# ON A SEQUENCE OF POLYNOMIALS WITH HYPOTHETICALLY INTEGER COEFFICIENTS

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## Abstract

The first author introduced a sequence of polynomials defined recursively. One of the main results of this study is proof of the integrality of its coefficients.

## 1. Introduction

In point of fact, there are only a few examples of sequences known where the question of the integrality of the terms is a difficult problem. In 1989, Somos [9] posed a problem on the integrality of sequences depending on parameter $k \geq 4$ which are defined by the recursion

$$a_n = \frac{\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} a_{n-j} a_{n-(k-j)}}{a_{n-k}}, \;\; n \geq k \geq 4,$$

with the initial conditions $a_i = 1, \;\; i = 1, \ldots, k-1$.

Gale [3] proved the integrality of Somos sequences when $k = 4$ and 5, attributing a proof to Malouf [4]. Hickerson and Stanley (see [6]) independently proved the integrality of the $k = 6$ case in unpublished work and Fomin and Zelevinsky (2002) gave the first published proof. Finally, Lotto (1990) gave an unpublished proof for the $k = 7$ case. These are sequences A006720-A006723 in [8]. It is interesting that, for $k \geq 8$, the property of integrality disappears (see sequence A030127 in [8]). In connection with this, note that in the so-called Göbel's sequence ([11]) defined by the recursion

$$x_n = \frac{1}{n}(1 + \sum_{i=0}^{n-1} x_i^2), \;\; n \geq 1, \;\; x_0 = 1,$$

the first non-integer term is $x_{43} = 5.4093 \times 10^{178485291567}$.

In this paper we study the Shevelev sequence of polynomials $\{P_n(x)\}_{n \geq 1}$ that are defined by the following recursion: $P_1 = 1$, $P_2 = 1$, and, for $n \geq 2$,

$$4(2x+n)P_{n+1}(x) = 2(x+n)P_n(x) +$$

$$(2x+n)P_n(x+1) + (4x+n)l_n(x), \text{ if } n \text{ is odd,} \qquad (1)$$

$$4P_{n+1}(x) = 4(x+n)P_n(x) +$$

$$2(2x+n+1)P_n(x+1) + (4x+n)l_{n-1}(x), \text{ if } n \text{ is even,} \qquad (2)$$

where

$$l_n(x) = (x + \frac{n-1}{2})(x + \frac{n-3}{2}) \cdots (x+1).$$

The first few polynomials are ([8], sequence A174531):

$$P_1 = 1,$$

$$P_2 = 1,$$

$$P_3 = 3x + 4,$$

$$P_4 = 2x + 4,$$

$$P_5 = 5x^2 + 25x + 32,$$

$$P_6 = 3x^2 + 19x + 32,$$

$$P_7 = 7x^3 + 77x^2 + 294x + 384,$$

$$P_8 = 4x^3 + 52x^2 + 240x + 384,$$

$$P_9 = 9x^4 + 174x^3 + 1323x^2 + 4614x + 6144,$$

$$P_{10} = 5x^4 + 110x^3 + 967x^2 + 3934x + 6144,$$

$$P_{11} = 11x^5 + 330x^4 + 4169x^3 + 27258x^2 + 90992x + 122880,$$

$$P_{12} = 6x^5 + 200x^4 + 2842x^3 + 21040x^2 + 79832x + 122880.$$

According to our observations, the following conjectures are natural.

1) The coefficients of all the polynomials are integers. Moreover, the greatest common divisor of all coefficients is $n/rad(n)$, where $rad(n) = \prod_{p|n} p$;

2) $P_n(0) = 4^{\lfloor \frac{n-1}{2} \rfloor} \lfloor \frac{n-1}{2} \rfloor!$;

3) For even $n$, $P_n(1) = (2^n - 1)(\frac{n}{2})!/(n+1)$, and for odd $n$, $P_n(1) = (2^n - 1)(\frac{n-1}{2})!$;

4) $P_n(x)$ has a real rational root if and only if either $n = 3$ or $n \equiv 0 \pmod 4$. In the latter case, such a unique root is $-\frac{n}{2}$;

5) Coefficients of $x^k$ increase when $k$ decreases;

6) If $n$ is even, then the coefficients of $P_n$ do not exceed the corresponding coefficients of $P_{n-1}$ and the equality holds only for the last ones; moreover, the ratios of coefficients of $x^k$ of polynomials $P_{n-1}$ and $P_n$ monotonically decrease to 1 when $k$ decreases;

7) All coefficients of $P_n$, except of the last one, are multiples of $n$ if and only if $n$ is prime.

The main results of our paper consist of the following two theorems.

