A FAST COMPUTATION OF DENSITY OF EXPONENTIALLY S-NUMBERS

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Abstract. The author [4] proved that, for every set S of positive integers containing 1 (finite or infinite) there exists the density $h = h(E(S))$ of the set $E(S)$ of numbers whose prime factorizations contain exponents only from S, and gave an explicit formula for $h(E(S))$. In this paper we give an equivalent polynomial formula for $\log h(E(S))$ which allows to get a fast calculation of $h(E(S))$.

1. Introduction

Let $\mathbf{G}$ be the set of all finite or infinite increasing sequences of positive integers beginning with 1. For a sequence $S = \{s(n)\}, n \geq 1$, from $\mathbf{G}$, a positive number N is called an exponentially S-number ($N \in E(S)$), if all exponents in its prime power factorization are in S. The author [4] proved that, for every sequence $S \in \mathbf{G}$, the sequence of exponentially S-numbers has a density $h = h(E(S)) \in [\frac{6}{\pi^2}, 1]$. More exactly, the following theorem was proved in [4]:

Theorem 1. *For every sequence $S \in \mathbf{G}$ the sequence of exponentially S-numbers has a density $h = h(E(S))$ such that*

$$\sum_{i \leq x,\ i \in E(S)} 1 = h(E(S))x + O(\sqrt{x} \log x e^{c\frac{\sqrt{\log x}}{\log\log x}}), \tag{1}$$

with $c = 4\sqrt{\frac{2.4}{\log 2}} = 7.443083...$ *and*

$$h(E(S)) = \prod_p \left(1 + \sum_{i \geq 2} \frac{u(i) - u(i-1)}{p^i}\right), \tag{2}$$

where the product is over all primes, $u(n)$ is the characteristic function of sequence S: $u(n) = 1$, if $n \in S$ and $u(n) = 0$ otherwise.

In case when S is the sequence of square-free numbers (see Toth [6]) Arias de Reyna [5,A262276], using the Wrench method of fast calculation [7], did the calculation of h with a very high degree of accuracy. In this paper, using Wrench's method for formula (2), we find a general representation of $h(E(S))$ based on a special polynomial over partitions of n which allows to get a fast calculation of $h(E(S))$ for every $S \in \mathbf{G}$. Note also that Wrench's

1991 *Mathematics Subject Classification.* 11A41.

method was successfully realized in a special case by Arias de Reyna, Brent and van de Lune in [2].

Everywhere below we write $\{h(E(S))\}$, understanding $\{h(E(S))\}|_{S\in\mathbf{G}}$.

2. A computing idea in Wrench's style

Consider function given by power series

$$F_S(x) = 1 + \sum_{i\geq 2}(u(i) - u(i-1))x^i, \ \ x \in (0, \frac{1}{2}]. \tag{3}$$

Since $u(n) - u(n-1) \geq -1$, then $F_S(x) \geq 1 - \frac{x^2}{1-x} > 0$. By (2), we have

$$h(E(S)) = \prod_p F_S\left(\frac{1}{p}\right). \tag{4}$$

and

$$\log h(E(S)) = \sum_p \log F_S(x)|_{x=\frac{1}{p}}. \tag{5}$$

Let

$$\log F_S(x) = \sum_{i\geq 2} \frac{f_i^{(S)}}{i} x^i. \tag{6}$$

Since $|u(n) - u(n-1)| \leq 1$, then by (3), $F_S(x) \leq 1 + \frac{x^2}{1-x}$ and $0 < \log F_S(x) \leq 2x^2, \ x \in (0, \frac{1}{2}]$. Thus the series (5) is absolutely convergent. Now, according to (5) - (6), we have

$$\log h(E(S)) = \sum_{n=2}^{\infty} \frac{f_n^{(S)}}{n} P(n), \tag{7}$$

where $P(n) = \sum_p \frac{1}{p^n}$ is the prime zeta function. The series (7) is fast convergent and very suitable for the calculation of $h(E(S))$.

