that this number is at most $O(n^2)$. The proof that we present below is somewhat different and is due to Andrzejak et al.
Let G be the graph as defined in the previous section. For a point $p \in P$ let $d(p)$ denote the degree of p in the graph G. We have:

Theorem 23.1.

$$Cr(G) + \sum_{p \in P} \binom{\frac{1}{2}(d(p)+1)}{2} = \binom{\frac{n}{2}}{2}.$$

Before we present the proof to the theorem we can see that in particular the theorem implies that $Cr(G) \leq \binom{\frac{n}{2}}{2} = O(n^2)$.

Proof. We can prove the theorem in the following way: We begin with the given sequence of points and we move the points continuously until we reach a configuration where all the points are in a convex position. In that position we have exactly $\binom{\frac{n}{2}}{2}$ crossings. Next we show that when we move the points, the left side of the identity above does not change. Consider three points x, y, z. Let us assume that (x, y) is a halving edge and we are moving the point z from one side of the this edge to the other (in this a way, sometime during the motion the points x, y, z will be collinear). Lets examine what happend after the motion of z:
a. Notice that (x, z) and (y, z) cannot be halving edges before the motion, after the motion, the edge (x, y) stops to be halving, and the edges (x, z) and (y, z) became halving, so the degree of z increases by 2.
b. Let $d(z) = 2r + 1$ and let k be a line passing through z and parallel to xy. There are more points of P on the side of k that contains x and y, and by lemma 1.2 there are $r + 1$ halving edges going from z to the larger side of k. So, after the motion we are lossing exactly $r + 1$ crossings.
Hence, on one hand the sum over all points incresed by $r + 1$ and on the other hand the number of crossings decreased by $r + 1$, therefore the left side of the identity remains unchanged.
There are other cases that we have to consider, one of them is the case when

we don't have the edge (x, y). In that case, if we have the edge (x, z) we must have the edge (y, z) too and then when we are moving z reversed process is happening. The other case is when we dont have any of the edges (x, z) or (y, z) than the motion leaves the graph unchanged. □

Next, let us show some other proof for the fact that $Cr(G) = O(n^2)$.

Proof. If G is antipodal graph, we can partition the set of the edges to a collection of pairwise disjoint sets $E_1, E_2, ..., E_t$ such that every E_i $(1 \leq i \leq t)$ is a convex x-monotone polygonal chain. We have that $t \leq n$, because at most one such chain can end in some point of P, for otherwise, it is a contradiction to the antipodality of G. Let C_i denote the chain we get from the edges in E_i. In a similar way, we can partition the set of the edges to concave chains $C'_1, C'_2, ..., C'_m$ $(m \leq n)$. Each crossing arises from a crossing between some convex chain and some concave chain. Such two chains can cross at most twice, therefore the number of croossings is at most $2n^2 = O(n^2)$. □

Summary

Bounds on the number of halving triangles in $\mathbb{R}^3$.

24 Halving triangles in $\mathbb{R}^3$

Let $P \in \mathbb{R}^3$ be a set of n points in general position, i.e. no 4 points are co-planar. Let T be the set of all *halving triangles*, i.e. all the triples $p, q, r \in P$ such the plane passing through them splits $\mathbb{R}^3$ into two halfspaces such that each of them contains exactly $\frac{n-3}{2}$ of points of P.

Theorem 24.1 (Dey- Edelsbrunner)**.** *Let t denote the size of T, then $t = O(n^{\frac{8}{3}})$.*

Proof. We will count the number of crossing pairs of triangles in T that share a common point of P, i.e. pairs of triangles T_1 and T_2 with a common vertex p and in which the edge of T_1 opposite to p intersects the interior of T_2. Let X denote this number. We will find upper and a lower bounds for X.
Consider two points $a, b \in P$, let $T_{a,b}$ denote the number of triangles T such that the segment ab crosses T. We have that $X \leq \sum_{a \neq b \in P} 3T_{a,b}$. Therefore, by the Lovász lemma we get that $X \leq \sum_{a \neq b \in P} 3T_{a,b} = \sum_{a \neq b \in P} O(n^2) =$

$O(n^4)$.
Next, we will find a lower bound for X. For every point $p \in P$ let t_p denote the number of triangles of T that are incident to p (i.e., that have p as one of their vertices). We draw a small two-dimensional sphere $\mathcal{S}$ centered at p and centrally project all triangles incident to p onto $\mathcal{S}$. In this projection we have a drawing of a graph G_p on a set of $n-1$ points (those are the projections of all the $n-1$ points of $S \setminus \{p\}$ onto $\mathcal{S}$) and a set of t_p great circular arcs (these are the projections of the triangles incident to p onto $\mathcal{S}$) connecting pairs of points on $\mathcal{S}$. Let X_p denote the number of crossings in G_p. Notice that $X = \sum_{p \in P} X_p$. So,

$$X = \sum_{p \in P} X_p = \sum_{p \in P} \left(\frac{t_p^3}{64(n-1)^2} - 4(n-1) \right) \geq \frac{1}{64n^2} \sum_{p \in P} t_p^3 - 4n^2$$

Next, we want to estimate the sum $\sum_{p \in P} t_p^3$. First observe that $\sum_{p \in P} t_p = 3t$. Next, by Hölder's inequality we have that $\sum_{p \in P} t_p \cdot 1 \leq (\sum_{p \in P} t_p^3)^{\frac{1}{3}} \sum_{p \in P} (1^{\frac{3}{2}})^{\frac{2}{3}}$. So, $3t \leq (\sum_{p \in P} t_p^3) n^{\frac{2}{3}} \Rightarrow \sum_{p \in P} t_p^3 \geq \frac{27t^3}{n^2}$. Therefore

$$X \geq \frac{1}{64n^2} \cdot \frac{27t^3}{n^2} - 4n^2 = \Omega(\frac{t^3}{n^4})$$

under the assumption that $\frac{27t^3}{n^4} \geq 8n^2$, i.e. $t \geq n^2$. Finally we get that $\frac{t^3}{n^4} \leq O(n^4)$, i.e. $t = O(n^{\frac{8}{3}})$ as required. □

Next, we will show a better lower bound.

Theorem 24.2 (Sharir, Smorodinsky, Tardos)**.** *Let T and t be as defined above. Then $t = O(n^{2.5})$.*

Proof. As above, we find a lower bound for X. We show that $X = \Omega(\frac{t^2}{n}) - O(nt)$, which together with the bound $X = O(n^4)$ implies that $t = O(n^{2.5})$. For a point $p \in P$, let T_p denote the set of triangles that have p as one of their vertices. Put $t_p = |T_p|$. Let h_p be the horizontal plane passing through p. Set π_p to be any horizontal plane above h_p. We project every triangle puv in T_p, such that u or v are above h_p, centrally from p onto π_p. The resulting set of projected triangles has the following structure. Each point $u \in P$ that lies above h_p is mapped to a point $u^* \in \pi_p$. Each triangle puv for which both u and v lie above h_p is mapped to a segment u^*v^*, and triangle puv for which u lies above h_p but v lies below h_p is mapped to a ray emanating from u^*.

Triangles puv for which both u and v are below h_p are excluded from the analysis. Let G_p denote this graph drawn on π_p, r_p will denote the number of rays in G_p and e_p will denote the number of edges that are not rays in G_p. We note the following simple facts:

- $\sum_{p \in P} t_p = 3t$
- $\sum_{p \in P} e_p = t$
- $\sum_{p \in P} r_p = t$

Every triangle contributes at most one edge and one ray. It is also easy to verify that every graph G_p is antipodal.

