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S-exponential numbers

by

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1. Introduction. For any finite or infinite increasing sequence $S = \{s(n)\}_{n\geq 1}$ of positive integers, let us call a positive number N an *S-exponential number*, written $N \in E(S)$, if all exponents in its prime power factorization are in S. Let us agree that $1 \in E(S)$. For example, if $S = \{1\}$, then the 1-exponential numbers form the sequence B of square-free numbers, and, as is well-known,

$$\sum_{i\leq x,\, i\in B} 1 = \frac{6}{\pi^2}x + O(x^{1/2}). \tag{1}$$

Note that if $s(1) > 1$, the density $h(E(S))$ of the set $E(S)$ is zero. Indeed, it is not more than the density of the sequence $E(\Upsilon)$, where $\Upsilon = \{2, 3, 4, \ldots\}$. Note that $E(\Upsilon)$ is also called the set of powerful numbers [6, sequence A001694]. Bateman and Grosswald [1] proved that

$$\sum_{i\leq x,\, i\in E(\Upsilon)} 1 = \frac{\zeta(3/2)}{\zeta(3)}x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)}x^{1/3} + O(x^{1/6}). \tag{2}$$

Thus $h(E(\Upsilon)) = 0$ and, in general, if $s(1) > 1$, then $h(E(S)) = 0$.

If $S = B$, the B-exponential numbers are also known as exponentially square-free numbers (for the first time this notion was introduced by M. V. Subbarao in 1972 [7], see [6, A209061]). These numbers were studied by many authors (for example, see [3], [7, Theorem 6.7], [8], [10]). In these papers, the authors analyzed the Subbarao asymptotic formula

$$\sum_{i\leq x,\, i\in E(B)} 1 = \prod_p\biggl(1 + \sum_{a=4}^{\infty}\frac{\mu^2(a) - \mu^2(a-1)}{p^a}\biggr)x + R(x), \tag{3}$$

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where the product is over all primes, and μ is the Möbius function. The best result of type $R(x) = o(x^{1/4})$ was obtained by Wu (1995) without assuming R.H. (for more details see [10]). In 2007, assuming R.H., Tóth [9] obtained $R(x) = O(x^{1/5+\varepsilon})$, and in 2010, Cao and Zhai [3] found more exactly that $R(x) = Cx^{1/5} + O(x^{38/193+\varepsilon})$, where C is a computable constant. Moreover, Tóth [8] also studied the exponentially k-free numbers for $k \geq 2$.

Let $\mathbf{G}$ be the set of all finite or infinite increasing positive integer sequences $S = \{s(n)\}_{n\geq 1}$ with $s(1) = 1$.

In this paper, without assuming R.H., we obtain a general formula with a remainder term $O(\sqrt{x}\log x e^{c\sqrt{\log x}/\log\log x})$ (c is a constant) not depending on $S \in \mathbf{G}$. More exactly, we prove the following.

Theorem 1. *For every sequence $S \in \mathbf{G}$ the S-exponential numbers have a density $h = h(E(S))$ such that*

$$\sum_{i\leq x,\, i\in E(S)} 1 = h(E(S))x + O(\sqrt{x}\log x e^{c\sqrt{\log x}/\log\log x}), \tag{4}$$

with $c = 4\sqrt{2.4/\log 2} = 7.443083\ldots$ and

$$h(E(S)) = \prod_p \left(1 + \sum_{i\geq 2} \frac{u(i) - u(i-1)}{p^i}\right), \tag{5}$$

where $u(\cdot)$ is the characteristic function of S: $u(n) = 1$ if $n \in S$ and $u(n) = 0$ otherwise.

In particular, for $S = B$ we obtain (3) with a slightly worse remainder term, but which is suitable for all sequences in $\mathbf{G}$.

Recall that a *perfect set* is a closed set with no isolated points.

