SET OF ALL DENSITIES OF EXPONENTIALLY S-NUMBERS

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Abstract. Let **G** be the set of all finite or infinite increasing sequences of positive integers beginning with 1. For a sequence $S = \{s(n)\}, n \geq 1,$ from **G** a positive number N is called an exponentially S-number $(N \in E(S))$, if all exponents in its prime power factorization are in S. The author [2] proved that, for every sequence $S \in \mathbf{G}$, the sequence of exponentially S-numbers has a density $h = h(E(S)) \in [\frac{6}{\pi^2}, 1]$. In this note we study the set $\{h(E(S)\}$ of all such densities.

1. Introduction

Let **G** be the set of all finite or infinite increasing sequences of positive integers beginning with 1. For a sequence $S = \{s(n)\}, n \geq 1$, from **G**, a positive number N is called an exponentially S-number $(N \in E(S))$, if all exponents in its prime power factorization are in S. The author [2] proved that, for every sequence $S \in \mathbf{G}$, the sequence of exponentially S-numbers has a density $h = h(E(S)) \in [\frac{6}{\pi^2}, 1]$. More exactly, the following theorem was proved in [2]:

Theorem 1. *For every sequence $S \in \mathbf{G}$ the sequence of exponentially S-numbers has a density $h = h(E(S))$ such that*

$$\sum_{i \leq x,\ \ i \in E(S)} 1 = h(E(S))x + O(\sqrt{x}\log x e^{c\frac{\sqrt{\log x}}{\log\log x}}), \tag{1}$$

with $c = 4\sqrt{\frac{2.4}{\log 2}} = 7.443083...$ and

$$h(E(S)) = \prod_p \left(1 + \sum_{i \geq 2} \frac{u(i) - u(i-1)}{p^i}\right), \tag{2}$$

where $u(n)$ is the characteristic function of sequence $S:\ \ u(n) = 1$, if $n \in S$ and $u(n) = 0$ otherwise.

Note that Sloane's Online Encyclopedia of Integer Sequences [3] contains some sequences of exponentially S-numbers, $S \in \mathbf{G}$. For example, A005117 $(S = \{1\})$, A004709 $(S = \{1, 2\})$, A138302 $(S = \{2^n\}|_{n \geq 0})$, A197680 $(S = \{n^2\}|_{n \geq 1})$, A209061 $(S = A005117)$, etc.

Everywhere below we write $\{h(E(S))\}$, understanding $\{h(E(S))\}|_{S \in \mathbf{G}}$. In [2] (Section 6) the author posed the question: is the set $\{h(E(S))\}$ dense

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in the interval $[\frac{6}{\pi^2}, 1]$? Berend [1] gave a negative answer by finding a gap in the set $\{h(E(S))\}$ in the interval

$$\left(\prod_p(1-\frac{p-1}{p^3}),\ \prod_p(1-\frac{1}{p^3})\right) \subset [\frac{6}{\pi^2}, 1]. \tag{3}$$

Berend's idea consists of the partition of $\mathbf{G}$ into two subsets - of those sequences which contain 2 and those that do not contain 2 - and applying Theorem 1. In our study of the set $\{h(E(S)\}$ we use this idea.

2. Cardinality

Lemma 1. $\mathbf{G}$ *is uncountable.*

Proof. Trivially $\mathbf{G}$ is equivalent to the set of all subsets of $\{2, 3, 4, ...\}$. □

Lemma 2. *For every two distinct* $A, B \in \mathbf{G}$, *we have* $h(E(A)) \neq h(E(B))$.

Proof. Let $A = \{a(i)\}|_{i\geq 1},\ B = \{b(i)\}|_{i\geq 1}$. Let $n \geq 1$ be maximal index such that $a(i) = b(i),\ i = 1, ..., n$, while $a(n+1) \neq b(n+1)$. Note that, if $A_n = \{a(1), ..., a(n)\},\ A^* = \{a(1), ..., a(n), a(n+1), a(n+1)+1, a(n+1)+2, ...\}$, then

$$h(E(A_{n+1})) \leq h(E(A)) \leq h(E(A^*)) \tag{4}$$

and analogously for sequence B.
Distinguish four cases:

$$(i)\ \ a(n+1) = a(n)+1,\ \ b(n+1) \geq a(n)+2;$$

$$(ii)\ \ for\ \ k \geq 2,\ \ a(n+1) \geq a(n)+k,\ \ b(n+1) = a(n)+1;$$

$$(iii)\ \ for\ \ k \geq 3,\ \ a(n+1) = a(n)+k,\ \ a(n)+2 \leq b(n+1) \leq a(n)+k-1;$$

$$(iv)\ \ for\ \ k \geq 2,\ \ a(n+1) = a(n)+k,\ \ b(n+1) \geq a(n)+k+1.$$

(i) By (2) and (4), we have

$$h(E(A)) \geq \prod_p \left(1 + \sum_{i=2}^{a(n)} \frac{u(i)-u(i-1)}{p^i}\right), \tag{5}$$

where $u(n)$ is the characteristic function of A. Since here $u(a(n+1)) - u(a(n+1)-1) = 0$, then in the right hand side we sum up to $a(n)$. On the other hand,

$$h(E(B^*)) \leq \prod_p \left(1 + \sum_{i=2}^{a(n)} \frac{u(i)-u(i-1)}{p^i} - \frac{1}{p^{a(n)+1}} + \frac{1}{p^{a(n)+2}}\right). \tag{6}$$

By (5)-(6), $h(E(B)) < h(E(A))$.