Next, we decompose the edges of each G_p into collection of convex chains. The number of such chains in the graph G_p is at least $\frac{r_p}{2}$, because each ray begins or ends a chain. Let X_p denote the number of intersections in G_p. We would like to bound the sum $\sum_{p \in P} X_p$. We have that $X_p \geq \binom{\frac{r_p}{2}}{2} - y_p$ where y_p is the number of pairs of chains that do not intersect. We will bound y_p from above. Let C_1 and C_2 be a non-crossing pair of chains. Then either (a) C_1 and C_2 meet at a vertex or (b) C_1 and C_2 are disjoint. In some vertex v at most $\frac{degree(v)}{2}$ chains can meet, so there are at most $n \sum_{v \in G_p} d(v) = n(2e_p + r_p)$ chains that can meet in a vertex. Two disjoint chains are uniquely defined by a ray or an edge and a point, so the number of disjoint chains is at most $n(r_p + e_p)$. So we have that $y_p = O(n(r_p + e_p))$, and therefore $X_p \geq \Omega(r_p^2) - O(n(r_p + e_p))$. Hence

$$X \geq \sum_{p \in P} \Omega(r_p^2) - O(n \sum_{p \in P} (r_p + e_p)) =$$

$$= \Omega(\frac{(\sum_{p \in P} r_p)^2}{n}) - O(2tn) =$$

$$= \Omega(\frac{t^2}{n}) - O(tn) = \Omega(\frac{t^2}{n})$$

for $t >> n^2$. □

Summary

Davenport-Schinzel sequences.

25 Davenport-Schinzel sequences

Let $f_1, f_2, ..., f_n : \mathbb{R} \to \mathbb{R}$ be n functions. We assume that every two of the functions intersect at most s times. Set $L(x) = min_i\{f_i(x)\}$, we want to find the combinatorial complexity of $L(x)$. We number the functions from 1 to n. Let $a_1, a_2, ..., a_l$ be the sequence of functions appearing on the lower envelope $L(x)$ from left to right. This sequence has the following properties:

(i) For all $1 \le i \le l$, $a_i \in \{1, ..., n\}$.

(ii) Every two sequential elements are different, i.e. $a_i \neq a_{i+1}$ $(1 \le i \le l-1)$.

(iii) There is no subsequence of the form

$$...a...b...a...b...a...$$

with $s+2$ letters a and b $(a \neq b)$. Indeed if we had such a sequence, we would have $s+1$ intersections between the functions a and b (between an occurrence of a function a and an occurrence of a function b, a and b must intersect).

Any finite sequence with the properties (i)-(iii) above is called a *Davenport-Schinzel sequence of order s over the alphabet* $\Sigma = \{1, ..., n\}$. Let $DS_s(n)$ denote the set of Davenport-Schinzel sequences of order s over n letters, also let $\lambda_s(n)$ denote the maximum possible length of a sequence in $DS_s(n)$. It is clear that the complexity of the lower envelope of n polynomials of degree s is at most than $\lambda_s(n)$. It is easy to see that $\lambda_1(n) = n$.

Proposition 25.1. *We have* $\lambda_2(n) = 2n - 1$.

Proof. The following sequence belongs to $DS_2(n)$

$$1, 2, 3, ..., n-1, n, n-1, ..., 3, 2, 1$$

so $\lambda_2(n) \ge 2n - 1$. On the other hand, we will show that $\lambda_2(n) \le 2n - 1$ by induction on n. Let $A = a_1, ..., a_l$ be a sequence in $DS_2(n)$. There is a symbol that appears at most once in our seqeunce. Indeed, assume that every symbol appears at least twice. Let x be the first element of the sequence A that appears twice, and let i and j be the indexes of the first and the second appearance of x. There is a symbol $y \neq x$ that appears at index

k when $i < k < j$. Since y appears once more at index l such that $l > j$ we get a forbidden sub-sequence $...x...y...x...y....$ Hence, let a be a symbol that appears only once in A. We delete the only appearance of a from the sequence and distinguish between two cases:

- The two symbols that appear just before and after a are different, i.e. we have $...xay....$ In that case, after deleting a we have a sequence in $DS_2(n-1)$, and by induction the length of the sequence we began with is at most $2(n-1)-1+1 = 2n-2 \leq 2n-1$.
- The symbols that appear before and after a are the same, i.e. we have $...yay....$ We delete a and one of the appearances of y and obtain a sequence in $DS_2(n-1)$. Again, by induction the length of the sequence we began with is at most $2(n-1)-1+2 = 2n-1$.

□

Let $\sigma(n)$ be the maximal complexity of the lower envelope of n line-segments in the plane. It is easy to see that $\sigma(n) \leq \lambda_3(n)$.

Proposition 25.2. *We have* $\lambda_3(n) = O(n \ln n)$.

Proof. Let w be a Davenport-Schinzel sequence of order 3 over n letters. Let l be the length of w, i.e. $l = |w|$. By the pigeonhole principle there is a symbol $'a'$ that appears at most $\frac{l}{n}$ times in the sequence w. We call an appearance of $'a'$ *dangerous* if the symbols that appear before $'a'$ and after are the same, i.e. $w = ...a...a...yay...a....$ Notice that only the first and the last appearances of a can be dangerous (otherwise we have the forbidden pattern $ayaya$). Hence by deleting all the appearances of a and at most two more symbols, we obtain a sequence in $DS_3(n-1)$. So we receive the recurrence

$$\lambda_3(n) \leq \lambda_3(n-1) + 2 + \frac{\lambda_3(n)}{n}$$

$$\Rightarrow \lambda_3(n)\Big(1 - \frac{1}{n}\Big) \leq \lambda_3(n-1) + 2$$

$$\Rightarrow \frac{\lambda_3(n)}{n} \leq \frac{\lambda_3(n-1)}{n-1} + \frac{2}{n-1}$$

Set $f(n) = \frac{\lambda_3(n)}{n}$, so

$$f(n) \leq f(n-1) + \frac{2}{n-1} \leq f(n-2) + \frac{2}{n-1} + \frac{2}{n-2} \leq$$

$$... \leq 1 + 2(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1}) = O(\ln n)$$

So, we get that $\frac{\lambda_3(n)}{n} = O(\ln n) \Rightarrow \lambda_3(n) = O(n \ln n)$. □

The exact known bounds for $\lambda_3(n)$ are $\frac{1}{2}n\alpha(n) - 2n \leq \lambda_3(n) \leq 2n\alpha(n) + O(\sqrt{n\alpha(n)})$, where α is the inverse function of the Ackerman function. The Ackerman function is defined in the following way:
$A_1(n) = 2n$
$A_k(1) = 1$
$A_k(n) = A_{k-1}(A_k(n-1))$
The main property of the Ackerman function is that it grows very fast, for example $A_2(n) = 2^n$ and $A_3(n) = 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}$, a tower of hight n. $A(n)$ is defined as $A_n(n)$ and α is defined as $\alpha(n) = min\{k \geq 1 : A(k) \geq n\}$.

26 An application of Davenport-Schinzel Circles

It can be shown that the length of $DS_2(n)$ circle is at most $2n-2$. Let $g_d(n)$ be the maximal number of geometric permutations of n convex and pairwise disjoint objects in $\mathbb{R}^d$. It is known that $\Omega(n^{d-1}) = g_d(n) = O(n^{2d-2})$. We will show that $g_2(n) \leq 2n-2$.

Proposition 26.1 (Edelsbrunner, Sharir)**.** *We have* $g_2(n) = 2n-2$.

Proof. Let $\mathcal{C}$ be the set of n pairwise disjoint convex sets $C_1, ..., C_n$. Every directed line transversal $\overrightarrow{l}$ that intersects all the sets of $\mathcal{C}$ induces some geometric permutation. Every such line $\overrightarrow{l}$ can be translated to a line $\overrightarrow{l'}$ such that $\overrightarrow{l'}$ is tangent to two sets from $\mathcal{C}$ from the same side and intersects the sets in $\mathcal{C}$ in the same order that $\overrightarrow{l}$ intersects. We call such a directed line transversal $\overrightarrow{l'}$ a *canonical line*. We can bound the number of canonical lines. Consider the circle of orientations. On that circle we can associate a segment that every line in this orientation, when we move it parallely from left to right there is some convex set that stops being intersected by that line first. In the orientation between some two segments a and b ($a, b \in \mathcal{C}$) there is a canonical line that tangent to the sets a and b. Notice that we recieve a $DS_2(n)$-circle, and that because on the circle we can not get a sequence of $..a...b...a...b..$ because to recieve such a sequence we need four lines that tangent to a and b on the same side, and we have just two of them. So, because the length of the $DS_2(n)$-circle is at most $2n-2$, there are at most $2n-2$ canonical lines, and therefore there are at most $2n-2$ geometric permutations. Also, we can find an example where the number of geometric permutations is $2n-2$. Hence, $g_2(n) = 2n-2$. □

Summary

Geometric hypergraphs. Transversals and Epsilon-nets.