Theorem 2. *The set $\{h(E(S)) : S \in \mathbf{G}\}$ is a perfect set with countably many gaps. All these gaps are left-sided neighborhoods of the densities of $E(S)$ corresponding to all finite S except for $S = \{1\}$.*

2. Proof of Theorem 1. For the proof Theorem 1 we need a lemma proved earlier (2007) by the author [5, pp. 200–202]. For a fixed square-free number r, denote by B_r the set of square-free numbers n for which $(n, r) = 1$, and set

$$b_r(x) = |B_r \cap \{1, \ldots, x\}|.$$

In particular, $B = B_1$ is the set of all square-free numbers.

Lemma 1.

$$b_r(x) = \frac{6r}{\pi^2}\prod_{p|r}(p+1)^{-1}x + R_r(x),$$

where for every $x \geq 1$ and every $r \in B$,

$$|R_r(x)| \leq \begin{cases} k\sqrt{x}, & r \leq N, \\ ke^{c\sqrt{\log r}/\log\log r}\sqrt{x}, & r \geq N+1, \end{cases}$$

where $k = 3.5\prod_{2\le p\le 23}(1+1/\sqrt{p}) = 57.682607\ldots$ (*in case* $r=1$, $k=3.5$), $c = 4\sqrt{2.4/\log 2} = 7.443083\ldots$, $N = 6469693229$.

Everywhere below, $r(n)$ denotes the product of all distinct prime divisors of n; we set $r(1)=1$.

Let $S \in \mathbf{G}$. Note that the set $E(\Upsilon)\cap E(S)$ contains all numbers of $E(S)$ whose exponents in their prime power factorizations are all more than 1. Evidently, every $y \in E(S)$ has a unique representation as the product of some $a \in E(\Upsilon)\cap E(S)$ and some $b \in B_{r(a)}$. Note that $E(\Upsilon)\cap E(S)$ is always non-empty (by definition, it contains 1). In particular, if y is square-free, then $a=1$, $b=y$ ($\in B_1$). For a fixed $a \in E(\Upsilon)\cap E(S)$, denote the set of $y = ab \in E(S)$ by $E(S)^{(a)}$. Then $E(S) = \bigsqcup_{a\in E(S)\cap E(\Upsilon)} E(S)^{(a)}$ (disjoint union). Consequently, by Lemma 1,

$$\sum_{i\le x,\, i\in E(S)} 1 = b_1(x) + \sum_{4\le a\le x,\, a\in E(S)\cap E(\Upsilon)} b_{r(a)}\left(\frac{x}{a}\right),$$

and since $r(a) = \prod_{p\mid r(a)} p$, we have

$$\sum_{i\le x,\, i\in E(S)} 1 = \frac{6}{\pi^2}\left(1 + \sum_{4\le a\le x,\, a\in E(S)\cap E(\Upsilon)} \prod_{p\mid r(a)}\left(1-\frac{1}{p+1}\right)\frac{1}{a}\right)x + R(x), \tag{6}$$

where

$$|R(x)| \le 3.5\sqrt{x} + \sum_{4\le a\le x,\, a\in E(S)\cap E(\Upsilon)}\left|R_{r(a)}\left(\frac{x}{a}\right)\right| \le 3.5\sqrt{x} + \sum_{\substack{4\le a\le x,\, r(a)\le N\\ a\in E(S)\cap E(\Upsilon)}}\left|R_{r(a)}\left(\frac{x}{a}\right)\right| + \sum_{\substack{4\le a\le x,\, r(a)\ge N+1\\ a\in E(S)\cap E(\Upsilon)}}\left|R_{r(a)}\left(\frac{x}{a}\right)\right| \tag{7}$$

with $N = 6469693229$.

Let $x > N$ go to infinity. We distinguish two cases.