**Theorem 1.** (Explicit formula for $P_n(k)$) *For an integer $k$ we have*

$$P_n(k) = \begin{cases} \left(\binom{(n-1)/2+k-1}{k-1} / \binom{n+2k-2}{k-1}\right)(\frac{n-1}{2})! \ T_n(k), & \text{if } n \geq 1 \text{ is odd} \\ \\ \left(\binom{n/2+k-1}{k} / \binom{n+2k-1}{k}\right)(\frac{n}{2} - 1)! \ T_n(k), & \text{if } n \geq 2 \text{ is even,} \end{cases} \tag{3}$$

$$= 2^{-(\lfloor \frac{n}{2} \rfloor + k - 1)} \frac{(n + k - 1)!}{(2\lfloor \frac{n}{2} \rfloor + 2k - 1)!!} T_n(k), \tag{4}$$

*where*

$$T_n(k) = \sum_{i=1}^{n} 2^{i-1} \binom{n + 2k - i - 1}{k - 1}. \tag{5}$$

Using Theorem 1, we prove Conjectures (2), (3) and the following main result.

**Theorem 2.** *For $n \geq 1$, $P_n(x)$ is a polynomial of degree $\lfloor \frac{n-1}{2} \rfloor$ with integer coefficients.*

Nevertheless, the subtle second part of Conjecture (1) remains open.

## 2. Representation of $P_n(k)$ Via a Polynomial in $n$ of Degree $k - 1$ with Integer Coefficients

**Theorem 3.** *For integers $k \geq 1$, $n \geq 1$, the following recursion holds*

$$P_n(k) = c_n(k)\left(2^{n+k-1} - \frac{R_k(n)}{(2k-2)!!}\right), \tag{6}$$

*where $R_k(n)$ is a polynomial in $n$ of degree $k - 1$ with integer coefficients and*

$$c_n(k) = \begin{cases} (\frac{n-1}{2})! \prod_{i=1}^{k-1} \frac{n+i}{n+2i}, & \text{if } n \text{ is odd,} \\ \\ \frac{1}{2}(\frac{n}{2} - 1)! \prod_{i=0}^{k-1} \frac{n+i}{n+2i+1}, & \text{if } n \text{ is even .} \end{cases} \tag{7}$$

*Proof.* Write (3) and (4) in the form

$$P_n(k+1) = -\frac{2f}{g}P_n(k) + 4P_{n+1}(k) - \frac{h}{g}\left(\frac{n-1}{2}\right)!\binom{\frac{g-1}{2}}{k}, \ \ if \ n \equiv 1 \ (\text{mod } 2); \quad (8)$$

$$P_n(k+1) = -\frac{2f}{g+1}P_n(k) + \frac{2}{g+1}P_{n+1}(k) - \frac{h}{2(g+1)}\left(\frac{n}{2}-1\right)!\binom{\frac{g}{2}-1}{k} \quad (9)$$

$$if \ n \equiv 0 \ (\text{mod } 2),$$

where $f = n + k, \ g = n + 2k, \ h = n + 4k$.

Let $n$ be odd. We use induction over $k$. For $k = 1$, (6) gives

$$R_1(n) = 2^n - \frac{P_n(1)}{c_n(1)} = Const(k). \quad (10)$$

Thus the base of induction is valid. Suppose the theorem is true for some value of $k$. Then, using this supposition and (6) to (9), we have

$$P_n(k+1) =$$

$$-\frac{2f}{g}\left(\frac{n-1}{2}\right)!\left(2^{n+k-1} - \frac{R_k(n)}{(2k-2)!!}\right)\prod_{i=1}^{k-1}\frac{n+i}{n+2i}+$$

$$2\left(\frac{n-1}{2}\right)!\left(2^{n+k} - \frac{R_k(n+1)}{(2k-2)!!}\right)\prod_{i=0}^{k-1}\frac{n+i+1}{n+2i+2}-$$

$$\frac{h}{g}\left(\frac{n-1}{2}\right)!\frac{\frac{g-1}{2}\frac{g-3}{2}\cdots\frac{n+1}{2}}{k!}.$$

Note that

$$\frac{f}{g}\prod_{i=0}^{k-1}\frac{n+i}{n+2i} = \prod_{j=1}^{k}\frac{n+j}{n+2j} = \prod_{i=0}^{k-1}\frac{n+i+1}{n+2i+2}.$$

Therefore,

$$P_n(k+1) = \left(\frac{n-1}{2}\right)!\left(-2^{n+k} + \frac{2R_k(n)}{(2k-2)!!} + 2^{n+k+1}-\right.$$

$$\left.\frac{2R_k(n+1)}{(2k-2)!!} - \frac{h}{g}\frac{\frac{g-1}{2}\frac{g-3}{2}\cdots\frac{n+1}{2}}{k!}\prod_{j=1}^{k}\frac{n+2j}{n+j}\right)\prod_{j=1}^{k}\frac{n+j}{n+2j}.$$

Here we note that

$$(g-1)(g-3)\cdots(n+1)\prod_{j=1}^{k}\frac{n+2j}{n+j} = (n+2k)_k,$$

where $(x)_k$ is a falling factorial. Hence

$$P_n(k+1) = c_n(k+1)\Big(2^{n+k} - 2\frac{R_k(n+1) - R_k(n)}{(2k-2)!!} - $$

$$\frac{4k+n}{(2k)!!}(n+2k-1)_{k-1}\Big) = c_n(k+1)\Big(2^{n+k} - \frac{R_{k+1}(n)}{(2k)!!}\Big),$$

where

$$R_{k+1}(n) = 4k(R_k(n+1) - R_k(n)) + (4k+n)(n+2k-1)_{k-1}. \qquad (11)$$

Since, by the inductive supposition, $R_k(n)$ is a polynomial of degree $k-1$ with integer coefficients, then, by (11), $R_{k+1}(n)$ is a polynomial of degree $k$ with integer coefficients. Note that the case of even $n$ is considered quite analogously, obtaining the *same* formula (11). $\qquad\square$