27 Geometric Hypergraphs

Hypergraphs are a generalization of graphs. A hypergraph is a pair $H = (V, E)$ where V is the vertex set and $E \subseteq 2^v$, i.e. a set of subsets of V (notice that in the case that all the elements of E are of size two, than we get a simple graph).

Definition. *A subset $T \subseteq V$ called a transversal or a hitting set of H if for each $S \in E$ we have that $S \cap T \neq \emptyset$.*

Hitting set is a generalization of the idea of vertex cover for hypergraphs. $\tau(H)$ will denote the minimal size of a hitting set of H.

Definition. *A subset $M \subseteq E$ called a packing or a matching if for every two sets $S_1, S_2 \in M$, we have that $S_1 \cap S_2 = \emptyset$.*

$\nu(H)$ will denote the maximal size of a packing in H.

Definition. *A fractional transversal for a graph H is a function $\varphi : V \to [0, 1]$ such that for every $S \in E$ we have that $\sum_{x \in S} \varphi(x) \geq 1$. The size of fractional transversal φ is $\sum_{x \in V} \varphi(x)$.*

$\tau^*(H)$ will denote the infimum of the sizes of the fractional transversals.

Definition. *A fractional packing for a graph H is a function $\psi : E \to [0, 1]$ such that for every $x \in V$ we have that $\sum_{x \in S} \psi(S) \leq 1$. The size of fractional transversal ψ is $\sum_{S \in E} \psi(S)$.*

$\nu^*(H)$ will denote the supremum of the sizes of all the fractional packings for H.
Notice that for hypergraph $H = (V, E)$, we have:

- $\nu(H) \leq \tau(H)$ (when H is a simple bipartite graph we have $\nu(H) = \tau(H)$ by the König theorem).
- $\tau^*(H) \leq \tau(H)$.
- $\nu^*(H) \geq \nu(H)$.

Theorem 27.1. *Let $H = (V, E)$ be a hypergraph, we have $\tau^*(H) = \nu^*(H)$.*

Proof. Set $n = |V|$ and $m = |E|$, and let A be a $m \times n$ incidence matrix of the hypergraph H (the entry in the matrix corresponding to a point p and a set S is 1 if $p \in S$ and 0 otherwise). It is easy to see that

$$\tau^*(H) = inf\{1_n^T x : x \geq 0, Ax \geq 1_m\}$$

and

$$\nu^*(H) = max\{y^T 1_m : y \geq 0, y^T A \leq 1_n^T\}$$

so, by the duality of linear programing we get that $\tau^*(H) = \nu^*(H)$. □

28 Epsilon-nets

Definition. *Let $H = (V, E)$ be a hypergraph with V finite and let $\epsilon \in [0, 1]$ be a real number. A subset $N \subseteq V$ is called an ϵ-net or $\frac{1}{r} - net$ (when $\frac{1}{r} = \epsilon$) if for every set $S \in E$ that $|S| \geq \epsilon |V|$ we have that $N \cap S \neq \emptyset$.*

We can also define an ϵ-net for an infinite set V:

Definition. *Let $H = (V, E)$ be a hypergraph, let μ be a probability measure on V such that the sets in E are μ-measurable, and let $\epsilon \in [0, 1]$ be a real number. A subset $N \subseteq V$ is called an $\epsilon - net$ if for every set $S \in E$ that $\mu(S) \geq \epsilon$ we have that $N \cap S \neq \emptyset$.*

For example, let P be a set of n points on the line, and $E = \{P \cap I : I \text{ is an interval}\}$. For $H = (P, E)$ the size of the minimal $\frac{1}{r}$-net is $r - 1$. Indeed, we can get a $\frac{1}{r}$-net of size $r-1$ just by taking the $i \cdot \frac{n}{r}$ $(i = 0, ..., r-1)$ points on the line. Next, we show that it can not be an $\frac{1}{r}$-net of size less than $r - 1$. If we assume by contradiction that there is a smaller $\frac{1}{r}$-net of size $x < r - 1$. In that case, there are $x + 1$ intervals between the points chosen to the $\frac{1}{r}$-net, so one of the intervals contain more than $\frac{n}{x+1} > \frac{n}{r}$ points from P, a contradiction to our assumption.

Summary

Epsilon-nets cont', VC-dimension.

29 Epsilon-nets cont'

Lets examine some other example for calculating the size of an $\frac{1}{r}$-net: We define a hypergraph $H = (P, E)$, where P is a set of n points in the plane and $E = \{P \cap h : h \text{ is a halfplane}\}$. For such hypergraphs the size of $\frac{1}{r}$-net is at least $\Omega(r)$, because in the hypergraph where the points arranged as clusters in convex position, such that every cluster has $\frac{n}{r}$ points, we will need at least r points to the $\frac{1}{r}$-net.

Proposition 29.1. *Let $H = (P, E)$ be a hypergraph defined above. There is an $\frac{1}{r}$-net of size at most $2r$.*

Proof. Let $P' \subseteq P$ be the set of points on the boundary of the convex hull of P. Notice that for every halfplane h such that $h \cap P \neq \emptyset$ we have that $h \cap P' \neq \emptyset$. We choose a minimal subset $N \subseteq P'$ such that N is an $\frac{1}{r}$-net. We claim that $|N| \leq 2r$. Indeed, let us choose a subset $N' \subseteq N$ such that $|N'| \geq \frac{|N|}{2}$ by choosing every second point on the boundary of the convex hull of N. For every $p \in N$ we have that $N \backslash \{p\}$ is not an $\frac{1}{r}$-net, i.e. for every $p \in N$ there is a halfplane h_p such that $|h_p \cap P| \geq \frac{n}{r}$ and $h_p \cap N = \{p\}$. Also notice that for two different points $p_1, p_2 \in N'$ we have that $h_{p_1} \cap h_{p_2} \cap P = \emptyset$ (otherwise there was a point in $N \backslash N'$ not on the CH(P)). So,

$$n \geq \sum_{p \in N'} |h_p| = |N'| \frac{n}{r} \Rightarrow |N'| \leq r$$

therefore $|N| \leq 2|N'| \leq 2r$. □

Pyrga and Ray showed that for the hypergraph $H = (P, E)$ where P is a set of n points in $\mathbb{R}^3$ and $E = \{P \cap h : \text{h is a halfspace in } \mathbb{R}^3\}$ there is a $\frac{1}{r}$-net of size $O(r)$. For higher dimensions it is not known if there is a $\frac{1}{r}$-net of size $O(r)$, but it is known that there is an $\frac{1}{r}$-net of size $O(r \log r)$.

30 VC-dimension

Definition. *Let $H = (V, E)$ be a hypergaph. We say that a subset $Y \subseteq V$ is shattered if $E|_Y = P(Y)$ where $E|_Y = \{S \cap Y : S \in E\}$ and $P(Y)$ is* *the power set of Y. The VC-dimension (Vapnik-Chervonenkis dimension) of H, denoted by $dim(H)$, is the supremum of the sizes of all finite shattered subsets of V.*

Lets begin with some examples:

- $H = (P, E)$ when P is a set of n points on the line, and $E = \{P \cap I :$ I is an interval$\}$. One can see that it is easy to find a subset of P of size two that can be shattered, but we can not find such susbset of size three, hence $dim(H) = 2$.
- $H = (\mathbb{R}^2, E)$ when $E = \{\mathbb{R}^2 \cap h :$ h is an halfplane$\}$. In that case $dim(H) \geq 3$, for example for points arranged in a triangle.
- In the general case, consider the hypergraphs $H = (\mathbb{R}^d, E)$ when $E = \{\mathbb{R}^d \cap h : h$ is a closed halfspace$\}$. On one hand, one can see that $dim(H) \geq d + 1$ by taking $d + 1$ affinely independent points. On the other hand, a set $P \subseteq \mathbb{R}^d$ such that $|P| = d + 2$ can not be shattered. Indeed, by the Radon Lemma we can partition such a set P into two nonempty disjoint sets P_1 and P_2 such that $CH(P_1) \cap CH(P_2) \neq \emptyset$. There is no halfspace h such that $h \cap P = P_1$, because if there was such a halfspace the intersection between $CH(P_1)$ and $CH(P_2)$ would be empty.