Case (i): $r(a)\le N$. Denote by $E(\Upsilon)(n)$ the nth powerful number (in increasing order). According to (2), $E(\Upsilon)(n) = (\zeta(3)/\zeta(3/2))^2 n^2(1+o(1))$. So, $\sum_{1\le n\le x} 1/\sqrt{E(\Upsilon)(n)} = O(\log x)$. Hence, by Lemma 1,

$$\sum_{\substack{4\le a\le x,\, r(a)\le N\\ a\in E(S)\cap E(\Upsilon)}}\left|R_{r(a)}\left(\frac{x}{a}\right)\right| \le k\sqrt{x}\sum_{4\le a\le x,\, a\in E(S)\cap E(\Upsilon)}\frac{1}{\sqrt{a}} = O(\sqrt{x}\log x).$$

Case (ii): $r(a) > N$. Then, by Lemma 1,

$$\sum_{\substack{4\le a\le x,\, r(a)\ge N+1\\ a\in E(S)\cap E(\Upsilon)}}\left|R_{r(a)}\left(\frac{x}{a}\right)\right| \le k\sqrt{x}\sum_{\substack{4\le a\le x,\, r(a)\ge N+1\\ a\in E(S)\cap E(\Upsilon)}}\frac{1}{\sqrt{a}}e^{c\sqrt{\log r(a)/\log\log r(a)}},$$

where the last sum does not exceed

$$\sum_{N+1\le a\le x,\, r(a)\ge N+1} \frac{1}{\sqrt{a}} e^{c\sqrt{\log a}/\log\log a} \le e^{c\sqrt{\log x}/\log\log x} O(\log x).$$

So, $R(x) = O(\sqrt{x}\log x e^{c\sqrt{\log x}/\log\log x})$, and by (6) we have

$$\sum_{i\le x,\, i\in E(S)} 1 =$$

$$\frac{6}{\pi^2}\Bigg(1+\sum_{4\le a\le x,\, a\in E(S)\cap E(\Upsilon)} \prod_{p|r(a)} \Big(1-\frac{1}{p+1}\Big)\frac{1}{a}\Bigg)x+O(\sqrt{x}\log x e^{c\sqrt{\log x}/\log\log x}).$$

Moreover, if we here replace $\sum_{a\le x,\, a\in E(S)\cap E(\Upsilon)}$ by $\sum_{a\in E(S)\cap E(\Upsilon)}$, then the error does not exceed $6x\pi^{-2}\sum_{n>x} 1/E(\Upsilon)(n) = 6x\pi^{-2}O(1/x) = O(1)$, so the result does not change. Finally,

$$(8)\qquad \sum_{i\le x,\, i\in E(S)} 1 =$$

$$\frac{6}{\pi^2}\Bigg(\sum_{a\in E(S)\cap E(\Upsilon)} \prod_{p|r(a)} \Big(1-\frac{1}{p+1}\Big)\frac{1}{a}\Bigg)x+O(\sqrt{x}\log x e^{c\sqrt{\log x}/\log\log x}).$$

Formula (8) shows that if $S\in \mathbf{G}$, then $E(S)$ has a density.

3. Completion of the proof of Theorem 1. It remains to evaluate the sum (8). For that we follow the scheme of [5, pp. 203–204]. For a fixed $l\in B$, denote by $C(l)$ the set of all $E(S)\cap E(\Upsilon)$-numbers a with $r(a)=l$. Recall that $r(1)=1$. By (8), we have

$$(9)\qquad \sum_{i\le x,\, i\in E(S)} 1 = \frac{6}{\pi^2}x\sum_{l\in B}\prod_{p|l}\Big(1-\frac{1}{p+1}\Big)\sum_{a\in C(l)}\frac{1}{a}+R(x).$$

Consider the function $A\colon \mathbb{N}\to\mathbb{R}$ given by

$$A(l)=\begin{cases}\sum_{a\in C(l)} 1/a, & l\in B,\\ 0, & l\notin B.\end{cases}$$