3. A recursion for coefficients

Denoting

$$v_n = u(n) - u(n-1), \ \ n \geq 2, \tag{8}$$

by (3) and (6), we have

$$F_S(x) = 1 + \sum_{n\geq 2} v_n x^n, \tag{9}$$

$$\log(1 + \sum_{n\geq 2} v_n x^n) = \sum_{i\geq 2} \frac{f_i^{(S)}}{i} x^i. \tag{10}$$

Lemma 1. *Coefficients $\{f_n^{(S)}\}$ satisfy the recurrence*

(11) $$f_{n+1}^{(S)} = (n+1)v_{n+1} - \sum_{i=1}^{n-2} v_{n-i} f_{i+1}^{(S)}, \ \ n \geq 1.$$

Proof. Differentiating (10), we have

$$\frac{\sum_{n\geq 2} n v_n x^{n-1}}{F_S(x)} = \sum_{j>=1} f_{j+1}^{(S)} x^j.$$

Hence,

$$\sum_{n\geq 2} n v_n x^{n-1} = (1 + \sum_{n\geq 2} v_n x^n)(\sum_{j>=1} f_{j+1}^{(S)} x^j).$$

Equating the coefficients of x^n in both sides, we get

$$(n+1)v_{n+1} = f_{n+1}^{(S)} + \sum_{j=1}^{n-2} v_{n-j} f_{j+1}^{(S)}$$

and the lemma follows. □

Corollary 1. *All* $\{f_n^{(S)}\}$ *are integers.*

Proof. For n=1,2,3, by the recurrence (11), we have

$$f_2^{(S)} = 2v_2, f_3^{(S)} = 3v_3, f_4^{(S)} = 4v_4 - 2v_2^2;$$

now the corollary follows by induction. □

4. Explicit polynomial formula

To apply (10) we need a fast way to generate the coefficients $f_i^{(S)}$. Since, for $x \in (0, \frac{1}{2}]$, $\sum_{n\geq 2} v_n x^n \leq \frac{x^2}{1-x} \leq \frac{1}{2}$, then

(12) $$\log(1 + \sum_{n\geq 2} v_n x^n) = \sum_{m\geq 1} \frac{(-1)^{m-1}}{m} (\sum_{n\geq 2} v_n x^n)^m.$$

Expanding these powers, we get a great sum of terms of type

(13) $$t_{\lambda_1, s_1} (v_{\lambda_1} x^{\lambda_1})^{s_1} ... t_{\lambda_r, s_r} (v_{\lambda_r} x^{\lambda_r})^{s_r}, \ \ s_i \geq 1, \lambda_i \geq 2.$$

When we collect all the terms with a fixed sum of exponents of x, say, n, we get a sum of terms (13) with $\lambda_1 s_1 + ... + \lambda_r s_r = n$, i.e., we have s_i parts λ_i in partition of n. Therefore, the considered expansion has the form

$$\log(1 + \sum_{n\geq 2} v_n x^n) = \sum_{n\geq 2} (\sum_{\sigma \in \Sigma_n} t_\sigma v_\sigma) \frac{x^n}{n} = \sum_{n\geq 2} \frac{f_n^{(S)}}{n} x^n,$$

where Σ_n is the set of the partitions $\{\sigma\}$ of n with parts $\lambda_i \geq 2$ and t_σ, v_σ are functions of partitions σ defined by (13) such that with every partition σ of n we associate the monomial

$$v_\sigma = \prod_{i=1}^{r} v_{\lambda_i}^{s_i} \ \ (\lambda_1 s_1 + ... + \lambda_r s_r = n, \ \ \lambda_i \geq 2). \tag{14}$$

So

$$f_n^{(S)} = \sum_{\sigma \in \Sigma_n} t_\sigma v_\sigma. \tag{15}$$

Substituting (15) in equation (11), we get

$$\sum_{\sigma \in \Sigma_{n+1}} t_\sigma v_\sigma = (n+1)v_{n+1} - \sum_{i=1}^{n-2} v_{n-i} \sum_{\sigma \in \Sigma_{i+1}} t_\sigma v_\sigma =$$

$$(n+1)v_{n+1} - \sum_{j=2}^{n-1} v_j \sum_{\sigma \in \Sigma_{n+1-j}} t_\sigma v_\sigma. \tag{16}$$