Theorem 30.1 (Hausler, Welzl : Epsilon-Net Theorem)**.** *Let $H = (V, E)$ be a hypergraph where V is a set equipped with a probability measure μ and E is a set of μ-measurable subsets of V. Assume that $dim(H) \leq d$. Then there is an absolute constant C such that for every $r \geq 2$ there is an $\frac{1}{r}$-net for H of size at most $Cdr \ln r = O(\frac{d}{\epsilon} \log \frac{1}{\epsilon})$.*

Before we show the proof for the Epsilon-net Theorem, we need a lemma bounding the number of distinct hyperedges in a hypergraph of a given VC-dimension. First we define a *shatter function* of a hypergraph $H = (V, E)$ by $\pi_H(m) = sup_{Y \subseteq V, |Y| = m}\{|E|_Y|\}$.

Lemma 30.2. *For any hypergraph $H = (V, E)$ with $dim(H) = d$, we have $\pi_H(m) \leq \Phi_d(m)$ where $\Phi_d(m) = \sum_{i=0}^{d} \binom{m}{i}$.*

Proof. Let $H = (V, E)$ be a hypergraph with $dim(H) = d$ and $|V| = n$. It is sufficient to show that $\pi_H(n) = |E| \leq \Phi_d(n)$ because when we pass to a subgraph the VC-dimension does not increase. The proof is by induction on n and d. Let x be some element of V. Put $E_1 = E|_{V\backslash\{x\}}$, we have that $|E_1| \leq \Phi_d(n-1)$ by the induction hypothesis. Put $E_2 = \{S \in E : x \notin S, S \cup \{x\} \in E\}$. We have that $|E| = |E_1| + |E_2|$, because the only edges we miscount in $|E_1|$ are the edges S such that $x \notin S$ and $S, S \cup \{x\} \in E$, for that two edges we add only one edge to E_1. So, every edge that we do not count in E_1 we count it in E_2 and receive the desirable equality. Notice that in the hypergraph $H' = (V\backslash\{x\}, E_2)$ we have $dim(H') \leq d-1$, that is true because if $A \subseteq V\backslash\{x\}$ is shattered in H' than $A \cup \{x\}$ is shattered in H. Therefore, again by the induction hypothesis, $|E_2| \leq \Phi_{d-1}(n-1)$. So $|E| \leq \Phi_d(n-1)+\Phi_{d-1}(n-1) = \Phi_d(n)$ (the equality $\Phi_d(n-1)+\Phi_{d-1}(n-1) = \Phi_d(n)$ we seen before). $\square$

From the lemma above we receive that if $H = (V, E)$ ($|V| = n$) is a hypergraph with a dimension at most d, than $|E| \leq \left(\frac{ne}{d}\right)^d = O(n^d)$, where e is the basis of the natural logarithm.
For the proof of the Epsilon-net Theorem we need a technical lemma concerning the binomial distribution.

Lemma 30.3. *Let $X = X_1 + X_2 +, ..., +X_n$ be a random variable where the X_i are independent Bernoulli random variables attaining the value 1 with probability p and value 0 with probability $1-p$. Then $Pr[X > \frac{1}{2}np] > \frac{1}{2}$ (we assume that $np \geq 8$).*

Proof. By the Chebyshev's inequality we have that $Pr[X - Exp[X] \geq t] \leq \frac{Var[X]}{t^2}$. In our case $Exp[x] = np$ and $Var[X] = np(1-p) \leq np$, therefore

$$Pr[X \leq \frac{1}{2}np] \leq Pr[|X - np| \geq \frac{1}{2}np] \leq \frac{Var[x]}{\frac{1}{4}(np)^2} \leq \frac{4}{np} \leq \frac{1}{2}$$

so, $Pr[X \geq \frac{1}{2}np] \geq \frac{1}{2}$. $\square$

Proof of the Epsilon-Net Theorem. Let us put $s = Cdr \ln r$. We pick a random sample $N \subseteq V$ of size at most s by s independent random draws, where each element is drawn from V according to the probability distribution μ (the selection of the elements is with replacements). We want to show that the probability that N is not an $\frac{1}{r}$-net is less than one, i.e. there is an $\frac{1}{r}$-net of size at most s with a positive probability.
We can assume that for every $e \in E$, $\mu(e) \geq \frac{1}{r}$ (we do not need to "hit" the edges with a smaller measure). Let $T \in E$ be some edge, $Pr[T \cap N = \emptyset] =$

$\left(1-\frac{1}{r}\right)^s \le e^{-\frac{s}{r}}$, i.e. the probability that we do not choose any of the vertices of some edge to N is smaller than $e^{-\frac{s}{r}}$. So, if s was at least $r \ln |E| + 1$ then N would be an $\frac{1}{r}$-net, but usually r is much smaller than $|E|$, so some more work need to be done.
Let A_0 be the event that N is not an $\frac{1}{r}$-net, i.e. we missed some edge of the edges in E. We want to bound the $Prob[A_0]$.
We pick another sample M of size at most s by s independent random draws. Set $k = \frac{s}{2r}$. We define another event A_1. A_1 is the event that N is not a $\frac{1}{r}$-net and there exists $T \in E$ such that $|M \cap T| \ge k$ (if an element from T repeated in M we will count it the appropriate number of times).
It is easy to see that $Pr(A_1) \le Pr(A_0)$, because for A_1 to happen A_0 need to happen. Next, we want to show that

(a) $Pr(A_0) \le 2Pr(A_1)$.

(b) $Pr(A_1) < \frac{1}{2}$.

First, we will show (a):
We know that $Pr(A_1) = Pr(A_1|A_0)Pr(A_0) + Pr(A_1|\overline{A_0})Pr(\overline{A_0})$, the $Pr(A_1|\overline{A_0}) = 0$ (if N is an $\frac{1}{r}$-net, A_1 can not occur) so, $Pr(A_1) = Pr(A_1|A_0)Pr(A_0)$. We want to show that $Pr(A_1|A_0) \ge \frac{1}{2}$. Let $T \in E$ be an edge such that $T \cap N = \emptyset$. We have that $Pr(A_1|A_0) \ge Pr(|M \cap T| \ge k)$, notice that $|M \cap T|$ is a sum of independent Bernoulli random variables as in lemma 2.3 with $n = s$ and $p = \frac{1}{r}$, so by the lemma $Pr(|M \cap T| \ge k) \ge \frac{1}{2}$. Hence $Pr(A_1) = Pr(A_1|A_0)Pr(A_0) \ge \frac{1}{2}Pr(A_0) \Rightarrow Pr(A_0) \le 2Pr(A_1)$.
Now, we show (b):
We choose N and M differently. First, we make $2s$ independent random draws from V, and we get a sequence $Z = z_1, z_2, ..., z_{2s}$. Then, we choose randomly s positions in Z and put the elements at these positions into N, and the remaining elements into M. In that way, we did not change the distributions. Next, we want to show that for every such a sequence Z, $Pr[A_1|Z] \le \frac{1}{2}$. So, let Z be fixed. First let $T \in E$ be an edge and set $P_T = Pr[N \cap T = \emptyset \wedge |M \cap T| \ge k|Z]$. If $|Z \cap T| < k$ then $P_T = 0$. Otherwise $P_T \le Pr[N \cap T = \emptyset|Z]$, so

$$P_s \le \frac{\binom{2s-k}{s}}{\binom{2s}{s}} \le \frac{(\frac{2s-k}{s})^s}{(\frac{2s}{s})^s} = \left(1 - \frac{k}{2s}\right)^s \le e^{-\frac{k}{2}} = e^{-\frac{Cd \ln r}{4}} = r^{-\frac{Cd}{4}}.$$

For a fixed Z, consider the hypergraph $H' = (Z, E|_Z)$, we know that

$dim(H') \leq d$, therefore by lemma 2.2 we have that $|E|_Z| \leq \Phi_d(2s) \leq \left(\frac{2se}{d}\right)^d$. So,

$$Pr[A_1|Z] \leq \Phi_d(2s)r^{-Cd/4} \leq \left(\frac{2se}{d}\right)^d r^{-Cd/4} = (2er\ln r \cdot r^{-C/4}) < \frac{1}{2}$$

for C large enough.
Hence $Pr(A_0) \leq 2Pr(A_1) < 1$, and there is a set N such that N is a $\frac{1}{r}$-net and $|N| \leq Cdr\ln r$. □

Komdos, Pach and Woginger showed that the above bound on the size of $\frac{1}{r}$-net is tight.