It is evident that if $l_1, l_2\in B$ and $(l_1,l_2)=1$, then

$$A(l_1l_2)=\sum_{a\in C(l_1l_2)}\frac{1}{a}=\sum_{a\in C(l_1)}\frac{1}{a}\sum_{a\in C(l_2)}\frac{1}{a}=A(l_1)A(l_2),$$

so $A(l)$ is a multiplicative function. Hence the function f defined by

$$f(l)=\prod_{p|l}\Big(1-\frac{1}{p+1}\Big)A(l)$$

is also multiplicative. Moreover,

$$\sum_{n=1}^{\infty} f(n) \le \sum_{n=1}^{\infty} A(n) \le \sum_{a\in E(\Upsilon)} \frac{1}{a} < \infty.$$

Consequently [4, p. 103],

$$\sum_{n=1}^{\infty} f(n) = \prod_p (1 + f(p) + f(p^2) + \cdots). \tag{10}$$

Since $f(p^k) = 0$ for $k \ge 2$, by (9) we obtain

$$\sum_{i\le x,\, i\in E(S)} 1 = \frac{6}{\pi^2} x \sum_{l=1}^{\infty} f(l) + R(x) = \frac{6}{\pi^2} x \prod_p (1 + f(p)) + R(x)$$
$$= \frac{6}{\pi^2} x \prod_p \left(1 + \left(1 - \frac{1}{p+1}\right)\left(\frac{1}{p^{s(2)}} + \frac{1}{p^{s(3)}} + \frac{1}{p^{s(4)}} + \cdots\right)\right) + R(x).$$

Now, since $\sum_{i\ge 2} \frac{1}{p^{s(i)}} = \sum_{i\ge 2} \frac{u(i)}{p^i}$ and $\frac{6}{\pi^2} = \prod_p (1 - \frac{1}{p^2})$, we have

$$\begin{aligned} h(E(S)) &= \frac{6}{\pi^2} \prod_p \left(1 + \left(1 - \frac{1}{p+1}\right) \sum_{i\ge 2} \frac{1}{p^{s(i)}}\right) \\ &= \frac{6}{\pi^2} \prod_p \left(1 + \left(1 - \frac{1}{p+1}\right) \sum_{i\ge 2} \frac{u(i)}{p^i}\right) \\ &= \prod_p \left(\left(1 - \frac{1}{p^2}\right) + \left(1 - \frac{1}{p^2}\right)\left(1 - \frac{1}{p+1}\right) \sum_{i\ge 2} \frac{u(i)}{p^i}\right) \\ &= \prod_p \left(\left(1 - \frac{1}{p^2}\right) + \left(1 - \frac{1}{p}\right) \sum_{i\ge 2} \frac{u(i)}{p^i}\right), \end{aligned} \tag{11}$$

and taking into account that $u(1) = 1$, we find

$$\begin{aligned} h(E(S)) &= \prod_p \left(1 - \frac{1}{p^2} - \left(1 - \frac{1}{p}\right)\frac{1}{p} + \left(1 - \frac{1}{p}\right) \sum_{i\ge 1} \frac{u(i)}{p^i}\right) \\ &= \prod_p \left(\left(1 - \frac{1}{p}\right) + \sum_{i\ge 1} \frac{u(i)}{p^j} - \frac{1}{p} \sum_{i\ge 1} \frac{u(i)}{p^j}\right) \\ &= \prod_p \left(\left(1 - \frac{1}{p}\right) + \frac{1}{p} + \sum_{i\ge 2} \frac{u(i)}{p^j} - \frac{1}{p} \sum_{j\ge 2} \frac{u(j-1)}{p^{j-1}}\right) \\ &= \prod_p \left(1 + \sum_{i\ge 2} \frac{u(i) - u(i-1)}{p^i}\right), \end{aligned}$$

which gives the required evaluation of the sum in (8) and completes the proof of the theorem.

