ON ERDŐS CONSTANT

VLADIMIR SHEVELEV

Abstract. In 1944, P. Erdős [1] proved that if n is a large highly composite number (HCN) and n_1 is the next HCN, then

$$n < n_1 < n + n(\log n)^{-c},$$

where $c > 0$ is a constant. In this paper, using numerical results by D. A. Corneth, we show that most likely $c < 1$.

1. Introduction

In 1915, Ramanujan [2] introduced *highly composite numbers* (HCN) as positive integers n such that $d(m) < d(n)$ for all $m < n$. (cf. A002182[3]). At the same time, he introduced a more wide sequence of *largely composite numbers* (LCN's) as positive integers n such that $d(m) \leq d(n)$ for all $m < n$. (cf. A067128[3]) In 1944, P. Erdős [1], strengthening inequality of Ramanujan, proved that if n is a large HCN and n_1 is the next HCN, then

$$n < n_1 < n + n(\log n)^{-c}, \tag{1}$$

where $c > 0$ is a constant. This result is equivalent to the following: the number of HCN's<= x is greater than $(\log x)^{1+c}$. At the beginning of the article he writes: "At present I cannot decide whether the number of HCN's not exceeding x is greater than $(\log x)^{\kappa}$ for every κ." In this paper, using numerical results by D. A. Cormeth, we show that most likely $c < 1$, or, the same, in the cited Erdős' question, only $\kappa < 2$.

2. Sequence A273379 [3]

Erdos [1] noted that every HCN is divisible by every prime less than its greatest prime divisor p. The author with P. J. C. Moses considered the sequence "LCN's n which are not divisible by all the primes $< p$, where p is the greatest prime divisor of n." The first few numbers of this sequence are

$$3, 10, 20, 84, 168, 336, 504, 660, 672, 3960, 4680, 32760, 42840, ... \tag{2}$$

Consider prime power factorization of these terms ≥ 10 :

$$2 \cdot 5,\ 2^2 \cdot 5,\ 2^2 \cdot 3 \cdot 7,\ 2^3 \cdot 3 \cdot 7,\ 2^4 \cdot 3 \cdot 7,\ 2^3 \cdot 3^2 \cdot 7,\ 2^2 \cdot 3 \cdot 5 \cdot 11,\ 2^5 \cdot 3 \cdot 7,$$
$$2^3 \cdot 3^2 \cdot 5 \cdot 11,\ 2^3 \cdot 3^2 \cdot 5 \cdot 13,\ 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13,\ 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 17, ...$$

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Note that among the first 12 terms ≥ 10, in ten terms there is missed only one prime between the greatest prime divisor and the second greatest prime divisor, while in two terms there missed two primes. D. A. Corneth (A273415[3]) found the smallest LCN's with i missed primes between the greatest prime divisor and the second greatest prime divisor, such that smaller primes all divide the terms, $i = 1, 2, ..., 10$:

$$10, 4680, 6585701522400, 193394747145600, 27377180785991836800,$$

$$29378941900252048776672000, 5384823686347760468943298225056000,$$

$$404593694258692410380118300618528000,$$

$$1714431214566179268370439406441900195214656000,$$

$$180656647480221782329653424360823828484237888000. \tag{3}$$

He asks, whether this sequence is infinite?

3. Infiniteness of sequence (3)

Let n be a large HCN. Let its greatest prime divisor be the k-th prime number p_k, $k = k(n)$. It is known [1], that $p_k || n$.

Lemma 1. [1]

$$c_1 \log n < p_k < c_2 \log n. \tag{4}$$

By Lemma 1 and prime number theorem, we have

$$k = O(\log n / \log \log n). \tag{5}$$

Lemma 2. *Let n_1 be the next HCN after n. Then all numbers in the interval $[n, n_1)$ of the form*

$$np_{k+1}/p_k, ..., np_{k+r}/p_k, \tag{6}$$

if they exist, are LCN's.

Proof. All numbers (6) have the same number of divisors as n, and between them there is no any HCN, since the smallest $HCN > n$ is n_1 which is larger every number (6). □

Lemma 3. *Let N be a term of sequence (3) with r missed primes between the greatest prime divisor p_{k+r} and the second greatest prime divisor p_{k-1} of N. Then together with N all numbers*

$$Np_{k+r-1}/p_{k+r}, Np_{k+r-2}/p_{k+r}, ..., Np_k/p_{k+r}$$

are LCN's (but not HCN's, except for the last number).

Proof. Indeed, in the opposite case, between these numbers there is a HCN $\leq N$, but since all they, including N, have the same number of divisors, it would contradict the condition, that N is LCN. $\square$

Lemma 3 means that every number of sequence (3) is building from an HCN (which always has not any gap between its prime divisors $2, ..., p_k$) by consecutive multiplication by $p_{k+1}/p_k, p_{k+2}/p_{k+1}, ..., p_{k+r}/p_{k+r-1}$ with the possible maximal r. By (1) and (6),

$$n < np_{k+r}/p_k < n(1 + (\log n)^{-c}).$$

In order to have a real chance to obtain the numbers (3), let require a stronger inequality

$$p_{k+r}/p_k < 1 + \frac{1}{2}(\log n)^{-c}. \tag{7}$$

Here $\frac{1}{2}$ could be changed by any smaller positive constant. Note that (7) means that

$$(1 + o(1))(k + r)\log(k + r)/(k \log k) < 1 + \frac{1}{2}(\log n)^{-c}, \tag{8}$$

where, by (5), $k = O(\log n/ \log\log n)$. Since, for $r < k$, $\log(k + r) = \log k + \log(1 + r/k) = \log k + r/k + O((r/k)^2)$, then $\log(k + r)/ \log k = 1 + O(r/(k \log k)$. So, by (8), we see that (7) yields

$$1 + r/k + O(r/(k \log k)) < 1 + \frac{1}{2}(\log n)^{-c},$$

or

$$r \leq \frac{k}{2(\log n)^c} = O((\log n)^{1-c}/ \log\log n).$$

Thus if and only if $c < 1$, the value of r could be arbitrary large for sufficiently large n. Moreover, the existence the numbers (3) allows to conjecture that, indeed, $0 < c < 1$.

References

[1] P. Erdos, On Highly composite numbers, J. London Math. Soc. 19 (1944), 130–133 MR7,145d; Zentralblatt 61,79.
[2] S. Ramanujan, Highly composite numbers, Proceedings of the London Mathematical Society, 2, XIV, 1915, 347 - 409.
[3] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences* http://oeis.org.

Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel. e-mail:shevelev@bgu.ac.il