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ON DIVISIBILITY OF $\binom{n-i-1}{i-1}$ by i

VLADIMIR SHEVELEV

Department of Mathematics
Ben-Gurion University of the Negev
Beer-Sheva 84105, Israel
shevelev@bgu.ac.il

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We investigate the function $b(n) = \sum 1$, where the summing is over all i for which $(n, i) > 1$, $i \mid \binom{n-i-1}{i-1}$.

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1. Introduction, Main Results and Problems

There are many papers devoted to arithmetical properties of the binomial coefficients (see [1–9, 13–18]). In this paper, we consider some other aspect of the divisibility problem of the binomial coefficients. The well-known combinatorial lemma by Kaplansky (cf. [12]), states that the number of ways of selecting i objects, no two consecutive, from n objects arrayed on a circle is $\frac{n}{i}\binom{n-i-1}{i-1}$. This means that for all n, i in the natural condition $i \le \frac{n}{2}$, the number $\frac{n}{i}\binom{n-i-1}{i-1}$ is an integer. Consequently, if $(i, n) = 1$ then $i \mid \binom{n-i-1}{i-1}$.

For $n \in \mathbb{N}$ consider the set

$$B_n = \left\{ i \in \left[2, \frac{n}{2}\right] : (n, i) > 1,\ i \,\middle|\, \binom{n-i-1}{i-1} \right\}.$$

The cardinality $b(n)$ of B_n we call the *binomial index of* n. The number n is called a *binomial prime* if $b(n) = 0$.

In particular, every prime is also a binomial prime. We have the following 24 binomial primes not exceeding 33:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 17, 19, 20, 21, 23, 24, 25, 29, 31, 33. \qquad (1)$$

At first sight, there is no visible regularity for the binomial primes sequence. Further calculations, however, convinced us more and more that the nine numbers

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of the sequence (1), namely, $1, 6, 8, 10, 12, 20, 21, 24, 33$, are an exception from a general rule. This conjecture is confirmed by the following theorem:

Theorem 1. *If* $n > 33$ *then* n *is a binomial prime if and only if either* n *is a prime or square of a prime or a product of twin primes* ($n = p(p+2)$).

Thus, for the sequence of the binomial primes, the prime number theorem is true.

Of course, it would have been interesting to split the binomial primes into three types according to Theorem 1. Let $\varphi(n)$ be the Euler phi-function. First of all, it is rather simple to establish the following statement: if $b(n) + \frac{1}{2}\varphi(n) = \frac{n-1}{2}$, then n is a prime.

However, the question about the validity of the following two natural conjectures remains open:

(a) if $b(n^2) + \frac{1}{2}\varphi(n^2) = \frac{1}{2}(n^2 - n)$, then n is a prime;
(b) if $b(4n^2-1) + \frac{1}{2}\varphi(4n^2-1) = 2(n^2-n)$, then $2n-1$ and $2n+1$ are prime twins.

Another open question, which is interesting in our opinion, is the following conjecture based on numerical experiments: if $p \geq 5$ is a prime, then $b(2p)$ is even for $p \equiv 1 \pmod 4$ and is odd for $p \equiv 3 \pmod 4$.

Further, we investigate the equation $b(n) = a$. The next result shows that for "almost all" $a \in \mathbb{N}$ this equation has infinitely many solutions.

Theorem 2. *The number of numbers* $a \leq x$ *for which the set of solutions of the equation* $b(n) = a$ *is finite or empty does not exceed* $C_N x/(\log x)^N$, *where* N *is an arbitrary large number*, C_N *depends on only* N.

In the proof of Theorem 2 we establish that the equation $b(n) = a$ has an infinite set of solutions, at least, in case of existence of infinitely many pairs of so-called "$(a+1)$-prime twins" $(p, p+2(a+1))$. Since the latter is an old very difficult unsolved problem, it would have been interesting to find weaker conditions for the existence of an infinite set of solutions of the considered equation. On the other hand, it would have been interesting to conduct a full investigation, in the form of Theorem 1, of the following concrete cases: $b(n) = 1,\ b(n) = 2,\ b(n) = 3$.

Notice furthermore that, evidently, $\liminf_{n\to\infty} \frac{b(n)}{\varphi(n)} = 0$. We prove the following result:

Theorem 3. *If the sequence of Mersenne primes is infinite, then* $\limsup_{n\to\infty} \frac{b(n)}{\varphi(n)} \geq \frac{1}{2}$.

Although there is no doubt that there exist infinitely many Mersenne primes, the question of finding of an unconditional proof of Theorem 3 is interesting. Notice that one of the ways to do that is to prove that $\limsup_{p\to\infty} \frac{\varphi(2^p-1)}{2^p} = 1$. It is a weakened form of the conjecture of infinity of the Mersenne primes. It seems that another way is to estimate $b(\prod_{i=1}^{n} p_i)$, where p_i is the ith prime. We conjecture that indeed

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$\limsup_{p\to\infty} \frac{b(n)}{\varphi(n)}$ is finite but more than $\frac{1}{2}$. Notice that using the Landau inequality for $\varphi(n)$ [11] we can only state that $b(n) \le \lfloor \frac{n}{2} \rfloor - \frac{1}{2}\varphi(n) = 0(\varphi(n)\log\log n)$.

In conclusion, we present some open questions regarding the set B_n;

(i) *Is it true that there is no n for which $|B_n| \ge 2$ and B_n contains only odd numbers?*
Notice that one can show (it follows from the proof of Theorem (2) that for pairs of 2-prime twins $(p, p+4)$, $B_{p(p+4)} = \{3p\}$.

(ii) *Are there infinitely many numbers n for which B_n contains a divisor of n?*
Notice that, for $n \le 100$, we have only seven such numbers: $18, 45, 48, 75, 84, 90, 100$.

(iii) It is easy to see that there are infinitely many pairs $(n, n+1)$, for which $B_n \cap B_{n+1} \ne \varnothing$, e.g. for $n = 72k+8$ $B_n \cap B_{n+1}$ contains the number 6.
It is interesting to estimate the number of all such pairs not exceeding x.

Now some words about the structure of the article.

Sections 2–4 are devoted to proofs of Theorems 1–3. Finally, in Sec. 5, we provide some numerical results.

2. Proof of Theorem 1

The proof of Theorem 1 follows from Lemmas 1–14.

Lemma 1. *Let n be a composite number, p be a prime divisor of n, and for some positive integer α suppose that $i = p^\alpha k \le \frac{n}{2}$ such that $(k, n) = 1$.*

$$\textit{If } p^\alpha \Big| \binom{n-i-1}{i-1} \quad \textit{then } i \in B_n.$$

Proof. Since $\frac{n}{i}\binom{n-i-1}{i-1} \in \mathbb{N}$ we have: $k \mid \binom{n-i-1}{i-1}$. Therefore by the condition $i \mid \binom{n-i-1}{i-1}$ and $(i, n) \ge p$. □

Everywhere below we denote for $m \in \mathbb{N}$ and a prime p via $\sigma_p(m)$ the number for which $p^{\sigma_p(m)} \,||\, m$.