4. A question of D. Berend. Berend [2] posed the following question. Let p_n be the nth prime. Let $\mathrm{A} = \{S_1, S_2, \ldots\}$ be an infinite sequence of sequences $S_i \in \mathbf{G}$. We say that a positive number N is an A-*exponential number* ($N \in E(\mathrm{A})$) if, for every n, whenever p_n divides N, then its exponent in the prime power factorization of N belongs to S_n. We agree that $1 \in E(\mathrm{A})$. How will Theorem 1 change for A-exponential numbers?

An analysis of the proof of Theorem 1 shows that also in this more general case, for every sequence A the density $h(\mathrm{A})$ of the A-exponential numbers exists, and

$$\sum_{i \le x,\, i \in E(\mathrm{A})} 1 = h(E(\mathrm{A}))x + R(x), \tag{12}$$

where $R(x)$ is as in Theorem 1 and

$$h(E(\mathrm{A})) = \prod_{n \ge 1} \left(1 + \sum_{i \ge 2} \frac{u_n(i) - u_n(i-1)}{{p_n}^i}\right), \tag{13}$$

where $u_n(\cdot)$ is the characteristic function of the sequence S_n: $u_n(k) = 1$ if $k \in S_n$, and $u_n(k) = 0$ otherwise.

Example 1. Let

$$\mathrm{A} = \bigl\{S_1 = \{1\}, S_2 = \{1,2\}, \ldots, S_n = \{1, \ldots, n\}, \ldots\bigr\}.$$

Then, by (13),

$$h(E(A)) = \prod_{n \ge 1} \left(1 - \frac{1}{{p_n}^{n+1}}\right) = 0.7210233\ldots.$$

5. The set $\{h(E(S)) : S \in \mathbf{G}\}$. Let $S \in \mathbf{G}$. Then $h(E(S)) \in [6/\pi^2, 1]$. Our question was the following: is the set $\{h(E(S))\}$ dense in this interval? D. Berend [2] gave a negative answer. Indeed, consider the set $\mathbf{G}^{(2)}$ of sequences $S \in \mathbf{G}$ containing 2. Then, evidently, $h(E(S)) \ge h(E(\{1,2\}))$, so that, by Theorem 1,

$$h(E(S)) \ge \prod_p \left(1 - \frac{1}{p^3}\right) \quad \text{for } S \in \mathbf{G}^{(2)}. \tag{14}$$

Now consider the set $\mathbf{G}^{(\overline{2})}$ of sequences $S \in \mathbf{G}$ not containing 2. Then $h(E(S)) \le h(E(\{1,3,4,5,6,\ldots\}))$, so that, by Theorem 1,

$$h(E(S)) \le \prod_p \left(1 - \frac{1}{p^2} + \frac{1}{p^3}\right) = \prod_p \left(1 - \frac{p-1}{p^3}\right) \quad \text{for } S \in \mathbf{G}^{(\overline{2})}. \tag{15}$$

Thus, by (14)–(15), we have a gap in the set $\{h(E(S))\}$ in the interval

$$\left(\prod_p \left(1 - \frac{p-1}{p^3}\right), \prod_p \left(1 - \frac{1}{p^3}\right)\right).$$

Below we use this idea of Berend. We write $\{h(E(S))\}$ instead of $\{h(E(S)) : S \in \mathbf{G}\}$.

We begin the study of $\{h(E(S))\}$ by considering its cardinality.

LEMMA 2. $\mathbf{G}$ *is uncountable.*

Proof. Trivially, $\mathbf{G}$ is equivalent to the set of all subsets of $\{2, 3, 4, \ldots\}$. ∎

LEMMA 3. *For any distinct* $A, B \in \mathbf{G}$, *we have* $h(E(A)) \neq h(E(B))$.

Proof. Let $A = \{a(i) : i \geq 1\}$, $B = \{b(i) : i \geq 1\}$. Let $n \geq 1$ be the maximal index such that $a(i) = b(i)$, $i = 1, \ldots, n$, while $a(n+1) \neq b(n+1)$ (the case $n = 0$ is impossible, since by assumption $a(1) = b(1) = 1$).