In (6) and (7), put $n = 1$. Then, for $k \geq 1$ we have

$$\Big(2^k - \frac{R_k(1)}{(2k-2)!!}\Big)\frac{k!}{(2k-1)!!} = 1,$$

from which

$$R_k(1) = (k-1)!\Big(2^{2k-1} - \binom{2k-1}{k}\Big). \qquad (12)$$

In particular, $R_1(1) = 1$ and, since $R_1(n)$ is of degree 0, $R_1(n) = 1$. Further, we find polynomials $R_k(n)$ using the recursion (11). The first polynomials $R_k(n)$ are

$$R_1(n) = 1,$$

$$R_2(n) = n + 4,$$

$$R_3(n) = n^2 + 11n + 32,$$

$$R_4(n) = n^3 + 21n^2 + 152n + 384,$$

$$R_5(n) = n^4 + 34n^3 + 443n^2 + 2642n + 6144,$$

$$R_6(n) = n^5 + 50n^4 + 1015n^3 + 10510n^2 + 55864n + 122880.$$

## 3. Proof of Conjectures (2) and (3)

We start with the proof of Conjecture (3) for $P_n(1)$.

*Proof.* Note that, since $R_1(n) = 1$, from (10) we find

$$P_n(1) = c_n(1)(2^n - 1). \tag{13}$$

Besides, by (7), we have

$$c_n(1) = \begin{cases} (\frac{n-1}{2})!, & if \ n \ is \ odd, \\ \\ \frac{1}{2}(\frac{n}{2} - 1)!\frac{n}{n+1} = (\frac{n}{2})!/(n+1), & if \ n \ is \ even, \end{cases} \tag{14}$$

and Conjecture (3) follows. $\square$

Let us now prove Conjecture (2).

*Proof.* Note that (8) and (9), as in (1) and (2), are valid for every nonnegative $k$. For $k = 0$ and odd $n \geq 1$, (8) gives

$$P_n(1) = -2P_n(0) + 4P_{n+1}(0) - \left(\frac{n-1}{2}\right)!,$$

or, using (13) and (14), we have

$$4P_{n+1}(0) - 2P_n(0) = 2^n \left(\frac{n-1}{2}\right)!$$

Analogously, for $k = 0$ and even $n \geq 1$, from (9), (13) and (14) we find

$$P_{n+1}(0) - nP_n(0) = 2^{n-1}\left(\frac{n}{2}\right)!$$

Thus

$$P_{n+1}(0) = \begin{cases} \frac{1}{2}P_n(0) + 2^{n-2}(\frac{n-1}{2})!, & \text{if } n \text{ is odd}, \\ \\ nP_n(0) + 2^{n-1}(\frac{n}{2})!, & \text{if } n \text{ is even} \end{cases}$$

with $P_1(0) = 1$, $P_2(0) = 1$. Since the difference equation

$$y(n+1) = \begin{cases} \frac{1}{2}y(n) + 2^{n-2}(\frac{n-1}{2})!, & \text{if } n \text{ is odd}, \\ \\ ny(n) + 2^{n-1}(\frac{n}{2})!, & \text{if } n \text{ is even} \end{cases}$$

with the initial values $y(1) = 1$, $y(2) = 1$ has an unique solution, it is sufficient to verify that $y(n) = P_n(0) = 4^{\lfloor \frac{n-1}{2} \rfloor} \lfloor \frac{n-1}{2} \rfloor!$ is a solution. $\square$

## 4. Explicit Formula for $R_k(n)$

Since from (11)

$$4kR_k(n+1) = 4kR_k(n)+$$

$$R_{k+1}(n) - (4k+n)(n+2k-1)_{k-1}, \tag{15}$$

we have a recursion in $n$ for $R_k(n)$ given by (12) and (15).
Our aim in this section is to find a generalization of (12) for an arbitrary integer
$n \geq 1$. Note that we can write (12) in the form

$$R_k(1) = 2(k-1)!4^{k-1} - \frac{(2k-1)!}{k!}.$$

Using (15) and (12), after some transformations, we find

$$R_k(2) = 2^2(k-1)!4^{k-1} - 2\frac{(2k-1)!}{k!} - \frac{(2k)!}{(k+1)!}.$$

The regularity is fixed in the following theorem.