Lemma 2. *A prime p divides $\binom{n-i-1}{i-1}$ if and only if there exists a positive integer t for which*

$$\Delta_t(n,i,p) = \left\lfloor \frac{n-i-1}{p^t} \right\rfloor - \left\lfloor \frac{i-1}{p^t} \right\rfloor - \left\lfloor \frac{n-2i}{p^t} \right\rfloor = 1. \tag{2}$$

Proof. As it is very well known,

$$\sigma_p(m!) = \left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots. \tag{3}$$

Since $\binom{n-i-1}{i-1} = \frac{(n-i-1)!}{(i-1)!(n-2i)!}$ the lemma follows directly from (3) and the trivial inequalities for real numbers α and β: $0 \le \lfloor \alpha+\beta \rfloor - \lfloor \alpha \rfloor - \lfloor \beta \rfloor \le 1$. □

Lemma 3. *For each twin pair $p,\ p+2$ the number $n = p\ (p+2)$ is a binomial prime.*

Proof. Consider $i_1 = pg$, $i_2 = (p+2)h$, such that $i_j \le \frac{p(p+2)}{2}$, $\ j = 1,2$. Thus, $1 \le g \le \frac{p+1}{2}$, $1 \le h \le \frac{p-1}{2}$. We have obviously

$$\sigma_p\left(\binom{p(p+2)-i_1-1}{i_1-1}\right) = \left\lfloor \frac{p(p+2)-pg-1}{p} \right\rfloor - \left\lfloor \frac{pg-1}{p} \right\rfloor - \left\lfloor \frac{p(p+2)-2pg}{p} \right\rfloor + \left\lfloor \frac{p(p+2)-pg-1}{p^2} \right\rfloor - \left\lfloor \frac{p(p+2)-2pg}{p^2} \right\rfloor = 0,$$

$$\sigma_{p+2}\left(\binom{p(p+2)-i_2-1}{i_2-1}\right) = \left\lfloor \frac{p(p+2)-(p+2)h-1}{p+2} \right\rfloor - \left\lfloor \frac{(p+2)h-1}{p+2} \right\rfloor - \left\lfloor \frac{p(p+2)-2(p+2)h}{p+2} \right\rfloor = 0,$$

$$\text{i.e.}\quad i_j \nmid \binom{p(p+2)-i_j-1}{i_j-1}, \quad j = 1,2.$$

□

Lemma 4. *For a prime p, the number $n = p^2$ is a binomial prime.*

Proof. Consider $i = ph$, $1 \le h \le \frac{p}{2}$. We have

$$\sigma_p\left(\binom{p^2-i-1}{i-1}\right) = \left\lfloor \frac{p^2-ph-1}{p} \right\rfloor - \left\lfloor \frac{ph-1}{p} \right\rfloor - \left\lfloor \frac{p^2-2ph}{p} \right\rfloor = 0,$$

$$\text{i.e.}\quad i \nmid \binom{p^2-i-1}{i-1}.$$

□

Lemma 5. *Let $n = p_1p_2$ where p_1, p_2 are primes so that $5 \le p_1 < p_2$, $\ p_2 - p_1 \ge 4$. Then n is not a binomial prime.*

Proof. We distinguish two cases:

Case 1. $p_1 - 1 < \frac{p_2-1}{2}$.
Let $p_2 = \lambda p_1 + r$, $1 \le r \le p_1 - 1$, $\lambda \ge 2$ $\ (\lambda, r \in \mathbb{N})$. Here we distinguish between three subcases:

1(a) $r \le p_1 - 3$,
1(b) $r = p_1 - 1$,
1(c) $r = p_1 - 2$.

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In subcase 1(a), put $i = p_1(p_1 - 1)$. Notice that $i \le p_1 \frac{p_2-1}{2} < \frac{n}{2}$. We have for $p_1 \ge 5$,

$$\begin{aligned}\Delta_2(n,i,p_1) &= \left\lfloor \frac{n-i-1}{p_1^2} \right\rfloor - \left\lfloor \frac{i-1}{p_1^2} \right\rfloor - \left\lfloor \frac{n-2i}{p_1^2} \right\rfloor \\ &= \left\lfloor \frac{p_1p_2 - p_1^2 + p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{p_1^2 - p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{p_1p_2 - 2p_1^2 + 2p_1}{p_1^2} \right\rfloor \\ &= \left\lfloor \frac{\lambda p_1^2 - p_1^2 + rp_1 + p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{\lambda p_1^2 - 2p_1^2 + rp_1 + 2p_1}{p_1^2} \right\rfloor \\ &= \lambda - 1 - (\lambda - 2) = 1,\end{aligned}$$

and by Lemmas 1 and 2, $i \in B_n$.

In subcase 1(b) put $i = p_1(p_1 - 2)$. We have for $p_1 \ge 5$,

$$\begin{aligned}&\Delta_2(n,i,p_1) \\ &\quad = \left\lfloor \frac{\lambda p_1^2 - p_1^2 + p_1(p_1-1) + 2p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{\lambda p_1^2 - 2p_1^2 + p_1(p_1-1) + 4p_1}{p_1^2} \right\rfloor = 1,\end{aligned}$$

and $i \in B_n$.

At last in subcase 1(c) put $i = p_1(p_1 - 3)$. We have for $p_1 \ge 5$,

$$\begin{aligned}&\Delta_2(n,i,p_1) \\ &\quad = \left\lfloor \frac{\lambda p_1^2 - p_1^2 + p_1(p_1-2) + 3p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{\lambda p_1^2 - p_1^2 + p_1(p_1-2) + 6p_1}{p_1^2} \right\rfloor = 1,\end{aligned}$$

and $i \in B_n$.

Case 2. $\frac{p_2-1}{2} \le p_1 - 1 < p_2 - 1$ or $p_1 < p_2 \le 2p_1 - 1$.
Using the condition, we have $4 \le p_2 - p_1 \le p_1 - 1$. Consequently, $p_1 \ge 7$ (if $p_1 = 5$ we have $p_2 - 5 = 4$ and p_2 is not a prime). Since

$$p_1 + 4 \le p_2 \le p_1 + (p_1 - 1),$$

suppose that $p_2 = p_1 + r,\ 4 \le r \le p_1 - 1$. Now put $i = (r-1)p_1$. Then by (2)

$$\begin{aligned}\Delta_2(n,i,p_1) &= \left\lfloor \frac{p_1(p_1+r) - (r-1)p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{(r-1)p_1 - 1}{p_1^2} \right\rfloor \\ &\quad - \left\lfloor \frac{p_1(p_1+r) - 2(r-1)p_1}{p_1^2} \right\rfloor \\ &= \left\lfloor \frac{p_1^2 + p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{p_1^2 - p_1r + 2p_1}{p_1^2} \right\rfloor = 1.\end{aligned}$$

□

Lemma 6. *Let* $n = p_1p_2\cdots p_k$, $k \ge 3$ *where* $p_1, p_2, \ldots, p_k$ *are primes so that* $5 \le p_1 \le p_2 \le \cdots \le p_k$. *Then* n *is not a binomial prime.*

Proof. Denote $p_2p_3\cdots p_k = P$ so that $n = p_1P$. It is clear that $p_1 - 1 < \frac{P-1}{2}$. Let $P = \lambda p_1 + r,\ 0 \le r \le p_1 - 1,\ \ \lambda \ge 2$. Here the case $r = 0$ is realized if $p_2 = p_1$.

Again we distinguish three subcases (cf. that of Lemma 5, Case 1):

(a) $r \leq p_1 - 3$,
(b) $r = p_1 - 1$,
(c) $r = p_1 - 2$.