Set

$$A_n = \{a(1), \ldots, a(n)\},$$
$$A_n^* = \{a(1), \ldots, a(n), a(n)+1, a(n)+2, \ldots\}.$$

Then

$$h(E(A_{n+1})) \leq h(E(A)) \leq h(E(A_{n+1}^*)), \tag{16}$$

and analogously for B.

We distinguish four cases:

(i) $a(n+1) = a(n)+1$, $b(n+1) \geq a(n)+2$;
(ii) $a(n+1) \geq a(n)+k$, $b(n+1) = a(n)+1$, $k \geq 2$;
(iii) $a(n+1) = a(n)+k$, $a(n)+2 \leq b(n+1) \leq a(n)+k-1$, $k \geq 3$;
(iv) $a(n+1) = a(n)+k$, $b(n+1) \geq a(n)+k+1$, $k \geq 2$.

(i) By (5) and (16), we have

$$h(E(A)) \geq \prod_p \left(1 + \sum_{i=2}^{a(n)} \frac{u(i)-u(i-1)}{p^i}\right), \tag{17}$$

where $u(\cdot)$ is the characteristic function of A. On the right-hand side we sum to $a(n)$ since here $u(a(n+1)) - u(a(n+1)-1) = 0$. On the other hand,

$$h(E(B_{n+1}^*)) \leq \prod_p \left(1 + \sum_{i=2}^{a(n)} \frac{u(i)-u(i-1)}{p^i} - \frac{1}{p^{a(n)+1}} + \frac{1}{p^{a(n)+2}}\right). \tag{18}$$

By (17)–(18), $h(E(B)) < h(E(A))$.

(ii) Symmetrically to (i), we have

$$h(E(B)) \geq \prod_p \left(1 + \sum_{i=2}^{a(n)} \frac{u(i)-u(i-1)}{p^i}\right). \tag{19}$$

On the other hand,

$$(20)\quad h(E(A^*_{n+1}))\le\prod_p\Big(1+\sum_{i=2}^{a(n)}\frac{u(i)-u(i-1)}{p^i}-\frac{1}{p^{a(n)+1}}+\frac{1}{p^{a(n)+2}}\Big).$$

Thus $h(E(A))<h(E(B))$.

(iii) Again, by (5) and (16), we have

$$(21)\quad h(E(B))\ge\prod_p\Big(1+\sum_{i=2}^{a(n)}\frac{u(i)-u(i-1)}{p^i}-\frac{1}{p^{a(n)+1}}+\frac{1}{p^{a(n)+k-1}}\Big),$$

while

$$(22)\quad h(E(A^*_{n+1}))\le\prod_p\Big(1+\sum_{i=2}^{a(n)}\frac{u(i)-u(i-1)}{p^i}-\frac{1}{p^{a(n)+1}}+\frac{1}{p^{a(n)+k}}\Big).$$

Hence, $h(E(A))<h(E(B))$.

(iv) Symmetrically,

$$(23)\quad h(E(B^*_{n+1}))\le\prod_p\Big(1+\sum_{i=2}^{a(n)}\frac{u(i)-u(i-1)}{p^i}-\frac{1}{p^{a(n)+1}}+\frac{1}{p^{a(n)+k+1}}\Big),$$

while

(24)

$$h(E(A))\ge\prod_p\Big(1+\sum_{i=2}^{a(n)}\frac{u(i)-u(i-1)}{p^i}-\frac{1}{p^{a(n)+1}}+\frac{1}{p^{a(n)+k}}-\frac{1}{p^{a(n)+k+1}}\Big)$$

and since $2/p^{a(n)+k+1}\le 1/p^{a(n)+k}$ with equality only for $p=2$, we have $h(E(A))>h(E(B))$. ∎