(ii) Symmetrically to (i), we have

$$h(E(B)) \geq \prod_p \left(1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i-1)}{p^i}\right). \tag{7}$$

On the other hand,

$$h(E(A^*)) \leq \prod_p \left(1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{a(n)+1}} + \frac{1}{p^{a(n)+2}}\right). \tag{8}$$

So, $h(E(A)) < h(E(B))$.
(iii) Again, by (2) and (4), we have

$$h(E(B)) \geq \prod_p \left(1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{a(n)+1}} + \frac{1}{p^{a(n)+k-1}}\right), \tag{9}$$

while

$$h(E(A^*)) \leq \prod_p \left(1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{a(n)+1}} + \frac{1}{p^{a(n)+k}}\right). \tag{10}$$

Hence, $h(E(A)) < h(E(B))$.
(iv) Symmetrically,

$$h(E(B^*)) \leq \prod_p \left(1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{a(n)+1}} + \frac{1}{p^{a(n)+k+1}}\right), \tag{11}$$

while

$$h(E(A)) \geq \prod_p \left(1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{a(n)+1}} + \frac{1}{p^{a(n)+k}} - \frac{1}{p^{a(n)+k+1}}\right) \tag{12}$$

and since $\frac{2}{p^{a(n)+k+1}} \leq \frac{1}{p^{a(n)+k}}$, where the equality holds only in case $p = 2$, then $h(E(A)) > h(E(B))$. □

Lemmas 1 and 2 directly imply

Theorem 2. *The set* $\{h(E(S)\}|_{S \in \mathbf{G}}$ *is uncountable.*

Denote by $\mathbf{G}(F)$ the subset of the finite sequences from $\mathbf{G}$. Since the set of all finite subsets of a countable set is countable, then $\mathbf{G}(F)$ is countable and then the set $\{h(E(S)\}|_{S \in \mathbf{G}(F)}$ is also countable.

3. Perfectness

Lemma 3. *Every point of the set* $h(E(S))$ *is an accumulation point.*

Proof. Distinguish two cases: a) S is finite set; b) S is infinite set.

a) Let $S=\{s(1),...,s(k)\}\in \mathbf{G}(F)$. Let $n\geq s(k)+2$. Denote by S_n the sequence $S_n=\{s(1),...,s(k),n\}$. Then, by (2),

$$h(E(S_n))-h(E(S))= \tag{13}$$
$$\prod_p\left(1+\sum_{i=2}^{s(k)}\frac{u(i)-u(i-1)}{p^i}-\frac{1}{p^{s(k)+1}}+\frac{1}{p^n}\right)-$$
$$\prod_p\left(1+\sum_{i=2}^{s(k)}\frac{u(i)-u(i-1)}{p^i}-\frac{1}{p^{s(k)+1}}\right).$$

For the first product $\prod_p(n)$,

$$\prod_p(n)=\exp\left(\sum_p\log\left(1+\sum_{i=2}^{s(k)}\frac{u(i)-u(i-1)}{p^i}-\frac{1}{p^{s(k)+1}}+\frac{1}{p^n}\right)\right),$$

the series over primes converges uniformly since

$$\sum_p\sum_{i\geq 2}\frac{|u(i)-u(i-1)|}{p^i}\leq\sum_p\sum_{i\geq 2}\frac{1}{p^i}=\sum_p\frac{1}{(p-1)p}.$$

Therefore, $\lim_{n\to\infty}(\prod_p(n))=\prod_p(\lim_{n\to\infty}(...))$ which coincides with the second product. So $\lim_{n\to\infty}h(E(S_n))=h(E(S))$.
b) Let $S=\{s(1),...,s(k),...\}\in\mathbf{G}$ be infinite sequence. Let $S_n=\{s(1),...,s(n)\}$ be the n-partial sequence of S. In the same way, taking into account the uniform convergence of the product for density of S_n, we find that $\lim_{n\to\infty}h(E(S_n))=h(E(S))$. □

Theorem 3. *The set $\{h(E(S))\}$ is a perfect set.*

A proof we give in Section 5.