As in the proof of Lemma 5 we put: in Case (a) $i = p_1(p_1 - 1)$ (then $i \leq p_1 \frac{P-1}{2} < \frac{n}{2}$); in Case (b) $i = p_1(p_1 - 2)$; in Case (c) $i = p_1(p_1 - 3)$, and verify that $\Delta_2(n, i, p_1) = 1$. □

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Lemma 7. *If p is a prime ≥ 13, then $n = 3p$ is not a binomial prime.*

Proof. We distinguish two cases:

Case 1. $p \equiv 1 \pmod 3$.
We have by modulo 18 three subcases:

1(a) $p \equiv 1$,
1(b) $p \equiv 7$,
1(c) $p \equiv 13$.

1(a) Let $p = 18k + 1,\ k \geq 1$. Put $i = 3\frac{p-5}{2} < \frac{n}{2}$. Notice that $\left(\frac{p-5}{2}, 3\right) = 1$. Then by (2)

$$\Delta_3(n, i, 3) = \left\lfloor \frac{27k + 8}{27} \right\rfloor - \left\lfloor \frac{27k - 7}{27} \right\rfloor = 1.$$

1(b) Let $p = 18k + 7,\ k \geq 2$. Put $i = 3\frac{p-15}{2}$. Then by (2)

$$\Delta_3(n, i, 3) = \left\lfloor \frac{27k + 32}{27} \right\rfloor - \left\lfloor \frac{27k - 13}{27} \right\rfloor - \left\lfloor \frac{51}{27} \right\rfloor = 1.$$

1(c) Let $p = 18k + 13,\ k \geq 0$. Put again $i = 3\frac{p-15}{2}$. Then

$$\Delta_3(n, i, 3) = \left\lfloor \frac{27k + 41}{27} \right\rfloor - \left\lfloor \frac{27k - 4}{27} \right\rfloor - \left\lfloor \frac{45}{27} \right\rfloor = 1.$$

Case 2. $p \equiv 2 \pmod 3$.
We have by modulo 18 three subcases:

2(a) $p \equiv 5$,
2(b) $p \equiv 11$,
2(c) $p \equiv 17$.

2(a) Let $p = 18k + 5,\ k \geq 1$. Put $i = 3\frac{p-7}{2}$. Notice that $\left(\frac{p-7}{2}, 3\right) = 1$. Now by (2)

$$\Delta_3(n, i, 3) = \left\lfloor \frac{27k + 17}{27} \right\rfloor - \left\lfloor \frac{27k - 4}{27} \right\rfloor = 1.$$

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2(b) Let $p = 18k + 11,\ k \geq 1$. Put $i = 3\frac{p-15}{2}$. Then

$$\Delta_3(n,i,3) = \left\lfloor \frac{27k+38}{27} \right\rfloor - \left\lfloor \frac{27k-7}{27} \right\rfloor - \left\lfloor \frac{45}{27} \right\rfloor = 1.$$

2(c) Let $p = 18k + 17,\ k \geq 0$. Put $i = 3\frac{p-7}{2}$. Notice that $\left(\frac{p-7}{2}, 3\right) = 1$. In this case

$$\Delta_3(n,i,3) = \left\lfloor \frac{27k+35}{27} \right\rfloor - \left\lfloor \frac{27k+14}{27} \right\rfloor = 1. \qquad \square$$

Lemma 8. *If* $n = 3p_1p_2\cdots p_k$, *where* $p_1, \ldots, p_k$ *are primes so that* $5 \leq p_1 \leq p_2 \leq \cdots \leq p_k,\ k \geq 2$, *then* n *is not a binomial prime.*

Proof. Denote $P = 3\prod_{i=2}^{k} p_i$ such that $n = p_1P$. Let

$$P = \lambda p_1 + r, \quad 0 \leq r \leq p_1 - 1, \quad \lambda \geq 2.$$

We distinguish two cases:

Case 1. $p_1 \equiv 2 \pmod 3$.
We consider the following subcases:

1(a) $0 \leq r \leq p_1 - 3$,
1(b) $r = p_1 - 2$,
1(c) $r = p_1 - 1,\ \ p_1 \geq 11$,
1(d) $p_1 = 5,\ r = p_1 - 1 = 4$.

1(a) Put $i = p_1(p_1 - 1)$. Notice that $(p_1 - 1, n) = 1$. We have

$$\begin{aligned}\Delta_2(n,i,p_1) &= \left\lfloor \frac{p_1P - p_1(p_1-1) - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{p_1(p_1-1)-1}{p_1^2} \right\rfloor \\ &\quad - \left\lfloor \frac{p_1P - 2p_1(p_1-1)}{p_1^2} \right\rfloor \\ &= \left\lfloor \frac{(\lambda-1)p_1^2 + (r+1)p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{(\lambda-2)p_1^2 + (r+2)p_1}{p_1^2} \right\rfloor = 1.\end{aligned}$$

1(b) Put $i = p_1(p_1 - 3)$. Then

$$\Delta_2(n,i,p_1) = \left\lfloor \frac{\lambda p_1^2 + p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{(\lambda-1)p_1^2 + 4p_1}{p_1^2} \right\rfloor = 1.$$

1(c) Again put $i = p_1(p_1 - 3)$. Then

$$\Delta_2(n,i,p_1) = \left\lfloor \frac{\lambda p_1^2 + 2p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{(\lambda-1)p_1^2 + 5p_1}{p_1^2} \right\rfloor = 1.$$

1(d) In this subcase $p_2 \geq 7$ and $n \geq 195$. Put $i = 40$. We have

$$\Delta_2(n,40,5) = \left\lfloor \frac{5(5\lambda+4) - 41}{25} \right\rfloor - \left\lfloor \frac{39}{25} \right\rfloor - \left\lfloor \frac{5(5\lambda+4) - 80}{25} \right\rfloor = 1.$$

Case 2. $p_1 \equiv 1 \pmod 3$, $\mathrm{p}_1 \geq 7$.
We consider the following subcases:

2(a) $0 \leq r \leq p_1 - 5$,
2(b) $r = p_1 - 1$ or $r = p_1 - 2$,
2(c) $r = p_1 - 4$,
2(d) $r = p_1 - 3,\ p_1 \geq 13$;
2(e) $p_1 = 7,\ r = p - 1 - 3 = 4$.

2(a) Put $i = p_1(p_1 - 2)$. Notice that $(p_1 - 2, n) = 1)$. We have

$$\Delta_2(n, i, p_1) = \left\lfloor \frac{p_1 P - p_1(p_1 - 2) - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{p_1 P - 2p_1(p_1 - 2)}{p_1^2} \right\rfloor$$
$$= \left\lfloor \frac{(\lambda - 1)p_1^2 + (r + 2)p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{(\lambda - 2)p_1^2 + (r + 4)p_1}{p_1^2} \right\rfloor = 1.$$

2(b) Put $i = p_1(p_1 - 3)$. Then

$$\Delta_2(n, i, p_1) = \left\lfloor \frac{(\lambda - 1)p_1^2 + (r + 3)p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{(\lambda - 2)p_1^2 + (r + 6)p_1}{p_1^2} \right\rfloor$$
$$= \begin{cases} \left\lfloor \frac{\lambda p_1^2 + 2p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{(\lambda - 1)p_1^2 + 5p_1}{p_1^2} \right\rfloor, & r = p_1 - 1, \\ \left\lfloor \frac{\lambda p_1^2 + p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{(\lambda - 1)p_1^2 + 4p_1}{p_1^2} \right\rfloor, & r = p_1 - 2, \end{cases}$$

whence $\Delta_2(n, i, p_1) = 1$.