**Theorem 4.** *For integer $k \geq 1$, $n \geq 1$, we have*

$$R_k(n) = 2^n(k-1)!4^{k-1} - \sum_{i=1}^{n} 2^{n-i}\frac{(2k+i-2)!}{(k+i-1)!}. \tag{16}$$

*Proof.* Taking into account that $\frac{(2k+i-2)!}{(k+i-1)!} = \binom{2k+i-2}{k-1}(k-1)!$, we prove (16) in the
following equivalent form:

$$R_k(n) = 2^n(k-1)!\left(4^{k-1} - \sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1}\right). \tag{17}$$

We use induction over $n$. Suppose that (17) is valid for some value of $n$ and an
arbitrary integer $k \geq 1$. By (15), we have

$$R_k(n+1) = 2^n(k-1)!\left(4^{k-1} - \sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1}\right)+$$

$$2^{n-2}(k-1)!\left(4^k - \sum_{i=1}^{n} 2^{-i}\binom{2k+i}{k}\right) - \frac{4k+n}{4k}(n+2k-1)_{k-1} =$$

$$2^n(k-1)!\left(4^{k-1} - \sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1}\right) + 2^n(k-1)!\left(4^{k-1} - \sum_{i=1}^{n} 2^{-i-2}\binom{2k+i}{k}\right) -$$

$$\frac{(n+2k-1)!}{(n+k)!} - \frac{n}{4k}\frac{(n+2k-1)!}{(n+k)!}.$$

Thus we should prove the identity

$$2^{n+1}(k-1)!4^{k-1} - 2^n(k-1)! \sum_{i=1}^{n} 2^{-i} \binom{2k+i-2}{k-1} -$$

$$2^{n-2}(k-1)! \sum_{i=1}^{n} 2^{-i} \binom{2k+i}{k} - \frac{n+4k}{4k} \frac{(n+2k-1)!}{(n+k)!} =$$

$$2^{n+1}(k-1)! \left( 4^{k-1} - \sum_{i=1}^{n+1} 2^{-i} \binom{2k+i-2}{k-1} \right),$$

which is easily reduced to the identity

$$4 \sum_{i=1}^{n} 2^{-i} \binom{2k+i-2}{k-1} - \sum_{i=1}^{n} 2^{-i} \binom{2k+i}{k} =$$

$$2^{-n} \frac{n+4k}{4k} \frac{(n+2k-1)!}{(n+k)!} - 4 \cdot 2^{-n} \binom{2k+n-1}{k-1}.$$

Note that, the right hand part is $\frac{n}{k2^n} \binom{2k+n-1}{k-1}$. Therefore, it is left to prove the identity

$$4 \sum_{i=1}^{n} 2^{-i} \binom{2k+i-2}{k-1} - \sum_{i=1}^{n} 2^{-i} \binom{2k+i}{k} = \frac{n}{2^n k} \binom{2k+n-1}{k-1}.$$

Since this is trivially satisfied for $n = 0$, it is sufficient to verify the equality of the first differences of the left and the right hand parts, which is reduced to the identity

$$2(n+2k-1) \binom{2k+n-2}{k-1} = n \binom{2k+n-1}{k-1} + k \binom{2k+n}{k},$$

which is verified directly.                                                  □

## 5. Proof of Theorem 1

Now we are able to prove Theorem 1.

*Proof.* According to (5), we have

$$T_n(k) = \sum_{i=1}^{n} 2^{i-1} \binom{n+2k-i-1}{k-1} = \sum_{j=1}^{n} 2^{n-j} \binom{2k+j-2}{k-1}. \tag{18}$$

Hence, by (17), we find

$$R_k(n) = 2^n(k-1)!(4^{k-1} - 2^{-n} T_n(k)) =$$

$$(k-1)!(2^{n+2k-2} - T_n(k)). \tag{19}$$

Now from (6) and (19) we have

$$P_n(k) = 2^{-(k-1)}c_n(k)T_n(k). \tag{20}$$

Let $n$ be odd. Note that, by (7),

$$2^{-(k-1)}c_n(k) =$$

$$2^{-(k-1)}\Big(\frac{n-1}{2}\Big)! \frac{(n+k-1)(n+k-2)\cdots(n+1)}{(n+2k-2)(n+2k-4)\cdots(n+2)} =$$

$$2^{-(k-1)}\Big(\frac{n-1}{2}\Big)! \frac{(n+k-1)!n!!}{n!(n+2k-2)!!}. \tag{21}$$

Taking into account that

$$n!! = \frac{n!}{(n-1)!!} = \frac{n!}{2^{\frac{n-1}{2}}\big(\frac{n-1}{2}\big)!}, \tag{22}$$

we find from (21)

$$2^{-(k-1)}c_n(k) = \frac{(n+k-1)!\big(\frac{n-1}{2}+k-1\big)!}{(n+2k-2)!} =$$

$$\frac{\big(\frac{n-1}{2}+k-1\big)!}{(k-1)!\binom{n+2k-2}{k-1}} = \frac{\binom{\frac{n-1}{2}+k-1}{k-1}}{\binom{n+2k-2}{k-1}}\Big(\frac{n-1}{2}\Big)!$$

and (3) follows from (20). Furthermore, since by (22) $\frac{n!!\big(\frac{n-1}{2}\big)!}{n!} = 2^{-\frac{n-1}{2}}$, from (20) and (21) we find

$$P_n(k) = 2^{-\big(\frac{n-1}{2}+k-1\big)}\frac{(n+k-1)!}{(n+2k-2)!!}T_n(k)$$

corresponds to (4) in the case of odd $n$. The case of even $n$ is considered quite analogously. $\qquad\square$

## 6. Bisection of Sequence $\{P_n(x)\}$

Note that $T_n(k)$, (5), has rather a simple structure, which allows us to find different relations for it. Using (3) and (4), we are able to find recursion relations for $P_n(x)$ which are simpler than the basis recursion (1) and (2). We start with the following simple recursions for $T_n(k)$.