Note that, using (16), one can proved that all coefficients t_σ are integer numbers. Let partition $\sigma = (b_2, ..., b_{n+1}) \in \Sigma_{n+1}$ contains b_2 elements 2, ..., b_{n+1} elements $n+1$ such that $2b_2 + ... + (n+1)b_{n+1} = n+1, \ \ b_i \geq 0$. In particular, evidently, $b_{n+1} = 0$ *or* 1 and in the latter case all other $b_i = 0$. We shall write $v_\sigma = v_2^{b_2} ... v_{n+1}^{b_{n+1}}$ and $t_\sigma = t(v_2^{b_2} ... v_{n+1}^{b_{n+1}})$. According to (16), the coefficient of the monomial $v_2^0 ... v_n^0 v_{n+1}^1$ equals $n+1$, i. e., for partition of $n+1$ with only part we have $t(\sigma) = n+1$. We agree that $0^0 = 1$.

Denote by Σ'_{n+1} the set of partitions of $n+1$ with parts ≥ 2 and $\leq n$. Then, by (16), we have

$$\sum_{\sigma \in \Sigma'_{n+1}} t_\sigma v_\sigma = -\sum_{j=2}^{n-1} v_j \sum_{\sigma \in \Sigma'_{n+1-j}} t_\sigma v_\sigma. \tag{17}$$

For every partition $(b_2, ..., b_{n+1}) \in \Sigma'_{n+1}$ we have $b_{n+1} = 0$ and $b_n = 0$ (the latter since all parts ≥ 2). Then (17) leads to the formula:

$$t(v_2^{b_2} ... v_{n-1}^{b_{n-1}} v_n^0 v_{n+1}^0) = -t(v_2^{b_2-1} v_3^{b_3} ... v_{n-1}^{b_{n-1}} v_n^0 v_{n+1}^0) -$$

$$t(v_2^{b_2} v_3^{b_3-1} ... v_{n-1}^{b_{n-1}} v_n^0 v_{n+1}^0) - ... - t(v_2^{b_2} v_3^{b_3} ... v_{n-1}^{b_{n-1}-1} v_n^0 v_{n+1}^0). \tag{18}$$

Using (18), we find an explicit formula for $f_n^{(S)}$.

Lemma 2. *Let, for* $n \geq 3$, $(b_2, ..., b_{n-1}, 0, 0) \in \Sigma'_{n+1}$. *Then*

$$t(v_2^{b_2} ... v_{n-1}^{b_{n-1}} v_n^0 v_{n+1}^0) = (-1)^{B_{n-1}-1} \frac{(B_{n-1}-1)!}{b_2! ... b_{n-1}!}(n+1), \tag{19}$$

where $B_{n-1} = b_2 + ... + b_{n-1}$.

Proof. Let $n = 3$. We saw that $f_4^{(S)} = 4v_4 - 2v_2^2$. So, $t(v_2^{b_2}) = -2$ with

$b_2 = 2$ and, by (19), we also obtain $t(v_2^{b_2}) = -2$. Let the lemma holds for $t(v_2^{c_2}...v_{n-1}^{c_{n-1}})$, $n \geq 3$, where all $c_i \leq b_i$ such that not all equalities hold. Then, by the relaion (18) and the induction supposition, we have

$$t(v_2^{b_2}...v_{n-1}^{b_{n-1}}) = -(-1)^{B_{n-1}-2}(\frac{(B_{n-1}-2)!}{(b_2-1)!b_3!...b_{n-1}!}(n+1-2)+$$

$$\frac{(B_{n-1}-2)!}{b_2!(b_3-1)!...b_{n-1}!}(n+1-3)+...+\frac{(B_{n-1}-1)!}{b_2!b_3!...(b_{n-1}-1)!}(n+1-(n-1)) =$$

$$(-1)^{B_{n-1}-1}\frac{(B_{n-1}-2)!}{b_2!...b_{n-1}!}(b_2(n+1-2)+b_3(n+1-3)+...+$$

$$b_{n-1}(n+1-(n-1)) = (-1)^{B_{n-1}-1}\frac{(B_{n-1}-2)!}{b_2!...b_{n-1}!}(B_{n-1}(n+1)-$$

