ALICE AND BOB GO TO DINNER: A VARIATION ON *MÉNAGE*

Vladimir Shevelev

Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva, Israel
shevelev@bgu.ac.il

Peter J. C. Moses

Moparmatic Company, Astwood Bank, Nr. Redditch, Worcestershire, England
mows@mopar.freeserve.co.uk

Abstract

We give a solution to a variation of the classic *ménage* problem where a fixed couple — Alice and Bob — are already seated.

1. Introduction

In 1891, Lucas [2] formulated the following "*ménage* problem":

Problem 1. To find the number M_n of ways of seating $n \geq 2$ male-female couples at a circular table, ladies and gentlemen in alternate positions, so that no husband sits next to his wife.

After seating the ladies in $2n!$ ways we have

$$M_n = 2n!U_n, \tag{1}$$

where U_n is the number of ways of seating the gentlemen.

Earlier, Muir [4] solved a problem posed by Tait (cf. [4]): to find the number H_n of permutations π of $\{1, ..., n\}$ for which $\pi(i) \neq i$ and $\pi(i) \neq i + 1 \pmod{n}$, $i = 1, ..., n$. Simplifying Muir's solution, Cayley [1] found a recursion for H_n : $H_2 = 0, H_3 = 1$, and for $n \geq 4$,

$$(n-2)H_n = n(n-2)H_{n-1} + nH_{n-2} + 4(-1)^{n+1}. \tag{2}$$

Thirteen years later, Lucas [2] gave the same formula for U_n. So, [1] and [2] imply

$$H_n = U_n \tag{3}$$

which became well known after the development of rook theory [5]. In Section 2 we also give a one-to-one correspondence between H_n and U_n. In 1934, the explicit formula

$$U_n = \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)! \qquad (4)$$

was found by Touchard [8]. One can find a beautiful proof of (4) with the help of the "rook technique" in [5].

The first terms of the sequence $\{U_n\}$, for $n \geq 2$, are ([2], A000179 in [7])

$$0, 1, 2, 13, 80, 579, 4738, 43387, 439792, 4890741, 59216642, ... \qquad (5)$$

Note that formulas for U_n in other forms are given by Wayman and Moser [9] and Shevelev [6].

In the present paper we study the following problem.

Problem 2. With no gentleman seated next to his wife, n male-female couples, including Alice and her husband Bob, are to be seated at $2n$ chairs around a circular table. After the ladies are seated at every other chair, Bob is the first gentleman to choose one of the remaining chairs. If Bob chooses to sit d seats clockwise from Alice, how many seating arrangements are there for the remaining gentlemen?

2. Equivalence of Tait's Problem with the *Ménage* Problem

Let A be an $n \times n$ $(0,1)$-matrix. For every permutation $\pi = \{i_1, ..., i_n\}$ of the numbers $\{1, ..., n\}$ there is a set of elements $\{(1, i_1), ..., (n, i_n)\}$, called a diagonal of A. Thus, A has $n!$ distinct diagonals. The number of diagonals of 1's of A is the permanent of A, denoted by $per A$. This is also the number of distinct ways of putting n non-attacking rooks in place of the 1's of A. Let $J = J_n$ be an $n \times n$ matrix which consists of 1's only, let $I = I_n$ be the identity matrix, and let $P = P_n$ be an $n \times n$ matrix with this diagonal of 1's: $(1, 2), ..., (n-1, n), (n, 1)$ and all other entries 0. In Tait's problem, we should find the number of permutations with the prohibited positions $(1, 1), ..., (n, n)$ and $(1, 2), ..., (n-1, n), (n, 1)$. Therefore, Tait's problem is the problem of calculating $H_n = per(J_n - I - P)$.