Corollary 1. *Let $A,B\in\mathbf{G}$ be distinct. Let $s^*=s^*(A,B)$ be the smallest number in the symmetric difference of A and B. If, say, $s^*\in A$, then $h(E(A))>h(E(B))$.*

Proof. Referring to the proof of Lemma 3, we have $s^*(A,B)=n+1$. We see that in all four cases in the proof of Lemma 3, the corollary is confirmed. ∎

Lemmas 2 and 3 directly imply

Lemma 4. *The set $\{h(E(S)\}$ is uncountable.*

Denote by $\mathbf{G}_{\text{fin}}$ the subset of all finite sequences in $\mathbf{G}$. Since the set of all finite subsets of a countable set is countable, $\mathbf{G}_{\text{fin}}$ is countable and so is the set $\{h(E(S)):S\in\mathbf{G}_{\text{fin}}\}$.

6. Proof of Theorem 2. Our proof is divided into three parts: 1) accumulation points; 2) gaps; 3) perfectness.

6.1. Accumulations

LEMMA 5. *Every point $h \in \{h(E(S))\}$ is an accumulation point.*

Proof. We distinguish two cases: when S is finite and when S is infinite.

Let $S = \{s(1), \ldots, s(k)\} \in \mathbf{G}_{\mathrm{fin}}$. Let $n \geq s(k) + 2$. Denote by S_n the sequence $\{s(1), \ldots, s(k), n\}$. Then, by (5),

$$h(E(S_n)) = \prod_p \biggl(1 + \sum_{i=2}^{s(k)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{s(k)+1}} + \frac{1}{p^n}\biggr) \tag{25}$$

$$= \exp\biggl(\sum_p \log\biggl(1 + \sum_{i=2}^{s(k)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{s(k)+1}} + \frac{1}{p^n}\biggr)\biggr).$$

The sum over all primes converges uniformly over n since it is majorized by the convergent series

$$\sum_p \sum_{i\geq 2} \frac{|u(i) - u(i-1)|}{p^i} \qquad \biggl(\leq \sum_p \sum_{i\geq 2} \frac{1}{p^i} = \sum_p \frac{1}{(p-1)p}\biggr).$$

Hence

$$\lim_{n\to\infty} h(E(S_n)) = \prod_p \biggl(1 + \sum_{i=2}^{s(k)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{s(k)+1}}\biggr),$$

and by (5), $\lim_{n\to\infty} h(E(S_n)) = h(E(S))$.

Let now $S = \{s(1), \ldots, s(k), \ldots\} \in \mathbf{G}$ be an infinite sequence. Let $S_n = \{s(1), \ldots, s(n)\}$ be the n-part of S. In the same way, taking into account the uniform convergence of the product for the density of S_n, we find that $\lim_{n\to\infty} h(E(S_n)) = h(E(S))$. ∎

6.2. Gaps. Let us show that, for every finite $S \in \mathbf{G}$ except $S = \{1\}$, there exists an $\varepsilon > 0$ such that no value of h is in $\bigl(h(E(S)) - \varepsilon, h(E(S))\bigr)$.

We need a lemma.

LEMMA 6. *Let*

$$S_1 = \{s(1), \ldots, s(k)\} \in \mathbf{G}_{\mathrm{fin}}, \qquad k \geq 2,$$
$$S_2 = \{s(1), \ldots, s(k-1), s(k) + 1, s(k) + 2, \ldots\}.$$

Then the interval

$$\bigl(h(E(S_2)), h(E(S_1))\bigr) \tag{26}$$

is a gap in the set $\{h(E(S)) : S \in \mathbf{G}\}$.