2(c) Put $i = p_1(p_1 - 5)$. Then

$$\Delta_2(n, i, p_1) = \left\lfloor \frac{\lambda p_1^2 + p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{(\lambda - 1)p_1^2 + 6p_1}{p_1^2} \right\rfloor = 1.$$

2(d) Again put $i = p_1(p_1 - 5)$. Then

$$\Delta_2(n, i, p_1) = \left\lfloor \frac{\lambda p_1^2 + 2p_1 - 1}{p_1^2} \right\rfloor - \left\lfloor \frac{(\lambda - 1)p_1^2 + 7p_1}{p_1^2} \right\rfloor = 1.$$

2(e) In this subcase $p_2 \geq 11$ and $n \geq 273$. Put $i = 70$. We have

$$\Delta_2(n, 70, 7) = \left\lfloor \frac{7(7\lambda + 4) - 71}{49} \right\rfloor - \left\lfloor \frac{69}{49} \right\rfloor - \left\lfloor \frac{7(7\lambda + 4) - 140}{49} \right\rfloor = 1. \qquad \square$$

Lemma 9. *Let* $n = 2^\alpha 3^\beta p_1 p_2 \cdots p_k,\ 5 \leq p_1 \leq p_2 \leq \cdots \leq p_k,\ k \geq 1,$ *with* $\alpha \geq 0$, $\beta \geq 2$. *Then* n *is not a binomial prime.*

Proof. Denote $P = 2^\alpha p_1 p_2 \cdots p_k$, so that $n = 3^\beta P$.

Put $i = 3t$, where

$$t = \begin{cases} P - 3, & \text{if } P \equiv 2 \pmod 3 \\ P + 1, & \text{if } P \equiv 1 \pmod 3 \end{cases}.$$

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Notice that $(b, n) = 1$; moreover, $t \equiv 2 \pmod 3$. Let $t = 3\tau + 2,\ \tau \geq 1$. We have:

$$\Delta_2(n, i, 3) = \left\lfloor \frac{3^\beta P - 9\tau - 7}{9} \right\rfloor - \left\lfloor \frac{9\tau + 5}{9} \right\rfloor - \left\lfloor \frac{3^\beta P - 18\tau - 12}{9} \right\rfloor = 1. \qquad \square$$

Lemma 10. *If $n = 2^\alpha 3^\beta > 32$, then n is not a binomial prime.*

Proof. Distinguish between three cases:

Case 1. $n = 2^\alpha,\ \alpha \geq 6$.
Put $i = 6$. We have

$$\Delta_3(n, 6, 2) = \left\lfloor \frac{2^\alpha - 7}{8} \right\rfloor - \left\lfloor \frac{2^\alpha - 12}{8} \right\rfloor = 1.$$

Case 2. $n = 3^\beta,\ \beta \geq 4$.
Again put $i = 6$. We have

$$\Delta_2(n, 6, 3) = \left\lfloor \frac{3^\beta - 7}{9} \right\rfloor - \left\lfloor \frac{3^\beta - 12}{9} \right\rfloor = 1.$$

Case 3. $n = 2^\alpha 3^\beta,\ \alpha \geq 1,\ \beta \geq 1$.
Distinguish between two subcases:

3(a) $\alpha \geq 1,\ \beta \geq 2$. Put $i = 15$. We have

$$\Delta_2(n, 15, 3) = \left\lfloor \frac{2^\alpha 3^\beta - 16}{9} \right\rfloor - \left\lfloor \frac{14}{9} \right\rfloor - \left\lfloor \frac{2^\alpha 3^\beta - 30}{9} \right\rfloor = 1.$$

3(b) $\beta = 1,\ \alpha \geq 4$ (since $n > 32$). Put $i = 10$. We have

$$\Delta_4(n, 10, 2) = \left\lfloor \frac{2^\alpha 3 - 11}{16} \right\rfloor - \left\lfloor \frac{2^\alpha 3 - 20}{16} \right\rfloor = 1. \qquad \square$$

Lemma 11. *If $n = 2^\alpha p_1 p_2 \cdots p_k,\ \ 5 \leq p_1 \leq p_2 \leq \cdots \leq p_k,\ \ k \geq 1$, and $n > 32$ then n is not a binomial prime.*

Proof. Denote $P = p_1 p_2 \cdots p_k$. We distinguish between: $\alpha \geq 3,\ \alpha = 1,\ \alpha = 2$.

(1) $\alpha \geq 3$. Put $i = 6$. we have

$$\Delta_3(n, 6, 2) = \left\lfloor \frac{2^\alpha P - 7}{8} \right\rfloor - \left\lfloor \frac{2^\alpha P - 12}{8} \right\rfloor = 1.$$

(2) $\alpha = 1$. Distinguish two subcases: 2(a) $P \equiv 1 \pmod 4$ and 2(b) $P \equiv 3 \pmod 4$.

2(a) Let $P = 4t + 1$. Since $n = 2P > 32$ we conclude that $t \geq 4$. Put $i = 6$. We have

$$\Delta_3(n, 6, 2) = \left\lfloor \frac{8t - 5}{8} \right\rfloor - \left\lfloor \frac{8t - 10}{8} \right\rfloor = 1.$$

2(b) Let $P = 4t + 3,\ t \geq 2$. Put $i = 4$. In this case evidently

$$4 \left| \binom{2P-5}{3} = \binom{8t+1}{3} \right. .$$

(3) $\alpha = 2$. Again distinguish the same subcases:

3(a) $P \equiv 1 \pmod 4$ and

3(b) $P \equiv 3 \pmod 4$.

3(a) Let $P = 4t+1,\ t \geq 3$ (since $n = 4P > 32$ and P is not divisible by 3) Notice that in this case

$$n \geq \begin{cases} 197, & \text{if } 7 \mid P \\ 52, & \text{else} \end{cases}.$$

Put

$$i = \begin{cases} 54, & \text{if } t \text{ is even, } t \geq 10 \\ 18, & \text{if } t \text{ is odd, } t \geq 3 \\ 14, & \text{if } t = 4; 6. \end{cases}$$

Notice that $i < \frac{n}{2}$ and $(i, P) = 1$.
We have

$$\Delta_5(n, 54, 2) = \left\lfloor \frac{16t-51}{32} \right\rfloor - \left\lfloor \frac{53}{32} \right\rfloor - \left\lfloor \frac{16t-104}{32} \right\rfloor$$

$$= \left\lfloor \frac{t-4}{2} \right\rfloor - 1 - \left\lfloor \frac{t-7}{2} \right\rfloor = 1, \quad \text{if } t \text{ is even} \geq 10.$$

$$\Delta_5(n, 18, 2) = \left\lfloor \frac{16t-15}{32} \right\rfloor - \left\lfloor \frac{16t-32}{32} \right\rfloor$$

$$= \left\lfloor \frac{t-1}{2} \right\rfloor - \left\lfloor \frac{t-2}{2} \right\rfloor = 1, \quad \text{if } t \text{ is odd} \geq 3.$$

$$\Delta_4(n, 14, 2) = \left\lfloor \frac{16t-11}{16} \right\rfloor - \left\lfloor \frac{16t-24}{16} \right\rfloor = 1, \quad \text{if } t = 4; 6.$$

3(b) Let $P = 4t + 3,\ t \geq 2$. Put

$$i = \begin{cases} 8, & \text{if } t \text{ is even,} \\ 24, & \text{if } t \text{ is odd.} \end{cases}$$

In the case of $t = 2j,\ j \geq 1$, it is easy to verify directly that

$$8 \left| \binom{n-9}{7} = \binom{32j+3}{7} \right., \quad \text{i.e. } 8 \in B_n;$$

in the case of $t = 2j+1,\ j \geq 1$, as well it is easy to verify directly that

$$8 \left| \binom{n-25}{23} = \binom{32j+3}{23} \right., \quad \text{i.e. } 24 \in B_n \text{ since } (n, 3) = 1. \qquad \square$$

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What is left is to consider n of the form

$$n = 2^{\alpha}3p_1p_2\cdots p_k, \quad k \geq 1, \quad 5 \leq p_1 \leq p_2 \leq \cdots \leq p_k.$$

Denote $P = 3p_1p_2\ldots p_k$.