Consider now the *ménage* problem. Denote the $2n$ chairs at a circular table by the symbols

$$1, \overline{1}, 2, \overline{2}, ..., n, \overline{n} \qquad (6)$$

going clockwise. Suppose the ladies occupy chairs $\{\overline{1}, ..., \overline{n}\}$. Number a gentleman i, if his wife occupies chair $\overline{i}$. Now the ith gentleman, for $i = 1, ..., n-1$, can occupy every chair except for chairs i and $i + 1$, while the nth gentleman cannot occupy chairs n and 1. If, in the corresponding $n \times n$ incidence matrix, the prohibited

positions are 0's and the other positions are 1's, we again obtain the matrix $J_n - I - P$. Every seating of the gentlemen corresponds to a diagonal of 1's in this matrix. This means that (cf. [3])

$$U_n = per(J_n - I - P), \tag{7}$$

and (3) follows. In particular, for $n = 2, 3, 4, 5, ...$, we have the following matricies, $(J_n - I - P)$:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}, ...$$

with permanents $0, 1, 2, 13, ...$.

There is a one-to-one correspondence between the diagonals of 1's of the matrix $J_n - I - P$ and arrangements of n male-female couples around a circular table, by the rules of the *ménage* problem, after the ladies $w_1, w_2, ..., w_n$ have taken the chairs numbered

$$2n, 2, 4, ..., 2n - 2 \tag{8}$$

respectively. Suppose we consider a diagonal of 1's of the matrix $J_n - I - P$:

$$(1, j_1), (2, j_2), ..., (n, j_n). \tag{9}$$

Then the gentlemen $m_1, m_2, ..., m_n$ took chairs numbered

$$2j_i - 3 \quad (\text{mod } 2n), \quad i = 1, 2, ..., n, \tag{10}$$

where the residues are chosen from the interval $[1, 2n]$.

Since $\{j_i\}$ is a permutation of $1, ..., n$, then $\{2j_i - 3\}$ (mod $2n$) is a permutation of the odd positive integers not greater than $2n - 1$. The distance between the places of m_i (10) and w_i (8) cannot be 1, indeed the equality $|2(j_i - i) - 1| = 1$ (mod $2n$) is possible if and only if either $j_i = i$ or $j_i = i+1$ (mod n), corresponding to the positions of the $0's$ in $J_n - I - P$. For example, in the case of $n = 5$ and $j_1 = 3, j_2 = 1, j_3 = 5, j_4 = 2, j_5 = 4$ in (9), then by (8) and (10), the chairs $1, 2, ..., 10$ are taken by $m_4, w_2, m_1, w_3, m_5, w_4, m_3, w_5, m_2, w_1$, respectively.

3. Equivalent Form of Problem 2

The one-to-one correspondence above suggests a solution to Problem 2. Let $(J_n - I - P)[1| \ r]$ be the matrix obtained by removing the first row and the rth column of $J_n - I - P$. Then, by expanding the permanent (7) over the first row, we have

$$U_n = \sum_{r=3}^{n} per((J_n - I - P)[1| \ r]). \tag{11}$$

In view of symmetry, the solution is invariant with respect to w_1, the chair that Alice occupies. Suppose Alice occupies chair $2n$ (or 0 (mod $2n$)). Then, by (10), in (11) the values of the distances corresponding to $r = 3, 4, 5, 6, ...$ are $d = 3, 5, 7, 9, ...$ clockwise, i.e., the distance

$$d = 2r - 3, \; r \geq 3, \tag{12}$$

clockwise between Alice and Bob. Thus for the solution of Problem 2 we should find the summands of (11). We can do this by the representation of the rook polynomials of each matrix $A_r = (J_n - I - P)[1| \; r], \; 3 \leq r \leq n$, as a product of rook polynomials of simpler matrices.

4. Lemmas

Let M be a rectangular $(0, 1)$-matrix.

Definition 1. The polynomial

$$R_M(x) = \sum_{j=0}^{n} \nu_j(M) x^j \tag{13}$$

where $\nu_0 = 1$ and ν_j is the number of ways of putting j non-attacking rooks on the positions of the 1's in M, is called a rook polynomial.