Corollary 30.4. *Let $H = (V, E)$ be a hypergraph such that $dim(H) \leq d$, then we have*

$$\tau^*(H) \leq \tau(H) \leq O(\tau^*(H)\ln\tau^*(H))$$

.

Proof. Let $r = \tau^*(H)$ and let $\varphi : V \to [0,1]$ be a fractional transversal such that $|\varphi| = r$. We define a probability measure $\mu : V \to [0,1]$ such that for every $x \in V$, $\mu(x) = \frac{\varphi(x)}{r}$. Notice, that because φ is a fractional transversal, for every $S \in E$ we have that $\mu(S) = \sum_{x\in S}\mu(x) = \frac{1}{r}\sum_{x\in S}\varphi(x) \geq \frac{1}{r}$. Hence, an $\frac{1}{r}$-net will be a transversal, and by the Epsilon-net Theorem we have $\frac{1}{r}$-net of size at most $O(dr\ln r) = O(d\tau^*(H)\ln\tau^*(H))$ □

Summary

Bounds on VC-Dimension. One more theorem about Epsilon nets.

31 Bounds on the VC-Dimension

We saw that in the hypergraphs $H = (\mathbb{R}^d, E)$ such that $E = \{\mathbb{R}^d \cap h : h \text{ is a closed halfspace}\}$ we have that $dim(H) = d + 1$. Notice that every halfspace in $\mathbb{R}^d$ can be described as the set $\{x \in \mathbb{R}^d | a_0 + \sum_{i=1}^d a_i x_i \geq 0\}$. In the general case we will show that we can bound the VC-dimension of hypergraphs where the edges defined by intersection of finite set in $\mathbb{R}^d$ with a polynomial in d variables of degree at most D. Let $\mathbb{R}[x_1, ..., x_d]_{\leq D}$ denote the set of all polynomials in d variables of degree at most D, and let $P_{d,D} = \{\{y \in \mathbb{R}^d | p(y) \geq 0\} | p \in \mathbb{R}[x_1, ..., x_d]_{\leq D}\}$.

Proposition 31.1. *We have that* $dim((\mathbb{R}^d, P_{d,D})) \leq \binom{d+D}{d}$.

Proof. For the proof of the proposition we will use a technique from algebriac geomtry called linearization. Let M be the set of all possible nonconstant monomials of degree at most D in $x_1, ..., x_d$, i.e. monomials of the form $x_1^{i_1} x_2^{i_2} ... x_d^{i_d}$ such that $i_1 + i_2 + ... + i_d \neq 0$ and $i_1 + i_2 + ... + i_d \leq D$. Put $m = |M|$. The size of M is the number of solutions $(i_1, i_2, ..., i_d) \neq 0$ such that $i_1 + i_2 + ... + i_d \leq D$, i.e. $m = \binom{d+D}{d} - 1$. Let the coordinates in $\mathbb{R}^m$ be indexed by the monomials in M. We define a transformation $\varphi : \mathbb{R}^d \to \mathbb{R}^m$ by $\varphi(y)_\mu = \mu(y)$. It is easy to see that φ is injective, so if we will show that if a set A is shattered in $(\mathbb{R}^d, P_{d,D})$, then $\varphi(A)$ is shattered by halfspaces in $\mathbb{R}^m$, in such a way, by what we know about the VC-dimension of halfspaces, we will get that $dim((\mathbb{R}^d, P_{d,D})) = m+1 \leq \binom{d+D}{d}$. So, let A be a set that can be shattered in $(\mathbb{R}^d, P_{d,D})$. Let $B \subseteq A$ be a set such that there is a polynomial $p \in \mathbb{R}[x_1, ..., x_d]_{\leq D}$ such that for each $y \in B$ we have that $p(y) \geq 0$ and for each $y \in A \backslash B$ we have that $p(y) < 0$. Let a_μ denote the coefficient of the monomial μ in p, then $p = a_0 + \sum_{\mu \in M} a_\mu \mu$. We define a halfspace h in $\mathbb{R}^m$ as $\{y \in \mathbb{R}^m | a_0 + \sum_{\mu \in M} a_\mu \mu(y) \geq 0\}$, so it is easy to see that for each $y \in \mathbb{R}^d$, if $y \in B$ then $\varphi(y) \in h$. Hence, for every $B \subseteq A$ there is a halfspace h such that $\varphi(A) \cap h = \varphi(B)$, and this completes the proof. □

Theorem 31.2. *Let* $F[X_1, X_2, ..., X_k]$ *be a expression over sets (using the operations: union, intersection and difference). Let* E *be a collection of subsets of a set* V *such that* $dim(V, E) \leq d$*, also let* $T = \{F[S_1, S_2, ..., S_k] | S_1, S_2, ..., S_k \in E\}$*. Then for* $H' = (V, T)$ *we have* $dim(H') = O(k \ln kd)$*.*

32 One more theorem about Epsilon nets

In this section we return to Epsilon nets and present a theorem of Matousek, Seidel and Welzl, and a recent simple proof by Pygra and Ray.

Theorem 32.1 (Matousek, Seidel and Welzl 90, Pyrga and Ray 08')**.** *Let* $H = (P, E)$ *be a hypergraph where* $P \subseteq \mathbb{R}^3$ *and* $E = \{\mathbb{R}^d \cap h : h \text{ is a closed halfspace in } \mathbb{R}^3\}$*. There is an* ϵ*-net* N *for* E*, such that* $|N| = O(\frac{1}{\epsilon})$*.*

Proof. Put $n = |P|$ and $k = \lfloor \epsilon n \rfloor$. We show that there is an ϵ-net N such that $|N| = O(\frac{n}{k})$. We will make two lenient assumptions, the first is that each halfspace contains exactly k points and the second is that every halfspace contains the origin. Let F be a maximal set of halfspaces with the following properties:

- For each halfspace $h \in F$ we have that $|h \cap P| = k$.
- For every $h_1, h_2 \in F$ we have that $|h_1 \cap h_2 \cap P| \leq \beta k$, where $\beta = \frac{1}{8}$.