**Lemma 5.**

$$T_n(k) - 2T_{n-1}(k) = \binom{n+2k-2}{k-1}, \quad k \geq 1; \tag{23}$$

$$T_n(k) - 4T_{n-2}(k) = \binom{n+2k-2}{k-1} + 2\binom{n+2k-3}{k-1}, \quad k \geq 2. \tag{24}$$

*Proof.* By (5), we have

$$T_n(k) - 2T_{n-1}(k) =$$

$$\sum_{i=1}^{n} 2^{i-1}\binom{n+2k-i-1}{k-1} - \sum_{j=1}^{n-1} 2^j\binom{n+2k-j-2}{k-1} =$$

$$\sum_{i=1}^{n} 2^{i-1}\binom{n+2k-i-1}{k-1} - \sum_{i=2}^{n} 2^{i-1}\binom{n+2k-i-1}{k-1}$$

and (23) follows; (24) is a simple corollary of (23). □

**Theorem 6.** (Bisection) *If $n \geq 3$ is odd, then*

$$(2x+n-2)P_n(x) = 2(x+n-1)(x+n-2)P_{n-2}(x)+$$

$$(4x+3n-4)(x+\frac{n-1}{2}-1)(x+\frac{n-1}{2}-2)\cdots x; \tag{25}$$

*if $n \geq 4$ is even, then*

$$(2x+n-1)P_n(x) = 2(x+n-1)(x+n-2)P_{n-2}(x)+$$

$$\frac{1}{2}(4x+3n-4)(x+\frac{n-2}{2}-1)(x+\frac{n-2}{2}-2)\cdots x. \tag{26}$$

*Proof.* According to (3), we have

$$T_n(k) = \begin{cases} \binom{n+2k-2}{k-1}/(\binom{(n-1)/2+k-1}{k-1}(\frac{n-1}{2})!) \ P_n(k), & \text{if } n \text{ is odd,} \\ \binom{n+2k-1}{k}/(\binom{n/2+k-1}{k}(\frac{n}{2}-1)!) \ P_n(k), & \text{if } n \text{ is even.} \end{cases} \tag{27}$$

Substituting this to (24), after simple transformations, we obtain (25) and (26), where $k$ is replaced by arbitrary $x$. □

Note that from (25) and (26), using a simple induction, we conclude that, for even $n \geq 4$, $P_n(x)$ is a polynomial of degree $\frac{n-2}{2}$, while, for odd $n \geq 3$, $P_n(x)$ is a polynomial of degree $\frac{n-1}{2}$. However, the structure of formulas (25) and (26) does not allow us to prove that all coefficients of $P_n(x)$ are integer. This will be done in the following section by the discovery of the special relationships with the required structure.

## 7. Proof of Theorem 2

**Lemma 7.** *For $n \geq 1$, we have*

$$T_n(k) - T_{n-2}(k+1) = \binom{n+2k-1}{k}. \tag{28}$$

*Proof.* By (18), we should prove that

$$\binom{2k+n-1}{k} = T_n(k) - T_{n-2}(k+1) =$$

$$\sum_{j=1}^{n} 2^{n-j}\binom{2k+j-2}{k-1} - \sum_{j=1}^{n-2} 2^{n-j-2}\binom{2k+j}{k} =$$

$$\sum_{j=1}^{n} 2^{n-j}\binom{2k+j-2}{k-1} - \sum_{i=1}^{n} 2^{n-i}\binom{2k+i-2}{k} +$$

$$2^{n-1}\binom{2k-1}{k} + 2^{n-2}\binom{2k}{k},$$

or

$$\sum_{j=1}^{n} 2^{-j}\left(\binom{2k+j-2}{k-1} - \binom{2k+j-2}{k}\right) =$$

$$2^{-n}\binom{2k+n-1}{k} - \frac{1}{2}\binom{2k-1}{k} - \frac{1}{4}\binom{2k}{k}. \tag{29}$$

It is verified directly that (29) is valid for $n = 1$. Therefore, it is sufficient to verify that the first differences over $n$ of the left hand side and the right hand side coincide. The corresponding identity

$$2^{-n}\left(\binom{2k+n-2}{k-1} - \binom{2k+n-2}{k}\right) =$$

$$2^{-n}\binom{2k+n-1}{k} - 2^{-n+1}\binom{2k+n-2}{k}$$

reduces to the equality $\binom{2k+n-2}{k-1} + \binom{2k+n-2}{k} = \binom{2k+n-1}{k}$.