In particular, if M is an $n \times n$-matrix, then $\nu_n(M) = per M$.   Now we formulate several results of the classic Kaplansky–Riordan rook theory (cf. [5], chaps. 7–8).

Lemma 1. *If M is a $(0, 1)$-matrix with rook polynomial* (13), *then*

$$per(J_n - M) = \sum_{j=0}^{n} (-1)^j \nu_j(M)(n - j)! \tag{14}$$

Definition 2. Two submatrices M_1 and M_2 of a $(0, 1)$-matrix M are called disjunct if no 1's of M_1 are in the same row or column as those of M_2.

From Definition 1, the following lemma is evident.

Lemma 2. *If a $(0, 1)$-matrix M consists of two disjunct submatrices M_1 and M_2, then*

$$R_M(x) = R_{M_1}(x) R_{M_2}(x). \tag{15}$$

Consider the position (i, j) of a 1 in M. Denote by $M^{(0(i,j))}$ the matrix obtained from M after replacing it by a 0. Denote by $M^{(i,j)}$ the matrix obtained from M by removing the ith row and jth column.

Lemma 3. *We have*

$$R_M(x) = xR_{M^{(i,j)}} + R_{M^{(0(i,j))}}. \tag{16}$$

Consider a staircase $(0,1)$-matrix. A $(0,1)$-matrix is called *staircase* if all its 1's form the endpoints of a polyline with alternate vertical and horizontal links of length 1. Every line (row or column) can have a maximum of 2 consecutive 1's. If a staircase matrix contains k 1's, then it is called a k-staircase matrix. For example, the following matrices are 5-staircase:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and the following matrices are 6-staircase:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Lemma 4. *For every $k \geq 1$, all k-staircase matrices M have the same rook polynomial*

$$R_M(x) = \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-i+1}{i} x^i. \tag{17}$$

Proof. Note that since all polylines with k alternate vertical and horizontal links of length 1 are congruent figures, then all k-staircase $(0,1)$-matricies have the same rook polynomial. So we consider the following k-staircase $(0,1)$-matrix with the configuration of the 1's of the form

$$\begin{matrix} 1 & 1 & . & . & . & . \\ . & 1 & 1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 1 & 1 & . \\ . & . & . & . & 1 & 1 \end{matrix}$$

The last 1 on the right is absent for odd k and is present for even k. In both cases, by Lemma 3, for the rook polynomial $R_k(x)$ we have

$$R_0(x) = 1, \quad R_1(x) = x + 1, \quad R_k(x) = R_{k-1}(x) + xR_{k-2}(x), \quad k >= 2.$$

This equation has solution (17) of Lemma 4 (see [5], chap. 7, eq. (27)). □

For example, let

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

M contains two disjoint staircase matricies:

$$M_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

So, by Lemmas 2 and 4,

$$R_M(x) = R_{M_1}(x) R_{M_2}(x) = \sum_{i=0}^{2} \binom{5-i}{i} x^i \sum_{j=0}^{2} \binom{4-j}{j} x^j =$$

$$(3x^2 + 4x + 1)(x^2 + 3x + 1) = 3x^4 + 13x^3 + 16x^2 + 7x + 1.$$

5. Solution of Problem 2

According to Lemma 1, in order to calculate the permanent of the matrix $(J_n - I - P)[1| \ r]$, we can find the rook polynomial of the matrix $J_{n-1} - (J_n - I - P)[1| \ r]$. We use the equation

$$J_{n-1} - (J_n - I - P)[1| \ r] = (I_n + P)[1| \ r], \tag{18}$$

which follows from $(A + B)[1| \ r] = A[1| \ r] + B[1| \ r]$ and the equality $J_{n-1} = J_n[1| \ r]$. Transforming from matrix $(I_n + P)$ to the matrix $(I_n + P)[1| \ r]$, we have (in this example, $n = 10$ and $r = 5$)