$$(2b_2+3b_3+...+(n-1)b_{n-1})$$

and, since $2b_2 + 3b_3 + ... + (n-1)b_{n-1} = n+1$, the lemma follows. □

Corollary 2. *Let, for $n \geq 3$, $(b_2, ..., b_{n+1}) \in \Sigma_{n+1}$. Then*

$$(20) \qquad t(v_2^{b_2}...v_{n+1}^{b_{n+1}}) = (\delta(b_{n+1,1}) + (-1)^{B_{n-1}-1}\frac{(B_{n-1}-1)!}{b_2!...b_{n-1}!})(n+1),$$

where $B_{n+1} = b_2 + ... + b_{n-1}$.

Proof. The statement follows from Lemma 2 and addition of the coefficient $n+1$ of v_{n+1} in equation (16) in case when $\delta(b_{n+1,1}) = 1$. □

Now, using (7), (15), Corollary 2 and the initial values of the coefficients $f_2^{(S)} = 2v_2$, $f_3^{(S)} = 3v_3$, and changing n by $n-1$, we get a suitable formula to compute $\log h(E(S))$.

Theorem 2. *We have*

$$(21) \qquad \log h(E(S)) = P(2)v_2 + P(3)v_3 + \sum_{n=4}^{\infty} P(n)(v_n + M(v_2, ..., v_{n-2})),$$

where $P(n)$ is the prime zeta function, M is the polynomial defined as

$$M(v_2, ..., v_{n-2}) = \sum_{2b_2+...+(n-2)b_{n-2}=n} (-1)^{B_{n-2}-1}\frac{(B_{n-2}-1)!}{b_2!...b_{n-2}!}v_2^{b_2}...v_{n-2}^{b_{n-2}},$$

where $B_{n-2} = b_2 + ... + b_{n-2}$, $b_i \geq 0$, $i = 2, ..., n-2$, $n \geq 4$.

In particular, for $n = 4, 5, 6, ...$, we have

$$M(v_2) = -\frac{v_2^2}{2}, M(v_2, v_3) = -v_2v_3, M(v_2, v_3, v_4) = -v_2v_4 - \frac{v_3^2}{2} + \frac{v_2^3}{3}, \ ...$$

For example, in case $n = 6$ the diophantine equation $2b_2 + 3b_3 + 4b_4 = 6$ has 3 solutions

a) $b_2 = 1, b_3 = 0, b_4 = 1$ with $B_4 = 2$;
b) $b_2 = 0, b_3 = 2, b_4 = 0$ with $B_4 = 2$;
c) $b_2 = 3, b_3 = 0, b_4 = 0$ with $B_4 = 3$.

Besides, using (11), for $M_n = M_n(v_2, ..., v_{n-2})$ we have the recursion

$$(22)\qquad M_2 = 0, M_3 = 0, M_n = -\frac{1}{n}\sum_{j=2}^{n-2} jv_{n-j}(v_j + M_j),\ \ n \geq 4$$

which, possibly, more suitable for fast calculations by Theorem 2.

5. Examples

1) As we already mentioned, in case when S is the sequence of square-free numbers, Arias de Reyna [5,A262276] obtained

$$h = \prod_p \left(1 + \sum_{i\geq 4} \frac{\mu(i)^2 - \mu(i-1)^2}{p^i}\right) = 0.955923015861902376 88...$$

By the results of [1], the coefficients $f_n^{(S)}$ (15) in this case (see A262400 [5]) have very interesting congruence properties.

2) The case of $S = 2^n$ was essentially considered by the author [3]. He found that $h = 0.872497...$ The author asked Arias de Reyna to get more digits. Using Theorem 2, he obtained

$$h = 0.87249717935391281355...$$

3) Among the other several calculations by Arias de Reyna, we give the following one. Let S be 1 and the primes (A008578 [5]). Then

$$h = 0.94671933735527801046...$$

6. Acknowledgement

The author is very grateful to Juan Arias de Reyna for an information of Wrench's method, useful discussions and his calculations by the formula of Theorem 2.

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