Lemma 12. *If* $n = 2^{\alpha}P,\ \alpha \geq 3$, *then* n *is not a binomial prime.*

Proof. Put

$$i = \begin{cases} 2(P-2), & \text{if } P = 4t+1, \\ 2(P-4), & \text{if } P = 4t+3. \end{cases}$$

Notice that $(i,P) = 1$ and $i = 8t - 2$. We have

$$\Delta_3(n,i,2) = \left\lfloor \frac{2^{\alpha}P - 8t + 1}{8} \right\rfloor - \left\lfloor \frac{8t+2}{8} \right\rfloor - \left\lfloor \frac{2^{\alpha}P - 16t + 4}{8} \right\rfloor = 1. \qquad \square$$

Lemma 13. *If* $n = 2P > 32$, *then* n *is not a binomial prime.*

Proof. Distinguish two cases:

Case 1. $P = 4t+1,\ t \geq 5$.
Put
$i = 2(\frac{P}{3} - 12)$. Then $i < \frac{n}{2}$ and $(i,P) = 1$. Notice that t has the form $3\tau + 2,\ \tau \geq 1$. Therefore $P = 12\tau + 9$ and $i = 8\tau - 18$. Now we have by (2)

$$\Delta_3(n,i,2) = \left\lfloor \frac{16\tau + 35}{8} \right\rfloor - \left\lfloor \frac{8\tau - 19}{8} \right\rfloor - \left\lfloor \frac{8\tau + 54}{8} \right\rfloor = 1.$$

Case 2. $P = 4t+3,\ t \geq 6$.
Put $i = 4$. Evidently in this case

$$4 \left| \binom{2P-5}{3} = \binom{8t+1}{3} \right. \qquad \square$$

Lemma 14. *If* $n = 4P > 32$, *then* n *is not a binomial prime.*

Proof. Denote $P_0 = \frac{4}{3}P$. Thus $n = 3P_0$, where

$$P_0 = 4p_1p_2\cdots p_k, \quad 5 \leq p_1 \leq p_2 \leq \cdots \leq p_k, \quad k \geq 1.$$

We split the proof in six cases modulo 9:

(1) $P_0 \equiv 1$,
(2) $P_0 \equiv 2$,
(3) $P_0 \equiv 4$,
(4) $P_0 \equiv 5$,
(5) $P_0 \equiv 7$,
(6) $P_0 \equiv 8$.

In each case we indicate $i \leq \frac{n}{2}$ such that $3\|i,\ (i, P_0) = 1$ and $\Delta_3(n, i, 3) = 1$, except of $n = 84,\quad P_0 = 28$ in Case 1. For this number, put $i = 21$. It is easy to verify that $21 \big| \binom{84-21-1}{21-1} = \binom{62}{20}$. Therefore in Case 1 we consider $n > 84$.

Case 1. Let $P_0 = 9t + 1,\ t > 3$.
Notice that $P_0 = 8a + 4$. Consequently, $9t = 8a + 3$ and $a = 3b$, whence $3t = 8b + 1$. This implies that b has the form $3c + 1$ and therefore $t = 8c + 3$. Thus $P_0 = 72c + 28,\ c \geq 1$. Put

$$i = \begin{cases} \frac{3}{8}(P_0 + 36), & \text{if } c \text{ is odd,} \\ \frac{3}{8}(P_0 - 36), & \text{if } c \text{ is even,} \end{cases}$$

or

$$i = \begin{cases} 27c + 24, & \text{if } c \text{ is odd,} \\ 27c - 3, & \text{if } c \text{ is even.} \end{cases}$$

Now for even $c \geq 2$ we have

$$\Delta_3(n, i, 3) = \left\lfloor \frac{189c + 59}{27} \right\rfloor - \left\lfloor \frac{27c + 23}{27} \right\rfloor - \left\lfloor \frac{162c + 36}{27} \right\rfloor = 1.$$

As well for odd c, we have

$$\Delta_3(n, i, 3) = \left\lfloor \frac{189c + 86}{27} \right\rfloor - \left\lfloor \frac{27c - 4}{27} \right\rfloor - \left\lfloor \frac{162c + 90}{27} \right\rfloor = 1.$$

Case 2. Let $P_0 = 9t + 2 = 8a + 4$.
Consequently, $a = 9c + 2$ and $P_0 = 72c + 20,\ c \geq 0$.

Put

$$i = \begin{cases} \frac{3}{8}(P_0 - 36), & \text{if } c \text{ is odd,} \\ \frac{3}{8}(P_0 + 36), & \text{if } c \text{ is even,} \end{cases}$$

or

$$i = \begin{cases} 27c - 6, & \text{if } c \text{ is odd,} \\ 27c + 21, & \text{if } c \text{ is even.} \end{cases}$$

Now for even c we have:

$$\Delta_3(n, i, 3) = \left\lfloor \frac{189c + 38}{27} \right\rfloor - \left\lfloor \frac{27c + 20}{27} \right\rfloor - \left\lfloor \frac{162c + 18}{27} \right\rfloor = 1.$$

As well for odd c, we have

$$\Delta_3(n, i, 3) = \left\lfloor \frac{189c + 65}{27} \right\rfloor - \left\lfloor \frac{27c - 7}{27} \right\rfloor - \left\lfloor \frac{162c + 72}{27} \right\rfloor = 1.$$

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Case 3. Let $P_0 = 9t + 4 = 8a + 4$.
We conclude that $a = 9c,\ t = 8c$ and $P_0 = 72c + 4,\ c \geq 1$. Put

$$i = \begin{cases} \frac{3}{8}(P_0 - 12), & \text{if } c \text{ is even,} \\ \frac{3}{16}(5P_0 + 36), & \text{if } c = 4j - 1\ j \geq 1, \\ \frac{3}{16}(5P_0 - 108), & \text{if } c = 4j + 1\ j \geq 0, \end{cases}$$

or

$$i = \begin{cases} 27c - 3, & \text{if } c \text{ is even,} \\ 270j - 57, & \text{if } c = 4j - 1, \\ 270j + 51, & \text{if } c = 4j + 1. \end{cases}$$