Now we are able to complete proof of Theorem 2. Considering even $n \geq 4$, by (27), we obtain the following relation for $P_n(k)$ corresponding to (28):

$$P_n(x) = (n+x-1)P_{n-2}(x+1) +$$

$$\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 2\right) \cdots (x+1). \tag{30}$$

On the other hand, using (23), for odd $n \geq 3$, we obtain the following relation

$$P_n(x) = 2(x + n - 1)P_{n-1}(x)+$$

$$(x + \frac{n-1}{2} - 1)(x + \frac{n-1}{2} - 2) \cdots x. \tag{31}$$

From (30), by simple induction, we see that, for even $n \geq 4$, $P_n(x)$ is a polynomial with integer coefficients. Then from (31) we find that $P_n(x)$, for odd $n$, is also a polynomial with integer coefficients. $\qquad\square$

## 8. Other Relations

Together with (25), (26), (30) and (31) there exist many other relations for $P_n(x)$. All of them are corollaries of the corresponding relations for $T_n(k)$. Below we give a few pairs of some such relations.

As we saw, for odd $n \geq 3$, (31) follows from (23). Let us consider even $n \geq 4$. Then we obtain the second component of the following recursion

$$P_n(x) = \begin{cases} 2(x + n - 1)P_{n-1}(x)+ \\ \\ ((x + n - 1)P_{n-1}(x)+ \end{cases}$$

$$\begin{cases} (x + \frac{n-1}{2} - 1)(x + \frac{n-1}{2} - 2) \cdots x, \text{ if } n \geq 3 \text{ is odd,} \\ \\ (x + \frac{n}{2} - 1)(x + \frac{n}{2} - 2) \cdots x)/(2x + n - 1), \text{ if } n \geq 4 \text{ is even.} \end{cases}$$

**Lemma 8.** *For $n \geq 1$, $k \geq 1$, we have*

$$T_n(k + 1) = 4T_n(k) - \frac{n}{k}\binom{n + 2k - 1}{k - 1}. \tag{32}$$

*Proof.* By (24) and (28), we have

$$T_n(k + 1) = T_{n+2}(k) - \binom{n + 2k + 1}{k} =$$

$$4T_n(k) + \binom{n + 2k}{k - 1} + 2\binom{n + 2k - 1}{k - 1} - \binom{n + 2k + 1}{k}.$$

It is left to note that

$$\binom{n + 2k}{k - 1} + 2\binom{n + 2k - 1}{k - 1} - \binom{n + 2k + 1}{k} = -\frac{n}{k}\binom{n + 2k - 1}{k - 1}.$$

$\qquad\square$

From Lemma 8 and (27) we find the following recursion

$$\begin{cases} (2x+n)P_n(x+1) = 2(x+n)P_n(x)- \\ \\ (2x+n+1)P_n(x+1) = 2(x+n)P_n(x)- \end{cases}$$

$$\begin{cases} n(x+\frac{n-1}{2})(x+\frac{n-1}{2}-1)\cdots(x+1), \text{ if } n \geq 3 \text{ is odd}, \\ \\ \frac{n}{2}(x+\frac{n}{2}-1)(x+\frac{n}{2}-2)\cdots(x+1), \text{ if } n \geq 4 \text{ is even}. \end{cases}$$

**Lemma 9.** *For $n \geq 2$, $k \geq 1$, we have*

$$(n+k-1)(T_n(k) - 4T_n(k-1)) = n(T_{n-1}(k) - 2T_n(k-1)). \tag{33}$$

*Proof.* By (32),

$$T_n(k) - 4T_n(k-1) = -\frac{n}{k-1}\binom{n+2k-3}{k-2}. \tag{34}$$

By (23),

$$T_n(k-1) = 2T_{n-1}(k-1) + \binom{n+2k-4}{k-2}.$$

Therefore,

$$T_{n-1}(k) - 2T_n(k-1) = T_{n-1}(k) - 4T_{n-1}(k-1) - 2\binom{n+2k-4}{k-2}.$$

Using again (32), we find

$$T_{n-1}(k) - 2T_n(k-1) = -(\frac{n-1}{k-1}+2)\binom{n+2k-4}{k-2}. \tag{35}$$

Now the lemma follows from (34) and (35) since $(n+k-1)\binom{n+2k-3}{k-2} = (n+2k-3)\binom{n+2k-4}{k-2}$. $\qquad\square$