For every $h \in F$ we choose an $\frac{1}{8}$-net N_h for the points $h \cap P$. Set $N = \cup_{h \in F} N_h$, we have that $|N| = O(|F|)$ (notice that the size of each N_h is at most some absolute constant). Next, we will prove the following:

(a) N is indeed an ϵ-net

(b) $|N| = O(\frac{n}{k})$

For (a), let h be a halfspace such that $|h \cap P| = \epsilon n = k$, if $h \in F$ then $|h \cap N| \neq \emptyset$ because $N_h \subseteq h \cap N$. Else, there is a halfspace $h' \in F$ such that $|h' \cap (h \cap P)| > \beta k$, so by definition of $N_{h'}$ we have that $h \cap N_{h'} \neq \emptyset$.
Next, we prove (b). We want to show that $x = |F| = O(\frac{n}{k})$. First, we count the incidences I between the points in P and the halfplanes in F. We have: $I = \sum_{h \in F} |h \cap P| = k \cdot |F| = k \cdot x$. Next, we define a duality transformation: for a point $a \in \mathbb{R}^3 \backslash \{0\}$ we define the plane $a^* = \{y \in \mathbb{R}^3 | \langle y, a \rangle = 1\}$, and for a non-vertical plane $\{y \in \mathbb{R}^3 | \langle y, b \rangle = 1\}$ we define the point $h^* = b$. It easy to check that a point $a \neq 0$ is above (under, respectively) a non-vertical plane h if and only if the plane a^* is above (under, respectively) the point h^*. Let P^* denote the set of the n dual planes to the points in P and let F^* denote the set of x points dual to the planes bounding the halfspaces in F. We define a graph G such that the vertices of G are the points F^* and the edges are the edges of the convex hull of F^*. We also add edges as follows: For every point p in the interior of the convex hull of F^*, we connect p to four points on the convex hull such that p is the convex combination of those four points (this is always possible due to Caratheodory's Theorem). So, the number of edges in the graph G is at most $4x$. One can see that for every halfspace h, the graph $G[h \cap F^*]$ is connected, hence $E(G[h \cap F^*]) \geq |h \cap F^*| - 1$ (one can check that if we look on the highest point p above h, then from every point that above h there is a path to p). Next, we count in two different ways the number of edges in every halfspace $h \in P^*$. So on one hand, $A = \sum_{h \in P^*} E(G[h \cap F^*]) \geq \sum_{h \in P^*} |h \cap F^*| - 1 = \sum_{h \in P^*} |h \cap F^*| - n = kx - n$. On the other hand, if we return to the primal hypergraph the number of of planes below some edge is the number of points in the primal that are below an intersection of

two halfspaces in F, i.e. at most $\beta \cdot k$, so $A \leq e(G) \cdot \beta \cdot k \leq 4x\beta k$. Hence, $kx - n \leq A \leq 4x\beta k \Rightarrow |F| = x \leq \frac{2n}{k}$, so finally $|N| = O(\frac{1}{\beta} \log \frac{1}{\beta} \frac{2n}{k}) = O(\frac{n}{k})$ as required.

□

Summary

Weak Epsilon nets, (p, q)-Condition.

33 Weak Epsilon nets

Consider a hypergraph $H = (\mathbb{R}^2, C)$ where C is a collection of all the convex sets in $\mathbb{R}^2$. For such a hypergraph we have that $dim(H) = \infty$ because for every n we can find a set of points $P \subset \mathbb{R}^2$ ($|P| = n$) that can be shattered. For example, let P be a set of n points on a circle, for every subset $P' \subseteq P$ we have that $P' = CH(P') \cap P$, moreover for every finite set V, if the points of V is in convex position then $C|_V = P(V)$, i.e. V can be shattered.

Definition. *Let $P \subseteq \mathbb{R}^d$ be a finite point set and $\epsilon > 0$ a parameter. A set $N \subseteq \mathbb{R}^d$ is called a weak ϵ-net if for every convex set $C \subseteq \mathbb{R}^d$ when ever $|C \cap P| \geq \epsilon n$ then $C \cap N \neq \emptyset$.*

For example, if we take $\epsilon = \frac{d}{d+1}$, so by the Helly theorem for every finite set $P \subseteq \mathbb{R}^d$ there is a weak ϵ-net of size one.

Theorem 33.1 (Alon, Bárány, Füredi, Kleitman)**.** *For every $\epsilon > 0$ and $d \geq 1$ there is a number $f(d, \epsilon)$ such that for every finite set $P \subset \mathbb{R}^d$ there is a weak ϵ-net N of size $|N| \leq f(d, \epsilon)$.*

Proof. We will show that $f(d, \frac{1}{r}) = O(r^{d+1})$. Let $P \subset \mathbb{R}^d$ be an n-point set. We construct the required weak $1/r$-net N by the following greedy algorithm:

Algorithm 1: Algorithm for constructing a weak $1/r$-net

Input: $P \in \mathbb{R}^d$ a n-point set
Output: A weak $1/r$-net N of size $O(r^{d+1})$
begin
  $i \longleftarrow 0$;
  $N \longleftarrow \emptyset$;
  while *N is not a weak $1/r$-net (i.e. there is a convex set C such that $C \cap N = \emptyset$ and $|C \cap P| \geq n/r$)* **do**
    take a point $x \in c_d\binom{n/r}{d+1} \approx \Omega(n^{d+1}/r^{d+1})$ $C \cap P$-simplicies (we have such a point by the first selection lemma);
    $N \longleftarrow N \cup \{x\}$;
  end
end

The size of the weak $1/r$-net that we obtain from the algorithm is bounded by the number of the iterations that the algorithm does. So lets examine how many iterations we do. We have $\binom{n}{d+1} = O(n^{d+1})$ P-simplicies. In every iteration we "kill" $\Omega(n^{d+1}/r^{d+1})$ such simplicies, so the number of the iterations is bounded by $O(r^{d+1})$. □

It is an open problem to find tight bounds on $f(d, \frac{1}{r})$. It is known that $\Omega(r \log r) = f(2, \frac{1}{r}) = O(r^2)$ and $f(d, \frac{1}{r}) = O(r^d (\log n)^{b(d)})$ with the suitable constant $b(d) > 0$.

Corollary 33.2. *Let μ be a discrete probability measure in $\mathbb{R}^d$. Then there is a weak ϵ-net for convex sets with respect to μ of size at most $f(d, \epsilon)$.*

Corollary 33.3. *Let F be a finite collection of convex sets in $\mathbb{R}^d$. Then $\tau(F) \leq f(d, \frac{1}{\tau^*(F)})$.*

34 (p, q)-Condition

Definition. *Let F be a finite set in $\mathbb{R}^d$. We say that F satisfies the (p, q)-condition when $d + 1 \leq q \leq p$ if for every p sets from F there are q sets with a non-empty intersection.*

Theorem 34.1 (Alon, Kleitman)**.** *For any three integers $d + 1 \leq q \leq p$ there is a constant number $C = C(p, q, d + 1)$ such that for every collection of convex sets F in $\mathbb{R}^d$ that satisfy the (p, q)-condition we have $\tau(F) \leq C$.*

By Helly's theorem we know that $C(d+1, d+1, d+1) = 1$.

Proof. We can assume that $q = d+1$, because if our claim is true for $q = d+1$, so it is true for all q such that $d+1 \leq q \leq p$ (check!).

Lemma 34.2. *Let F be a collection satisfying the $(p, d+1)$ condition. There is a constant $\alpha = \alpha(p, d)$ such that at least $\alpha\binom{n}{d+1}$ of $d+1$ sets from F have a non-empty intersection.*

Proof. We will prove the lemma by double counting. In each collection of p sets there is a collection of $d+1$ sets that have nonempty intersection, and each such a collection of $d+1$ sets is contained in at most $\binom{n-(d+1)}{p-(d+1)}$ collections of p sets. Therefore there are at least $\frac{\binom{n}{p}}{\binom{n-(d+1)}{p-(d+1)}} = \alpha\binom{n}{d+1}$ collections of $d+1$ sets with a nonempty intersection. So, by the Fractional Helly Theorem there is a constant $\beta = \beta(\alpha)$ and a point a contained in at least βn sets from F. □

We already know that $\tau(F) \leq f(d, \frac{1}{\tau^*(F)})$ and because $\nu^*(F) = \tau^*(F)$ it is enough to show that there is a constant bound on $\nu^*(F)$. Let $\psi : F \to [0, 1]$ be an optimal fractional packing, i.e. $\nu^*(F) = \sum_{S \in F} \psi(S)$. We may assume that the values of ψ are rational numbers. Therefore we can write $\psi(S) = \frac{m(S)}{D}$ where $m(S)$ and D are integers and also D is the common denominator of the values of ψ. Notice that $\sum_{S \in F} m(S) = D \sum_{S \in F} \frac{m(S)}{D} = D\nu^*(F)$. Next, we form a multi-set of sets F_m by putting $m(S)$ copies of each S into F_m. Let N be the size of F_m, i.e. $N = |F_m| = \sum_{S \in F} m(S) = D\nu^*(F)$. If we could conclude the existence of a point a lying in at least βN sets of F_m, then

$$1 \geq \sum_{a \in S; S \in F} \psi(S) = \sum_{a \in S; S \in F} \frac{m(s)}{D} = \frac{1}{D} \sum_{a \in S; S \in F} m(S) \geq \frac{1}{D}\beta N = \frac{1}{D}\beta(D\nu^*(F))$$

and so $\nu^*(F) \leq \frac{1}{\beta}$.
The existence of a point a in at least βN sets of F_m follows from the Fractional Helly Theorem. Remember that the family F satisfies the $(p, d+1)$-condition, but the family F_m does not satisfy that condition. However, fortunately, the family F_m satisfy the $(p', d+1)$-condition with $p' = d(p-1)+1$. Indeed, if we choose a collection C of p' sets we have two options: Either there is a set $S \in F$ that appears $d+1$ times in C or there are p distinct sets in C. Now, as before by the Fractional helly theorem for F_m there is $\beta' = \beta(p, d)$

such that $\beta' N$ sets from F_m have nonempty intersection as asserted. This completes the proof of the theorem. □

Summary

Hypergraph Coloring.