Now for even c, we have

$$\Delta_3(n, i, 3) = \left\lfloor \frac{189c + 10}{27} \right\rfloor - \left\lfloor \frac{27c - 4}{27} \right\rfloor - \left\lfloor \frac{162c + 18}{27} \right\rfloor = 1,$$

for $c = 4j - 1$,

$$\Delta_3(n, i, 3) = \left\lfloor \frac{594j - 148}{27} \right\rfloor - \left\lfloor \frac{270j - 58}{27} \right\rfloor - \left\lfloor \frac{324j - 90}{27} \right\rfloor = 1,$$

and for $c = 4j + 1$,

$$\Delta_3(n, i, 3) = \left\lfloor \frac{594j - 256}{27} \right\rfloor - \left\lfloor \frac{270j + 58}{27} \right\rfloor - \left\lfloor \frac{324j - 306}{27} \right\rfloor = 1.$$

Case 4. Let $P_0 = 9t + 5 = 8a + 4$.
Consequently, $t = 8c + 7,\ P_0 = 72c + 68,\ c \geq 0$. Put

$$i = \begin{cases} \frac{3}{8}(P_0 - 36), & \text{if } c \text{ is odd,} \\ \frac{3}{8}(P_0 + 36), & \text{if } c \text{ is even,} \end{cases}$$

or

$$i = \begin{cases} 27c + 12, & \text{if } c \text{ is odd,} \\ 27c + 39, & \text{if } c \text{ is even.} \end{cases}$$

Now for even c, we have

$$\Delta_3(n, i, 3) = \left\lfloor \frac{189c + 164}{27} \right\rfloor - \left\lfloor \frac{27c + 38}{27} \right\rfloor - \left\lfloor \frac{162c + 126}{27} \right\rfloor = 1.$$

As well for odd c, we have

$$\Delta_3(n, i, 3) = \left\lfloor \frac{189c + 191}{27} \right\rfloor - \left\lfloor \frac{27c + 11}{27} \right\rfloor - \left\lfloor \frac{162c + 180}{27} \right\rfloor = 1.$$

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Case 5. Let $P_0 = 9t+7 = 8a+4$. Consequently, $a = 3b$ and $3t = 8b-1$. This implies that b has the form $3c+2$ and therefore $t = 8c+5$. Thus $P_0 = 72c+52,\ c \geq 0$. Put

$$i = \begin{cases} \frac{3}{8}(P_0 - 12), & \text{if } c \text{ is even,} \\ \frac{3}{16}(5P_0 + 36), & \text{if } c = 4j+1\ j \geq 0, \\ \frac{3}{16}(5P_0 - 108), & \text{if } c = 4j-1+1\ j \geq 1, \end{cases}$$

or

$$i = \begin{cases} 27c + 15, & \text{if } c \text{ is even,} \\ 270j - 12, & \text{if } c = 4j+1, \\ 270j + 96, & \text{if } c = 4j-1. \end{cases}$$

Now for even c, we have

$$\Delta_3(n,i,3) = \left\lfloor \frac{189c+140}{27} \right\rfloor - \left\lfloor \frac{27c+14}{27} \right\rfloor - \left\lfloor \frac{162c+126}{27} \right\rfloor = 1,$$

for $c = 4j+1$,

$$\Delta_3(n,i,3) = \left\lfloor \frac{594j+383}{27} \right\rfloor - \left\lfloor \frac{270j-13}{27} \right\rfloor - \left\lfloor \frac{324j+396}{27} \right\rfloor = 1$$

and for $c = 4j-1$,

$$\Delta_3(n,i,3) = \left\lfloor \frac{594j-157}{27} \right\rfloor - \left\lfloor \frac{270j+95}{27} \right\rfloor - \left\lfloor \frac{324j-252}{27} \right\rfloor = 1.$$

Case 6. Let $P_0 = 9t+8 = 8a+4$. Consequently, $t = 8c+4,\ P_0 = 72c+44,\ c \geq 0$. Put

$$i = \begin{cases} \frac{3}{8}(P_0 + 12), & \text{if } c \text{ is even,} \\ \frac{3}{16}(5P_0 + 108), & \text{if } c = 4j+1,\ j \geq 0, \\ \frac{3}{16}(5P_0 - 36), & \text{if } c = 4j-1,\ j \geq 1, \end{cases}$$

or

$$i = \begin{cases} 27c + 21, & \text{if } c \text{ is even,} \\ 270j - 129, & \text{if } c = 4j+1, \\ 270j - 33, & \text{if } c = 4j-1. \end{cases}$$

Now for even c, we have

$$\Delta_3(n,i,3) = \left\lfloor \frac{189c+110}{27} \right\rfloor - \left\lfloor \frac{27c+20}{27} \right\rfloor - \left\lfloor \frac{162c+90}{27} \right\rfloor = 1,$$

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for $c = 4j + 1$,

$$\Delta_3(n, i, 3) = \left\lfloor \frac{594j + 218}{27} \right\rfloor - \left\lfloor \frac{270j + 128}{27} \right\rfloor - \left\lfloor \frac{324j + 90}{27} \right\rfloor = 1,$$

for $c = 4j - 1$,

$$\Delta_3(n, i, 3) = \left\lfloor \frac{594j - 52}{27} \right\rfloor - \left\lfloor \frac{270j - 34}{27} \right\rfloor - \left\lfloor \frac{324j - 18}{27} \right\rfloor = 1.$$

This completes the proof of Lemma 14 and at the same time the proof of Theorem 1. □

3. Proof of Theorem 2

For a prime $p \geq 5$ and an odd number h from the interval $[1, p-4]$, such that $2p-h$ is a prime, let us consider $n = p(2p - h)$. At first we show that for every number $l,\ 1 \leq l \leq \frac{p-h-2}{2}$, the number $i = p\big(\frac{p-h}{2} + l\big) \in B_n$.

Indeed,

$$\begin{aligned}\Delta_2 &= \left\lfloor \frac{n-i-1}{p^2} \right\rfloor - \left\lfloor \frac{i-1}{p^2} \right\rfloor - \left\lfloor \frac{n-2i}{p^2} \right\rfloor \\ &= \left\lfloor \frac{2p - h - \frac{p-h}{2} - l - 1}{p} \right\rfloor - \left\lfloor \frac{\frac{p-h}{2} + l - 1}{p} \right\rfloor - \left\lfloor \frac{p-2l}{p} \right\rfloor \\ &\geq \left\lfloor \frac{\frac{3}{2}p - \frac{h}{2} - l - 1}{p} \right\rfloor - \left\lfloor \frac{\frac{p-h}{2} + \frac{p-h-2}{2} - 1}{p} \right\rfloor - \left\lfloor \frac{p-2l}{p} \right\rfloor \\ &= \left\lfloor \frac{\frac{3}{2}p - \frac{h}{2} - l - 1}{p} \right\rfloor \geq \left\lfloor \frac{\frac{3}{2}p - \frac{h}{2} - \frac{p-h-2}{2} - 1}{p} \right\rfloor = 1.\end{aligned}$$

Thus, $i \in B_n$ and $b(n) \geq \frac{p-h-2}{2}$. Let us show further that indeed we already have found all the elements of B_n.

Notice that for elements $i \in B_n$ of the form $i = p\big(\frac{p-h}{2} + l\big)$ we have

$$p \leq p\left(\frac{p-h}{2} + l\right) \leq \frac{n}{2} = \frac{p}{2}(2p - h),$$

whence

$$-\frac{p-h-2}{2} \leq l \leq \frac{p-1}{2}.$$

We distinguish between the rest of three cases:

Case 1. $-\frac{p-h-2}{2} \leq l \leq 0$,

Case 2. $\frac{p-h}{2} \leq l \leq \frac{p-1}{2}$,

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Case 3. $i = (2p - h)j$, where $1 \le j \le \frac{p-1}{2}$.