Proof. Consider in addition to S_1, S_2 any sequence $S \in \mathbf{G}$, $S \neq S_1, S_2$, which contains $s^*(S_1, S)$. By Corollary 1, $h(E(S)) > h(E(S_1))$. Thus $h(E(S))$ is not in the interval (26). Now consider in addition to S_1, S_2 any sequence $S \in \mathbf{G}$ which does not contain $s^*(S_1, S)$. Then S_2 contains $s^*(S, S_2)$. Indeed, 1) S cannot contain all terms $s(1), \ldots, s(k)$ (since S differs from S_1, it should contain additional terms, the smallest of which is $s^*(S, S_1) \in S$, contrary to assumption); 2) if i, $1 \le i \le k$, is the smallest for which S misses $s(i)$, then, by assumption, all terms of S are larger than $s(i)$. So $s^*(S, S_2) = s(i) \in S_2$ if $i < k$, while if $i = k$, since S differs from S_2, we have $s^*(S, S_2) = s(k)+j \in S_2$, where j is the smallest for which $s_k + j$ is not in S. Hence, by Lemma 6, $h(E(S_2)) > h(E(S))$, and again $h(E(S))$ is not in the interval (26). ∎

LEMMA 7. *All gaps of the set $\{h(E(S))\}$ are left-sided neighborhoods of the densities of $E(S)$ corresponding to all finite S except for $S = \{1\}$.*

Proof. By Lemma 6, for every $S = S_1 \in \mathbf{G}_{\text{fin}}$ except $S = \{1\}$, we have a gap of type (26) which is a left neighborhood of $h(E(S))$. Let us show that it is the only situation when a gap appears. Suppose a sequence $S \in \mathbf{G}$ does not contain an infinite set L of positive integers. Adding $l \in L$ to S, which goes to infinity, we obtain a set S_l such that $h(E(S_l)) > h(E(S))$ and $h(E(S_l)) \to h(E(S))$. So, in a right neighborhood of $h(E(S))$ there cannot be a gap of $\{h(E(S))\}$. In the opposite case, when $S \in \mathbf{G}$ only misses a finite set of positive integers, in a right neighborhood of $h(E(S))$ a gap of $\{h(E(S))\}$ is possible, but in this case S has the form of S_2 in Lemma 6. Also, if $S \in \mathbf{G}$ is infinite, then in a left neighborhood of $h(E(S))$ there cannot be a gap of $\{h(E(S))\}$, since $h(E(S))$ is a limiting point of $\{h(E(S_n))\}$, where S_n is the n-part of S. ∎

It is easy to see that for distinct sequences S_1, the gaps (26) are disjoint. From Lemmas 6 and 7 we deduce

LEMMA 8. *The set $\{h(E(S))\}$ has countably many gaps.*

6.3. Perfectness

LEMMA 9. *The set $\{h(E(S))\}$ is perfect.*

Proof. By Lemma 5, the set $\{h(E(S))\}$ has no isolated points. For a set $A \subseteq [6/\pi^2, 1]$, let $\overline{A} = [6/\pi^2, 1] \setminus A$. Let further $\{g\}$ be the set of all gaps of $\{h(E(S))\}$. Then

$$\{h(E(S))\} = \overline{\bigcup g} = \bigcap \overline{g}.$$

Since a gap g is an open interval, $\overline{g}$ is closed. But arbitrary intersections of closed sets are closed. Thus the set $\{h(E(S))\}$ is closed without isolated points. So it is a perfect set. ∎

This completes the proof of Theorem 2.

7. An open problem. It is natural to conjecture that the sum of lengths of all gaps equals the length of the whole interval $[6/\pi^2, 1]$, or equivalently, the set $\{h(E(S))\}$ has measure zero. This question we leave open.

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Abstract (will appear on the journal's web site only)

We prove that, for every set S of positive integers containing 1 (finite or infinite), the density $h(E(S))$ of the set $E(S)$ of numbers that have prime factorizations with exponents only from S exists, and we give an explicit formula for it. Further, we study the set of such densities for all S and prove that it is a perfect set with a countable set of gaps which are some left-sided neighborhoods of the densities corresponding to all finite S except for $S = \{1\}$.