4. Gaps

Let us show that, for every finite $S\in\mathbf{G}$, with the exception of $S=\{1\}$, there exists an $\varepsilon>0$ such that the image of h is disjoint from the interval $(h(E(S))-\varepsilon, h(E(S)))$.

We need a lemma.

Lemma 4. *Let $A,B\in\mathbf{G}$ be distinct sequences. Let $s^*=s^*(A,B)$ be the smallest number which is a term of one of them, but not in another. If, say, $s^*\in A$, then $h(E(A))>h(E(B))$.*

Proof. In fact, the lemma is a corollary of the proof of Lemma 2. Comparing with the proof of Lemma 2, we have $s^*(A,B)=n+1$. We see that in all four cases in the proof of Lemma 2, the statement of Lemma 4 is confirmed. □

Proposition 1. *Let* $S_1 = \{s(1), ..., s(k)\} \in \mathbf{G}(F),\ k \geq 2,$ *and* $S_2 = \{s(1), ..., s(k-1), s(k)+1, s(k)+2, ...\}$. *Then the interval*

$$(h(E(S_2)),\ \ h(E(S_1))) \tag{14}$$

is a gap in the set $\{h(E(S)) : S \in \mathbf{G}\}$.

Proof. Consider other than S_1, S_2 any sequence $S \in \mathbf{G}$ which contains $s^*(S_1, S)$. By Lemma 4, $h(E(S)) > h(E(S_1))$. So, $h(E(S))$ is not in interval (14). Now consider other than S_1, S_2 any sequence $S \in \mathbf{G}$ which does not contain $s^*(S_1, S)$. Then S_2 contains $s^*(S, S_2)$. Indeed, 1) S cannot contain all terms $s(1), ..., s(k)$ (since S differs from S_1, it should contain additional terms, the smallest of which is $s^*(S, S_1) \in S$ that contradicts the condition); 2) if $i,\ 1 \leq i \leq k$, is the smallest for which S misses $s(i)$, then, by the condition, all terms of S are more than than $s(i)$. So $s^*(S, S_2) = s(i) \in S_2$, if $i < k$, while, if i=k, since S differs from S_2, $s^*(S, S_2) = s(k) + j \in S_2$, where j is the smallest for which $s_k + j$ is not in S. Hence, by Lemma 4, $h(E(S_2)) > h(E(S))$ and again $h(E(S))$ is not in interval (14). □

Lemma 5. *Every gap in* $\{h(E(S))\}$ *has the form described in Proposition* 1.

Proof. Indeed, the gap (14) is in a right neighborhood of $h(E(S_2))$. Let a sequence $S \in \mathbf{G}$ do not contain any infinite set of positive integers K. Adding to S $k \in K$, which goes to infinity, we obtain set S_k such that $h(E(S_k)) > h(E(S))$ and $h(E(S_k)) \to h(E(S))$. So, in a right neighborhood of $h(E(S))$ cannot be a gap of $\{h(E(S))\}$. In opposite case, when $S \in \mathbf{G}$ does not contain only a finite set of positive integers, in a right neighborhood of $h(E(S))$ a gap of $\{h(E(S))\}$ is possible, but in this case S has the form of S_2 in Proposition 1. Also, if $S \in \mathbf{G}$ is infinite, then in a left neighborhood of $h(E(S))$ cannot be a gap of $\{h(E(S))\}$, since $h(E(S))$ is a limiting point of $\{h(E(S_n))\}$, where S_n is the n-partial sequence of S. □

It is easy to see that, for distinct sequences S_1, the gaps (14) are disjoint. From Propositions 1 and Lemma 5 we have the statement:

Theorem 4. *The set* $\{h(E(S))\}$ *has countably many gaps.*

5. Proof of Theorem 3

Proof. By Lemma 3, the set $\{h(E(S))\}$ does not contain isolated points. For a set $A \subseteq [\frac{6}{\pi^2}, 1]$, let $\overline{A}$ be $[\frac{6}{\pi^2}, 1] \backslash A$. Let, further, $\{g\}$ be the set of all gaps of $\{h(E(S))\}$. Then we have

$$\{h(E(S))\} = \overline{\bigcup g} = \bigcap \overline{g}.$$

Since a gap g is an open interval, then $\overline{g}$ is a closed set. But arbitrary intersections of closed sets are closed. Thus the set $\{h(E(S))\}$ is closed without isolated points. So it is a perfect set. □

6. Conclusion

Thus, by Theorems 2-4, the set $\{h(E(S))\}$ is a perfect set with a countable set of gaps which associate with some left-sided neighborhoods of the densities of all exponentially finite S-sequences, $S \in \mathbf{G}$, except for $S = \{1\}$. It is natural to conjecture that the sum of lengths of all gaps equals the length of the whole interval $[\frac{6}{\pi^2}, 1]$, or, the same, that the sequence $\{h(E(S))\}$ has zero measure. This important question we remain open.

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