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

Now let us use Lemma 3 on the latter matrix in the case $i = n - 1, \ j = 1$. Write

$$A = ((I_n + P)[1| \ r])^{(n-1,1)}, \ \ B = ((I_n + P)[1| \ r])^{(0(n-1,1))}. \tag{20}$$

According to (16), we have

$$R_{(I_n+P)[1| \ r]}(x) = xR_A(x) + R_B(x). \tag{21}$$

Note that A has the form (following the example, $n = 10$ and $r = 5$)

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{22}$$

which is an $(n-2) \times (n-2)$ matrix with $(2n-6)$ 1's. This matrix consists of two disjunct matrices: an $(r-2) \times (r-2)$ matrix A_1 of the form (here $r = 5$)

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{23}$$

which is a $(2r-5)$-staircase matrix, and an $(n-r) \times (n-r)$ matrix (again $n = 10$ and $r = 5$)

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{24}$$

which is a $2(n-r) - 1$-staircase matrix. Thus, by Lemmas 2 and 4, we have

$$R_A(x) = \sum_{i=0}^{r-2} \binom{2r-i-4}{i} x^i \sum_{k=0}^{n-r} \binom{2(n-r)-k}{k} x^k.$$

It is convenient to put $k = j - 1$:

$$R_A(x) = \sum_{i=0}^{r-2} \binom{2r-i-4}{i} x^i \sum_{j=0}^{n-r+1} \binom{2(n-r)-j+1}{j-1} x^{j-1}. \tag{25}$$

Note that, since $\binom{n}{-1} = 0$, we can write the lower limit in the sum over j as $j = 0$. Furthermore, the $(n-1) \times (n-1)$ matrix $B = B(r)$ (20) has the form (here $n = 10$ and $r = 5$)

$$B = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{26}$$

and contains $(2n - 5)$ 1's. This matrix consists of two disjunct matrices: an $(r - 2) \times (r - 1)$ matrix B_1 of the form (here $r = 5$)

$$B_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{27}$$

which is a $(2r-5)$-staircase matrix, and an $(n-r+1) \times (n-r)$ matrix (here $n = 10$ and $r = 5$)

$$B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{28}$$

which is a $2(n - r)$-staircase matrix. Thus, by Lemmas 2 and 4, we have

$$R_B(x) = \sum_{i=0}^{r-2} \binom{2r - i - 4}{i} x^i \sum_{j=0}^{n-r+1} \binom{2(n - r) - j + 1}{j} x^j. \tag{29}$$

Note that, since $\binom{n-r}{n-r+1} = 0$, we can write the upper limit in the second sum as $j = n - r + 1$. Now from (21), (25) and (29) we find

$$R_{(I_n+P)[1| \ r]}(x) = \sum_{i=0}^{r-2} \binom{2r - i - 4}{i} x^i \sum_{j=0}^{n-r+1} \binom{2(n - r) - j + 2}{j} x^j.$$

Set $i + j = k$. Then $k <= r - 2 + n - r + 1 = n - 1$; besides $i <= k$ and since we have $\binom{2r-i-4}{i}$, then $2r - i - 4 >= i, i <= r - 2$. Thus

$$R_{(I_n+P)[1| \ r]}(x) = \sum_{k=0}^{n-1} x^k \sum_{i=0}^{\min(k, \ r-2)} \binom{2r - i - 4}{i} \binom{2(n - r) - k + i + 2}{k - i}. \tag{30}$$

Note that for the interior sum of (30), it is sufficient to take the summation over interval $[\max(r + k - n - 1, 0), \min(k, r - 2)]$. Thus, by Lemma 1 and (18), we have

$$per((J_n - I - P)[1|\ r]) =$$

$$\sum_{k=0}^{n-1}(-1)^k(n - k - 1)! \sum_{i=\max(r+k-n-1,\ 0)}^{\min(k,\ r-2)} \binom{2r - i - 4}{i}\binom{2(n - r) - k + i + 2}{k - i}. \quad (31)$$

Formula (31) solves Problem 2 for $r = (d + 3)/2 \geq 3$ (by 12) and naturally $n > (d + 1)/2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \diamond$

For sequences corresponding to $d = 3, 5, 7, 9$ and 11, see A258664–A258667 and A258673 in [7].