35 Hypergraph Coloring

Let $H = (V, E)$ be a hypergraph. A *k-coloring* of the hypergraph H is a function $\varphi : V \to \{1, ..., k\} = [k]$. We call a k-coloring *proper* or *not monochromatic* if for every $S \in E$ such that $|S| \geq 2$ there is $x \neq y \in S$ such that $\varphi(x) \neq \varphi(y)$. A k-coloring φ is called *conflict-free* or *CF-coloring* if for every hyperedge $S \in E$ there is $x \in S$ such that $\varphi(x) \neq \varphi(y)$ for every $y \neq x \in S$. Namely, in every hyperedge there is at least one color that appears uniquely.
Motivation for studying conflict-free coloring arises in several applications. One such application is frequency assignment to cellular antennae. A cellular network consists of several antennae that are broadcasting in some frequencies. We wish that in any given point in the range of the antennae, there will be some antenna with some unique frequency, so we can receive the broadcast at that point without interference from other antennae. The range of an antenna can be modeled as a disc centered at the antenna. Let D denote the set of the discs that represent all the antenna, and let $H = (D, E)$ be a hypergraph where $E = \{D_p | p \in \mathbb{R}\}$ and $D_p = \{d \subset D | p \in d\}$. Hence, we are interested in a conflict-free coloring for the hypergraph H. Another motivation to study conflict-free coloring of hypergraphs is RFID-networks (radio frequency identification tags netowrks).
Let $H = (P, E)$ be a hypergraph where P is an n-point set in $\mathbb{R}^2$ and E is the family of all subsets $P \cap C \subset P$ where C is some disc. A natural question is what is the minimal number $f(n)$ such that every n-point set P can be conflict-free colored with respect to discs in at most $f(n)$ colors.
First lets examine a private case where all the points of P are on x-axis. It easy to see that $f(n) \leq \lfloor \log_2 n \rfloor + 1$. Indeed, we can assign an unique color to the median point p, and recursively assign colors to the two disjoint sets P_l, P_r where P_l is the set of all points to the left of p and p_r is the set

of all points to the right of p. In the recursive procedure we use the same set of colors for P_l and P_r but we keep this colors disjoint from the color of p. In this way it is easy to see that the number of colors $f(n)$ used satisfy the inequality $f(n) \leq 1 + f(n/2)$ which imply the above inequality. On the other hand, we also have that $f(n) \geq \Omega(\log n)$. Indeed, let χ be any valid CF-coloring for P. There is at least one point that has a unique color in the disc containing all points of P. Denote this uniquely colored point by p_1. Either to the left or right of p_1, there are at least $\frac{n}{2}$ points. Denote this set of points by P_1. Since there is a disc containing only P_1 and not p_1, there is a point p_2 which is uniquely colored in P_1 and this color is different than the one assigned to p_1. Repeating this argument, it is easy to see that we will need at least $\log n$ distinct colors.

Theorem 35.1 (Even, Lotker, Ron, Smorodinsky)**.** *For a n-point set in $\mathbb{R}^2$ there is a CF-coloring (w.r.t discs) which uses at most $\log_{4/3} n$ colors.*

Before we show the proof we need a definition of a Delaunay graph.

Definition. *Let $H = (V, E)$ be some hypergraph. The* Delaunay graph *of H is the graph $G(H) = (V, E^2)$ such that $E^2 = \{S \in E \| |S| = 2\}$.*

Proof. Let P be an n-point set. We provide an algorithm for finding a conflict free coloring for P with at most $\log_{4/3} n$ colors:

Algorithm 2: Algorithm for finding a conflict free coloring

```
Input: P ⊂ ℝ² such that |P| < ∞
Output: A conflict free coloring φ : P → {1, ..., log_{4/3} n}
begin
    i ⟵ 0;
    while P ≠ ∅ do
        find P' ⊆ P (try to make P' as large as possible) s.t. P' forms
        an independent set in G(P) (i.e. the Delaunay graph of P);
        ∀x ∈ P' set φ(x) = i;
        P ⟵ P\P';
        i++;
    end
end
```

First we count how many colors we use. Consider iteration i of the algorithm and let n_i be the number of remaining non-colored points after iteration

$i-1$. We claim that at least $\frac{n_i}{4}$ points are colored by i in iteration i. This follows from the fact that the Delaunay graph in our case is a planar graph, so by the Four-Color theorem and the pigeon-hole principle it has an independent set of of size at least $\frac{n_i}{4}$. So, after each iteration we remain with $\frac{3}{4}$ of the points that we began with. Hence, the number of points in iteration i is $n \cdot (\frac{3}{4})^i$. After $\log_{\frac{4}{3}} n$ iterations we will remain with only one point, so the number of the iterations is at most $\log_{\frac{4}{3}} n + 1$ as asserted.
Next, we show that the coloring above is indeed a conflict free coloring. In fact, we will show the stronger property that for every possible disc d, the maximal indexed color in $C \cap P$ appears only once. Assume to the contradiction that there is disc d such that there are at least two points $p, q \in P$ with the maximal color i in d. The points p and q are independent in the Delaunay graph of the point set P_i of iteration i. Therefore, in the disc d, there is at least one more point $r \in d \cap P_i$. Since i is maximal in d then r must be colored i as well. By shrinking arguments, one can show that there is another disc d' such that: (i)$|d' \cap P_i| = 2$ and (ii) $d' \cap P_i \subset d \cap P_i$. This is a contradiction for the fact that the points in $d \cap P_i$ are independent in the Delaunay graph of P_i. □

The running time of the algorithm shown in the proof is $O(\sum_{i=1}^{\log_{\frac{4}{3}} n} n_i \log n_i) = O(\log n \sum_{i=1}^{\log_{\frac{4}{3}} n} n_i) = O(\log n \sum_{i=1}^{\log_{\frac{4}{3}} n} n(\frac{3}{4})^i) = O(n \log n \sum_{i=1}^{\log_{\frac{4}{3}} n} (\frac{3}{4})^i) = O(n \log n)$ where n_i is the number of points that remain after the i'th iteration.

The $O(\log n)$ bound provided above is tight as we know that there exists point set which require $\Omega(\log n)$ colors in any cf-coloring. In fact, Pach and Toth proved that one can not avoid using $\Omega(\log n)$ colors for any n-point set.

Theorem 35.2 (Pach, Toth)**.** *Any n-point set in $\mathbb{R}^2$ require at least $\Omega(\log n)$ colors in any cf-coloring w.r.t discs.*

Summary

Conflict-free coloring of circles.