Case 1. Consider for $i = p(\frac{p-h}{2} + l)$,

$$\Delta_t(n,i,p) = \left\lfloor \frac{n-i-1}{p^t} \right\rfloor - \left\lfloor \frac{i-1}{p^t} \right\rfloor - \left\lfloor \frac{n-2i}{p^t} \right\rfloor$$

$$= \left\lfloor \frac{\frac{3}{2}p - \frac{h}{2} - l - 1}{p^{t-1}} \right\rfloor - \left\lfloor \frac{\frac{p-h}{2} + l - 1}{p^{t-1}} \right\rfloor - \left\lfloor \frac{p-2l}{p^{t-1}} \right\rfloor.$$

Notice that for $t = 1$ and $t \ge 3$, $\Delta_t = 0$. For $t = 2$, we have:

$$\Delta_2 = \left\lfloor \frac{\frac{3}{2}p - \frac{h}{2} - l - 1}{p} \right\rfloor - \left\lfloor \frac{p-2l}{p} \right\rfloor = \left\lfloor \frac{\frac{3}{2}p - \frac{h}{2} - l - 1}{p} \right\rfloor - 1$$

$$\le \left\lfloor \frac{\frac{3}{2}p - \frac{h}{2} + \frac{p}{2} - 1}{p} \right\rfloor - 1 = \left\lfloor \frac{2p - \frac{h}{2} - 1}{p} \right\rfloor - 1 = 0.$$

Case 2. Again for $t = 1$ and $t \ge 3$, $\Delta_t = 0$.

For $t = 2$, we have

$$\Delta_2 = \left\lfloor \frac{\frac{3}{2}p - \frac{h}{2} - l - 1}{p} \right\rfloor - \left\lfloor \frac{\frac{p-h}{2} + l - 1}{p} \right\rfloor = \left\lfloor \frac{\frac{3}{2}p - \frac{h}{2} - l - 1}{p} \right\rfloor$$

$$\le \left\lfloor \frac{\frac{3}{2}p - \frac{h}{2} - \frac{p-h}{2} - 1}{p} \right\rfloor = 0.$$

Case 3. We have

$$\Delta_t(n,i,2p-h) = \left\lfloor \frac{p-j-1}{(2p-h)^{t-1}} \right\rfloor - \left\lfloor \frac{j-1}{(2p-h)^{t-1}} \right\rfloor - \left\lfloor \frac{p-2j}{(2p-h)^{t-1}} \right\rfloor.$$

Again for $t = 1$ and $t \ge 3$, $\Delta_t = 0$. For $t = 2$, we have

$$\Delta_2 = \left\lfloor \frac{p-j-1}{2p-h} \right\rfloor - \left\lfloor \frac{j-1}{2p-h} \right\rfloor - \left\lfloor \frac{p-2j}{2p-h} \right\rfloor = 0.$$

Thus, $b(n) = \frac{p-h-2}{2} = \frac{q-p}{2} - 1$, where $q = 2p - h$ is a prime. Notice that for $h = 1, 3, \ldots, p-4$, the prime q assumes all values of primes for which $p+4 \le q \le 2p-1$.

Let us consider the equation $b(n) = a,\ n \in \mathbb{N}$. We conclude that at least in the case $2(a+1) = q-p$ for $p \ge 2a+3$, this equation has a solution of the form: $n = pq$. In this case p and q are $(a+1)$-twins. It is well known [10], that the number of $(a+1)$-twin pairs not exceeding x for $a \le \frac{x}{(\log x)^\alpha}$, except, perhaps, of $O(\frac{x}{(\log x)^N})$ from them, where α is an arbitrary small constant and N is an arbitrary large constant, is the magnitude of the order $C_a \frac{x}{\log^2 x}$, where $C_a = C_{tw} \prod_{3 \le p | (a+1)} \frac{p-1}{p-2}$ and $C_{tw} = 2 \prod_{p \ge 3} (1 - \frac{1}{(p-1)^2}) = 1,32032362\ldots$. Hence, for a fixed number a, the number of solutions $n \le x$ of the form $n = pq$, where p, q are $(a+1)$-twin primes, has the order $C_a \frac{\sqrt{x}}{\log x}$. It is a lower bound for the number of all the solutions not exceeding x of the equation $b(n) = a$ for "almost all" a.

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4. Proof of Theorem 3

Let for a prime q the number $p = 2^q - 1$ is a Mersenne prime. Notice that if $i \in B_{2p}$ then i has the form

$$i = 2^a j, \quad 1 \le a \le q-1, \quad j \text{ is odd}, \tag{4}$$

such that $4 \le i \le 2^q - 4$. Suppose that

$$2^{b-1} \le i < 2^b, \quad 3 \le b \le q-1. \tag{5}$$

In the supposition (5) we have in (4): $a \le b - 1$.

Notice that, for real positive numbers α, β, we have: $\lfloor \alpha + \beta \rfloor - \lfloor \alpha \rfloor - \lfloor \beta \rfloor = 1$ if and only if $\{\alpha\} + \{\beta\} \ge 1$. It immediately follows from the equality

$$\lfloor \alpha + \beta \rfloor - \lfloor \alpha \rfloor - \lfloor \beta \rfloor = \{\alpha\} + \{\beta\} - \{\alpha + \beta\}.$$

In connection with (2) denote

$$\delta_t(n, i, p) = \left\{ \frac{i-1}{p^t} \right\} + \left\{ \frac{n-2i}{p^t} \right\}. \tag{6}$$

We distinguish three cases in (6):

Case a. $a = 1,\ j = 2^{b-1} - 1$,

Case b. $a = 1,\ j < 2^{b-1} - 1$,

Case c. $a \ge 2$.

Case a. According to (6), consider

$$\delta_t(2^{q+1} - 2,\ 2^b - 2,\ 2) = \left\{ \frac{2^b - 3}{2^t} \right\} + \left\{ \frac{2^{q+1} - 2^{b+1} + 2}{2^t} \right\}, \quad t = 1, 2, \ldots, q.$$

We have evidently

$$\delta_t(2^{q+1} - 2,\ 2^b - 2,\ 2) = \begin{cases} \dfrac{1}{2}, & t = 1, \\[2ex] \dfrac{2^t - 1}{2^t}, & 2 \le t \le b, \\[2ex] \dfrac{2^t - 2^{b+1} + 2}{2^t}, & b + 1 \le t \le q. \end{cases}$$

Thus, $\delta_t(2^{q+1}-2,\ 2^b-2,\ 2)<1$ and $i\notin B_{2p}$.

Case b. $i=2j,\ 2^{b-2}+1\le j\le 2^{b-1}-3,\ b\ge 4.$

By (7), consider for $t=b$,

$$\delta_b(2^{q+1}-2,\ 2j,\ 2)=\left\{\frac{2j-1}{2^b}\right\}+\left\{\frac{2^{q+1}-2-4j}{2^b}\right\}.$$

Notice that

$$2^{b-1}+1\le 2j-1\le 2^b-7,\quad b\ge 4, \tag{7}$$

consequently,

$$\left\{\frac{2j-1}{2^b}\right\}=\frac{2j-1}{2^b}.$$

Further,

$$2^b+6\le 4j+2\le 2^{b+1}-10.$$

Therefore,

$$\left\{\frac{2^{q+1}-2-4j}{2^b}\right\}=\frac{2^{b+1}-(4j+2)}{2^b}.$$

Thus, using (7) we have

$$\delta_b(2^{q+1}-2,\ 2j,\ 2)=\frac{2^{b+1}-2j-3}{2^b}\ge\frac{2^b+3}{2^b}>1,$$

and we conclude that $i\in B_{2p}$.