Remark 1. The prohibited values $d = 1$ and $d = 2n - 1$ correspond to the case when Alice and Bob are seated at neighboring chairs. Let us calculate the number of ways of seating the remaining gentlemen after all the ladies have occupied their chairs, so that ladies and gentlemen are in alternate chairs, but Alice and Bob are the only couple seated next to each other. Thus we have a classic *ménage* problem for $n - 1$ couples for a one-sided linear table, after the ladies have already occupied their chairs. So, by [5], chap. 8, Thm. 1, $t = 0$, the solution V_n of this problem is

$$V_n = \sum_{k=0}^{n-1}(-1)^k\binom{2n - k - 2}{k}(n - k - 1)!, \ \ n > 1. \quad (32)$$

One can verify that this result can be obtained from (31) for both $r = 2$ and $r = n + 1$. It could also be proved independently. Cf. also A259212 in [7].

6. Enumeration of Arrangements

A simple method of finding the arrangements is to cycle through the permutations of the $n - 1$ remaining gentlemen and weed out the ones that do not follow the "no gentleman next to his wife" rule. This Mathematica (version 7) snippet works in that manner.

```
enumerateSeatings[couples_,d_]:=
 If[d==1||d>=2 couples-1||EvenQ[d],{},Map[#[[1]]&,
 DeleteCases[Map[{#,Differences[#]}&,
 Flatten[Map[{Riffle[Range[couples],
 RotateRight[Insert[#,1,1],(d-1)/2]]}&,
 Permutations[Range[couples-1],{couples-1}]+1],1]],
 {{___},{___,0,___}}]]];
```

Also, we can code for (31) by

```
numberOfSeatings[couples_,d_]:=
 If[couples<=#-1||EvenQ[d]||d==1,0,
 Sum[((-1)^k)*Factorial[(couples-k-1)]Sum[Binomial[2#-j-4,j]*
 Binomial[2(couples-#)-k+j+2,k-j],
 {j,Max[#+k-couples-1,0],Min[k,#-2]}],{k,0,couples-1}]]&[(d+3)/2];
```

For 6 couples with Bob sitting 3 seats clockwise from Alice, who is in chair 1, to find how many possible possible seating arrangements there are, the command `numberOfSeatings[6,3]` will return 20. Similarly, to enumerate those possible seatings, `enumerateSeatings[6,3]` will give

$$\{1,3,2,1,3,2,4,6,5,4,6,5\} \quad \{1,3,2,1,3,5,4,6,5,2,6,4\}$$
$$\{1,3,2,1,3,5,4,6,5,4,6,2\} \quad \{1,3,2,1,3,6,4,2,5,4,6,5\}$$
$$\{1,4,2,1,3,2,4,6,5,3,6,5\} \quad \{1,4,2,1,3,5,4,6,5,2,6,3\}$$
$$\{1,4,2,1,3,5,4,6,5,3,6,2\} \quad \{1,4,2,1,3,6,4,2,5,3,6,5\}$$
$$\{1,4,2,1,3,6,4,3,5,2,6,5\} \quad \{1,5,2,1,3,2,4,6,5,3,6,4\}$$
$$\{1,5,2,1,3,2,4,6,5,4,6,3\} \quad \{1,5,2,1,3,6,4,2,5,3,6,4\}$$
$$\{1,5,2,1,3,6,4,2,5,4,6,3\} \quad \{1,5,2,1,3,6,4,3,5,2,6,4\}$$
$$\{1,5,2,1,3,6,4,3,5,4,6,2\} \quad \{1,6,2,1,3,2,4,3,5,4,6,5\}$$
$$\{1,6,2,1,3,5,4,2,5,3,6,4\} \quad \{1,6,2,1,3,5,4,2,5,4,6,3\}$$
$$\{1,6,2,1,3,5,4,3,5,2,6,4\} \quad \{1,6,2,1,3,5,4,3,5,4,6,2\}.$$