Going from (33) to the corresponding formula for $P_n(x)$ in the case of odd $n \geq 3$ unexpectedly leads to a very simple homogeneous relation

$$P_n(x) = P_n(x-1) + nP_{n-1}(x) \tag{36}$$

which we use in Sections 9 and 12. The corresponding relation for even $n \geq 4$ is

$$(2x+n-1)P_n(x) = (2x+n-2)P_n(x-1) + \frac{n}{2}P_{n-1}(x). \tag{37}$$

**Lemma 10.** *For $n \geq 1$, $k \geq 2$, we have*

$$2T_n(k) - T_{n-1}(k+1) = \binom{n+2k-1}{k}.$$

*Proof.* By (23), we have

$$2T_n(k) - T_{n-1}(k+1) = 4T_{n-1}(k) + 2\binom{n+2k-2}{k-1} - T_{n-1}(k+1).$$

Furthermore, by (28),

$$T_{n-1}(k+1) = T_{n+1}(k) - \binom{n+2k}{k}.$$

Hence,

$$2T_n(k) - T_{n-1}(k+1) =$$
$$4T_{n-1}(k) - T_{n+1}(k) + 2\binom{n+2k-2}{k-1} + \binom{n+2k}{k}. \qquad (38)$$

Finally, by (24),

$$T_{n+1}(k) - 4T_{n-1}(k) = \binom{n+2k-1}{k-1} + 2\binom{n+2k-2}{k-1}$$

and the lemma follows from (38). □

Using Lemma 10 and (27), for even $n \geq 4$, we find

$$2P_n(x) = P_{n-1}(x+1) + (x + \frac{n}{2} - 1)(x + \frac{n}{2} - 2)\cdots(x+1), \qquad (39)$$

while, for odd $n \geq 3$,

$$P_n(x) = (2x + n)P_{n-1}(x+1) + (x + \frac{n-1}{2})(x + \frac{n-1}{2} - 1)\cdots(x+1).$$

**Proposition 11.** *For odd $n \geq 3$, we have $P_n(k) \equiv P_n(0) \pmod{n}$.*

*Proof.* From (36) we find that $\sum_{i=1}^{k} P_{n-1}(i) = (P_n(k) - P_n(0))/n$, and the proposition follows. □

## 9. On the Coefficients of $P_n(x)$

Using formulas (25) and (26), we give a recursion for the calculation of the coefficients of $P_n(x)$ with a fixed parity of $n$. Let

$$P_n(x) = a_0(n)x^m + a_1(n)x^{m-1} + \cdots + a_{m-1}(n)x + a_m(n),$$

where $m = \lfloor \frac{n-1}{2} \rfloor$. We prove the following.

**Theorem 12.** *For $n \geq 1$, we have*

$$a_0(n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ \\ \frac{n}{2}, & \text{if } n \text{ is even;} \end{cases} \tag{40}$$

$$a_1(n) = \begin{cases} \frac{1}{24}(7n^3 - 12n^2 + 5n) \\ \\ \frac{1}{48}(7n^3 - 18n^2 + 8n) \end{cases} = \begin{cases} \frac{1}{24}n(n-1)(7n-5), & \text{if } n \text{ is odd,} \\ \\ \frac{1}{48}n(n-2)(7n-4), & \text{if } n \text{ is even.} \end{cases} \tag{41}$$

*In general, for a fixed $i$, $a_i(n) = U_i(n)$, if $n$ is odd, and $a_i(n) = V_i(n)$, if $n$ is even, where $U_i$ and $V_i$ are polynomials in $n$ of degree $2i+1$.*

*Proof.* `Case 1).` Let $n$ be even. Then, using (26), for integer $x$ and $m = \frac{n-2}{2}$, we have

$$(2x + n - 1)(a_0(n)x^m + a_1(n)x^{m-1} + \cdots) =$$

$$2(x + n - 1)(x + n - 2)(a_0(n-2)x^{m-1} + a_1(n-2)x^{m-2} + \cdots) +$$

$$\frac{1}{2}\left(\frac{n-2}{2}\right)!(4x + 3n - 4)\binom{x - 1 + \frac{n-2}{2}}{\frac{n-2}{2}}. \tag{42}$$

Comparing the coefficient of $x^{m+1}$ on both sides, we find

$$a_0(n) = a_0(n-2) + 1, \quad n \geq 4, \quad a_0(4) = 2.$$

Thus $a_0(6) = 3, a_0(8) = 4, \ldots, a_0(n) = n/2$.

Furthermore, comparing the coefficient of $x^m$ on both sides of (42), we have

$$2a_1(n) + (n-1)a_0(n) = 2a_1(n-2) + 2(2n-3)a_0(n-2) +$$

$$Coef[x^m](\frac{1}{2}(4x + 3n - 4)(x + \frac{n-4}{2})(x + \frac{n-6}{2}) \cdots (x+1)x). \tag{43}$$

Note that

$$Coef[x^m](\frac{1}{2}(4x + 3n - 4)(x + \frac{n-4}{2})(x + \frac{n-6}{2}) \cdots (x+1)x) =$$

$$\frac{3n-4}{2} + 2(\frac{n-4}{2} + \frac{n-6}{2} + \cdots + 1) =$$

$$\frac{3n-4}{2} + \sum_{i=2}^{m}(n - 2i) = \frac{n^2}{4}.$$

Therefore, by (43),

$$a_1(n) - a_1(n-2) = \frac{(2n-3)(n-2)}{2} - \frac{(n-1)n}{4} + \frac{n^2}{8} = \frac{7n^2 - 26n + 24}{8}.$$