Case c. $i=2^aj,\ a\ge 2.$

Here we prove that $2^a\mid\binom{2p-i-1}{i-1}$. To do this, we must find a values of t for which $\delta_t(2^{q+1}-2,\ i,2)\ge 1$. Show that they are $t=2,3,\ldots,a+1$. At first, let $2\le t\le a$. We have

$$\delta_t(2^{q+1}-2,\ 2^aj,\ 2)=\left\{\frac{2^aj-1}{2^t}\right\}+\left\{\frac{2^{q+1}-2^{a+1}j-2}{2^t}\right\} \tag{8}$$

$$=\frac{2^t-1}{2^t}+\frac{2t-2}{2^t}=2-\frac{3}{2^t}\ge\frac{5}{4}. \tag{9}$$

Now let $t=a+1$. Notice that

$$\frac{2^aj-1}{2^{a+1}}=\frac{j-1}{2}+\frac{2^a-1}{2^{a+1}},$$

thus

$$\left\{\frac{2^aj-1}{2^{a+1}}\right\}=\frac{2^a-1}{2^{a+1}}.$$

Therefore, since $a\ge 2$, we have

$$\delta_{a+1}(2^{q+1}-2,\ 2^aj,\ 2)=\frac{2^a-1}{2^{a+1}}+\frac{2^{a+1}-2}{2^{a+1}}=1+\frac{2^a-3}{2^{a+1}}>1.$$

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Hence, we conclude that $2^a \mid \binom{2p-i-1}{i-1}$, and by Lemma 1, $i \in B_{2p}$. In all, we proved that in the segment $[1, p]$, all the even numbers belong to B_{2p}, except the numbers of the form $i = 2^k - 2$, $k \geq 2$. This means, that $b(2p) = \frac{p+1}{2} - \log_2(p+1)$. Since $\varphi(2p) = p - 1$, this completes the proof. □

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5. Numerical Results

1. Below we show all n, $1 \leq n \leq 100$, with given binomial index.

$b(n) = 0.\, n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 17, 19, 20, 21, 23, 24,$
$25, 29, 31, 33, 35, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.$

$b(n) = 1.\, n = 14, 16, 18, 22, 27, 28, 39, 55, 65, 77, 85.$

$b(n) = 2.\, n = 26, 30, 36, 40, 42, 44, 95.$

$b(n) = 3.\, n = 32, 38, 45, 51, 52, 54, 56, 57, 63, 69, 87, 91.$

$b(n) = 4.\, n = 34, 68, 75, 84, 93.$

$b(n) = 5.\, n = 46, 48, 60, 76, 81.$

$b(n) = 6.\, n = 50, 88, 99.$

$b(n) = 7.\, n = 72, 80.$

$b(n) = 8.\, n = 58, 64, 66, 74, 78, 92.$

$b(n) = 9.\, n = 70, 86.$

$b(n) = 10.\, n = 82, 90, 100.$

$b(n) = 11.\, n = 62, 96.$

$b(n) = 15.\, n = 94.$

$b(n) = 16.\, n = 98.$

2. Here we describe the sets B_n for $n \leq 100$, that are not binomial primes.

$B_{14} = \{4\},\ B_{16} = \{6\},\ B_{18} = \{6\},\ B_{22} = \{4\},\ B_{26} = \{6, 8\}, B_{27} = \{6\},$

$B_{28} = \{10\},\ B_{30} = \{4, 8\},\ B_{32} = \{6, 10, 14\},\ B_{34} = \{6, 10, 12, 14\},$

$B_{36} = \{14, 15\},\ B_{38} = \{4, 12, 14\},\ B_{39} = \{6\},\ B_{40} = \{6, 14\},\ B_{42} = \{8, 9\},$

$B_{44} = \{8, 10\},\ B_{45} = \{6, 12, 15\},\ B_{46} = \{4, 8, 10, 12, 20\},$

$B_{48} = \{10, 12, 14, 15, 22\},\ B_{50} = \{6, 12, 14, 15, 16, 22\},$

$B_{51} = \{15, 18, 21\},\ B_{52} = \{12, 14, 18\},\ B_{54} = \{4, 14, 15\},$

$B_{55} = \{20\},\ B_{56} = \{6, 18, 22\},\ B_{57} = \{18, 21, 24\},$

$B_{58} = \{6, 8, 14, 16, 18, 20, 22, 24\}$,

$B_{60} = \{10, 18, 21, 22, 26\}$, $B_{62} = \{4, 8, 10, 12, 16, 18, 20, 22, 24, 26, 28\}$,

$B_{63} = \{6, 15, 24\}$,

$B_{64} = \{6, 10, 14, 18, 22, 26, 28, 30\}$, $B_{65} = \{10\}$,

$B_{66} = \{9, 10, 14, 20, 21, 24, 26, 28\}$,

$B_{68} = \{14, 22, 28, 30\}$, $B_{69} = \{9, 12, 24\}$, $B_{70} = \{4, 12, 14, 15, 20, 22, 24, 26, 30\}$,

$B_{72} = \{6, 14, 15, 22, 26, 28, 33\}$, $B_{74} = \{6, 8, 14, 22, 24, 26, 28, 30\}$,

$B_{75} = \{12, 15, 18, 20\}$,

$B_{76} = \{8, 10, 24, 26, 30\}$, $B_{77} = \{21\}$, $B_{78} = \{4, 8, 10, 15, 20, 21, 26, 28\}$,

$B_{80} = \{6, 12, 14, 22, 26, 28, 38\}$, $B_{81} = \{6, 15, 21, 24, 33\}$,

$B_{82} = \{4, 6, 10, 12, 14, 16, 22, 26, 28, 30\}$, $B_{84} = \{16, 18, 21, 28\}$, $B_{85} = \{20\}$,

$B_{86} = \{4, 12, 14, 16, 18, 20, 28, 30, 36\}$, $B_{87} = \{18, 21, 24\}$,

$B_{88} = \{6, 14, 18, 20, 30, 38\}$,

$B_{90} = \{6, 8, 14, 16, 21, 22, 24, 33, 35, 38\}$, $B_{91} = \{28, 35, 42\}$,

$B_{92} = \{10, 16, 18, 20, 22, 24, 26, 42\}$,

$B_{93} = \{9, 21, 24, 36\}$, $B_{94} = \{4, 8, 10, 12, 16, 18, 20, 22, 24, 26, 28, 36, 40, 42, 44\}$,

$B_{95} = \{15, 40\}$, $B_{96} = \{9, 10, 14, 20, 21, 22, 26, 28, 38, 39, 46\}$,

$B_{98} = \{6, 10, 12, 14, 18, 20, 22, 24, 26, 28, 30, 32, 38, 42, 44, 46\}$,

$B_{99} = \{6, 12, 15, 24, 39, 42\}$, $B_{100} = \{14, 15, 20, 22, 24, 26, 28, 34, 45, 46\}$.

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