36 Conflict-free coloring of disks

We are interested in the following problem: let $D = \{d_1, d_2, ..., d_n\}$ be a set of n disks in $\mathbb{R}^2$. Let $H(D) = (D, E)$ be the hypergraph where $E = \{D_p | p \in \mathbb{R}\}$

and $D_p = \{d \in D | p \in d\}$. How many colors suffice for conflict-free coloring (CF-coloring in short) of $H(D)$. We will show that this can be reduced to the following problem: let P be an n-point set in $\mathbb{R}^3$ and let $H(P) = (P, S)$ be a hypergraph where S is the collection of all subsets of points of P that can be cut-off by some negative halfspace. We first study the problem of conflict-free coloring the hypergraph $H(P)$. We provide a transformation from the first problem to the second showing that the number of colors that suffice for the later is an upper bound on the number of colors that suffice for the former problem. The transformation is as follows:

- A point p in the plane is transformed to a plane:

$$p = (a, b) \in \mathbb{R}^2 \rightarrow z = -2ax - 2by + a^2 + b^2$$

- A disk in the plane is transformed to a point in $\mathbb{R}^3$:

  a disk d with center at (x, y) and radius $r > 0 \rightarrow d^* = (x, y, r^2 - x^2 - y^2)$

This transformation has the following properties which preserve incidences:

Lemma 36.1. *Let p be some point and d be some disk. Then*

- *$p \in int(d)$ if and only if the plane p^* is below the point d^*.*

- *The circle ∂d contains the point p if and only if the plane p^* contains the point d^*.*

- *The point p lies outside the disk d if and only if p^* is above the point d^*.*

The above properties are easily verified: We verify the first property: Let $p = (a, b)$ be a point in the interior of the disk d with a center at (x, y) and a radius r. So, $(a - x)^2 + (b - y)^2 < r^2 \Leftrightarrow -2ax - 2by + a^2 + b^2 < r^2 - x^2 - y^2$ i.e., the value of the z coordinate of the point (x, y) in the plane p^* is smaller than the the value of the z coordinate of the point d^*. In other words, the plane p^* is below the point d^*.

hence, if we can color every n-point set in $\mathbb{R}^3$ in such a way that above every plane π, there will be a point with a unique color and the total number of colors will be bounded by, say $O(\log n)$, then, we can also obtain a CF-coloring in $O(\log n)$ colors for $H(D)$.

Theorem 36.2. *Let P be a n-point set in $\mathbb{R}^3$. Let $H = (P, E)$ be a hypergraph as above. Then there is a CF-coloring with $O(\log n)$ colors for H.*

Proof. Let $G(H)$ be the Delaunay graph of the hypergraph H, i.e. $G(H) = (P, E^2)$ where $E^2 = \{\{p, q, \}|p, q \in P$ and there is a lower halfspace h such that $h \cap P = \{p, q\}\}$.

Lemma 36.3. *$G(H)$ is a planar graph.*

proof sketch. We draw the graph $G(H)$ in the plane by projecting it on the plane. Indeed, we show that this projection is planar. Assume to the contrary that the graph is not planar, i.e. there are two segments $\{p', q'\}$ and $\{r', s'\}$ that cross (where p', q', r', s' are the projection of the points p, q, r, s). Let l be the line orthogonal to the xy-plane and passing through the crossing point of the segments $\{p', q'\}$ and $\{r', s'\}$. Let $h_{r,s}$ be a halfspace witnessing the edge (r, s) and let $h_{p,q}$ be the halfspace witnessing the edge (p, q). That is: $h_{p,q} \cap P = \{p, q\}$ and $h_{r,s} \cap P = \{r, s\}$. Let x_1 be the intersection point of l with the plane bounding $h_{r,s}$ and let x_2 be the intersection point of l with the plane bounding $h_{p,q}$. Assume, without loss of generality, that x_1 is above x_2. The points p and q must be above the plane bounding $h_{r,s}$. By convexity, also the segment $\overline{pq}$ must lie above that plane and therefore the point $\overline{pq} \cap l$ is above that plane. Hence the point x_2 must be above the point x_1, a contradiction. This completes the proof. □

We now present an algorithm for CF-coloring a finite set $P \subset \mathbb{R}^3$ with respect to lower halfspaces:

Algorithm 3: Algorithm for CF-coloring points w.r.t halfspaces in $\mathbb{R}^3$

Input: a finite set $P \subset \mathbb{R}^3$
Output: A conflict free coloring $\varphi : P \rightarrow \{1, ..., O(\log n)\}$
begin
  $i \longleftarrow 0$;
  while $P \neq \emptyset$ **do**
    $i + +$;
    P':=a "large" independent set in $G(H)$;
    $\forall x \in P'$ set $\varphi(x) = i$;
    $P \longleftarrow P \backslash P'$;
  end
end

We show that the coloring obtained by the algorithm is conflict-free, namely, we show that for every lower halfspace h, there is a point in $P \cap h$ colored with a unique color. In fact, we show that maximal color that appears in $P \cap h$ is unique. Assume to the contrary that there is a lower halfspace h such that the maximal color that appears in $P \cap h$ is i and there are at least two points $p, q \in P \cap h$ colored in the color i. Let P_i be the set of points that remain after the $i-1$'th iteration of the algorithm. Let G_i be the Delaunay graph defined over the set of points P_i. Since the two points p and q are colored with the same color, namely i, they are independent in $G(H_i)$. Hence there is at least one more point $r \in P_i \cap h$. r is colored with a color greater or equal to i. However, by the maximality of i in h r must be colored with i. One can easily verify that there exists a another halfspace h' such that $h' \cap P \subset h \cap P$ and $|h' \cap P_i| = 2$. This is a contradiction to the fact that the two points in $h' \cap P_i$ are independent in G_i.
By the Four-Color theorem for planar graphs and by the pigeon-hole principle, at every iteration of the above algorithm we color at least $\frac{1}{4}$-fraction of the uncolored points. Thus the total number of iterations (and hence the number of colors) is bounded by $O(\log n)$. □

The bound of $O(\log n)$ on the number of colors needed for CF-coloring n disks in the plane is asymptotically tight. We can place n unit disks such that their centers are all on a line and the distance from the first center to the last center is less than one. One can see that for every interval there is a point contained only in the disks whose centers are in the interval, i.e. we have a hyperedge that contains only those disks. As we saw for the case of coloring points on a line with respect to intervals, the same lower bound $\log n$ will hold verbatim.
The presented algorithm is not good for every hypergraph, for example we can take the hypergraph $H = (V, E)$ where $V = \{1, 2, 3\}$ and $E = \{V\}$, in H the algorithm will color all the verticies in the same color. So, now we will present an improved algorithm (Smorodinsky):

Algorithm 4: A better algorithm for finding a conflict free coloring

Input: Hypergraph $H = (V, E)$
Output: A conflict free coloring φ of H
begin
  $i \longleftarrow 0$;
  while $V \neq \emptyset$ **do**
    $i++$;
    find a proper not-monochromatic coloring $\chi : V \rightarrow \{1, ..., l\}$ (hopefully make l as small as possible) ;
    $V' \leftarrow$ largest color class of χ. $\forall v \in V'$ set $\varphi(v) = i$;
    $V \longleftarrow V \backslash V'$;
  end
end

Proposition 36.4. *Let $S \in E$ be some hyperedge. Let m be the maximal color among the colors that are assigned by the coloring φ to the verticies in S. There is only one vertex $v \in S$ such that $\varphi(v) = m$. In other words, the maximal color that appears in S is unique.*

Proof. Let $S \in E$ be a hyperedge and let i be the maximal color among the colors assigned by the coloring φ to the vertices in S. Let $v \in S$ be the vertex such that $\varphi(v) = i$. Assume to the contrary that there is a another vertex $u \neq v \in S$ such that $\varphi(u) = i$. Let V_i be the set of vertices that remain with after the $i-1$'th iteration, one can easily check that the set V_i is also the set of vertices that are assigned colors $\geq i$. Let $S' = S \cap V_i$. At the i'th iteration we have a non-monochromatic auxiliary coloring $\chi : V_i \rightarrow \{1, ..., l\}$. The vertices u and v are assigned the same color, i.e. $\chi(v) = \chi(u)$. Since χ is a non-monochromatic coloring there is a vertex $w \in S'$ with a different color, i.e. $\chi(w) \neq \chi(v)$. Therefore, the vertex w gets its final color in a later iteration of the algorithm. This is a contradiction to our assumption that i is the maximal color in S. □