Symbolizing Alice by ① and Bob by ①, the nth lady by Ⓝ and her husband by Ⓝ, we can show this in a graphical manner as in Figure 1, below. Also, Table 1 shows a triangular table of the number of seatings as given by (31). The rows of Table 1 appear to be unimodal for which we have no proof. The row sums, $1, 2, 13, 80, ...$, in Table 1 are equal to the solutions to the classic *ménage* problem.
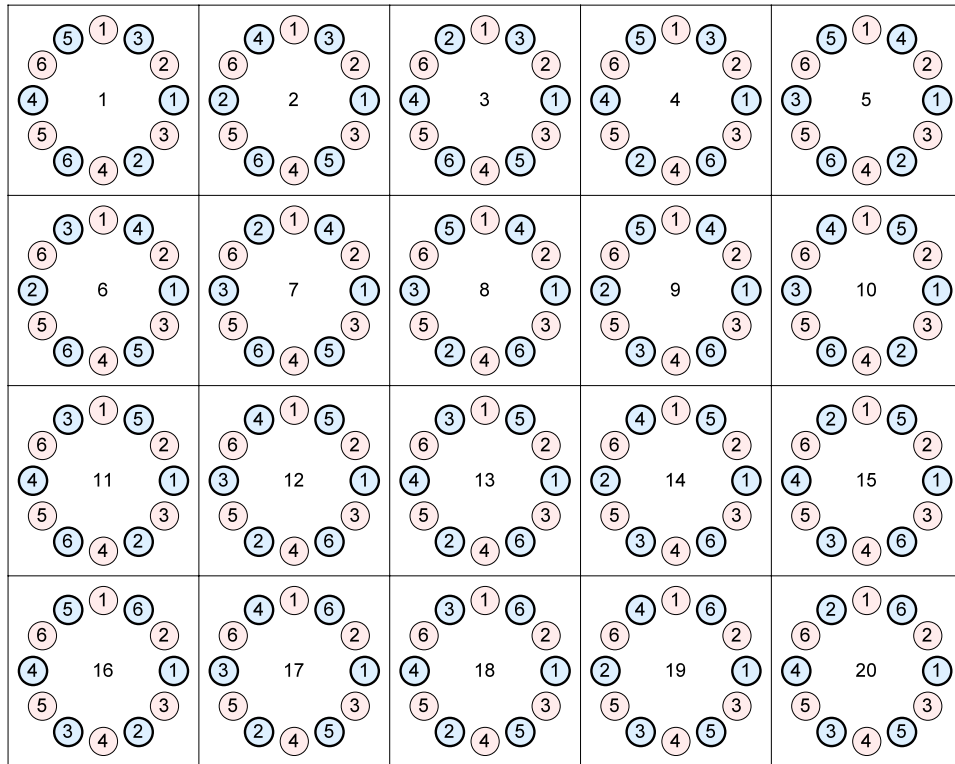
Figure 1: The 20 possible seatings of the 5 remaining gentlemen if Bob sits 3 seats clockwise from Alice.

n								
3	1							
4	1	1						
5	4	5	4					
6	20	20	20	20				
7	115	116	117	116	115			
8	787	791	791	791	791	787		
9	6184	6203	6204	6205	6204	6203	6184	
10	54888	55000	55004	55004	55004	55004	55000	54888
	3	5	7	9	11	13	15	17
				d				

Table 1: Number of arrangements for n couples with Bob seated d seats clockwise from Alice.

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