Hence

$$a_1(n) = \sum_{i=4,6,\cdots,n} (a_1(i) - a_1(i-2)) = \frac{1}{8} \sum_{i=4,6,\dots,n} (7i^2 - 26i + 24) =$$

$$\frac{1}{2} \sum_{j=2}^{n/2} (7j^2 - 13j + 6) = \frac{1}{48}(7n^3 - 18n^2 + 8n).$$

Finally, comparing the coefficient of $x^{m-i}$ on both sides of (42), we find

$$2a_{i+1}(n) + (n-1)a_i(n) = 2a_{i+1}(n-2)+$$

$$2(2n-3)a_i(n-2) + 2(n-1)(n-2)a_{i-1}(n-2)+$$

$$\frac{1}{2}Coef[x^{m-i}]((4x+3n-4)(x+\frac{n-4}{2})(x+\frac{n-6}{2})\cdots(x+1)(x)). \quad (44)$$

Note that, polynomial $(4x + 3n - 4)(x + \frac{n-4}{2})(x + \frac{n-6}{2}) \cdots (x+1)x$ has degree $m + 1$. Therefore, in order to calculate $Coef[x^{m-i}]$ in (44), we should choose, in all possible ways, in $m - i$ brackets (from $m + 1$ ones) $x's$, and for the other $i + 1$ brackets we choose linear forms of $n$. Thus $\frac{1}{2}Coef[x^{m-i}]$ in (44) is a polynomial $r_i(n)$ of degree $i + 1$. Further we use induction over $i$ with the formulas (40) and (41) as the inductive base. Write (44) in the form

$$2(a_{i+1}(n) - a_{i+1}(n-2)) =$$

$$2(2n-3)a_i(n-2) - (n-1)a_i(n) + 2(n-1)(n-2)a_{i-1}(n-2) + r_i(n). \quad (45)$$

By the inductive supposition, $a_{i-1}(n)$ and $a_i(n)$ are polynomials of degree $2i - 1$ and $2i+1$ respectively. Thus $a_{i+1}(n) - a_{i+1}(n-2)$ is a polynomial of degree $2i+2$. This means that $a_{i+1}(n)$ is a polynomial of degree $2i + 3$.

**Case 2).** Let $n$ be odd. By (25), for integer $x$ and $m = \frac{n-1}{2}$, we have

$$(2x + n - 2)(a_0(n)x^m + a_1(n)x^{m-1} + \cdots) =$$

$$2(x+n-1)(x+n-2)(a_0(n-2)x^{m-1} + a_1(n-2)x^{m-2} + \cdots)+$$

$$\left(\frac{n-1}{2}\right)!(4x+3n-4)\binom{x+\frac{n-3}{2}}{\frac{n-1}{2}}. \quad (46)$$

Hence, comparing the coefficient of $x^{m+1}$ on both sides, we find

$$a_0(n) = a_0(n-2) + 2, \ n \geq 3, \ a_0(1) = 1.$$

Thus $a_0(3) = 3, a_0(5) = 5, \ldots, a_0(n) = n$.

Furthermore, comparing the coefficient of $x^m$ on both sides of (46), using the same arguments as in 1), we have

$$a_1(n) = a_1(n-2) + \frac{7n^2 - 22n + 19}{4}, \quad n \geq 3, \quad a_1(1) = 0.$$

Since $a_1(n) = \sum_{i=3,5,\ldots,n}(a_1(i) - a_1(i-2))$, we find

$$a_1(n) = \frac{1}{4} \sum_{i=3,5,\ldots,n}(7i^2 - 22i + 19) = \frac{1}{24}(7n^3 - 12n^2 + 5n).$$

Finally, comparing the coefficient of $x^{m-i}$ on both sides of (46), we find

$$2(a_{i+1}(n) - a_{i+1}(n-2)) =$$

$$2(2n-3)a_i(n-2) - (n-2)a_i(n) +$$

$$2(n-1)(n-2)a_{i-1}(n-2) + s_i(n), \tag{47}$$

where

$$s_i(n) = Coef[x^{m-i}]((4x + 3n - 4)(x + \frac{n-3}{2})(x + \frac{n-5}{2}) \cdots (x+1)x)$$

and, as in 1), the statement is proved by induction over $i$. $\qquad\square$

A few such polynomials are the following:

For odd $n$ :

$$U_0(n) = n,$$

$$U_1(n) = \frac{1}{24}(n-1)n(7n-5),$$

$$U_2(n) = \frac{1}{640}(n-3)(n-1)n(29n^2 - 44n + 7),$$

$$U_3(n) = \frac{1}{322560}(n-5)(n-3)(n-1)n(1581n^3 - 3775n^2 + 1587n + 223);$$

For even $n$ :

$$V_0(n) = \frac{1}{2}n,$$

$$V_1(n) = \frac{1}{48}(n-2)n(7n-4),$$

$$V_2(n) = \frac{1}{3840}(n-4)(n-2)n(87n^2 - 98n + 16),$$

$$V_3(n) = \frac{1}{645120}(n-6)(n-4)(n-2)n(1581n^3 - 2686n^2 + 936n + 64).$$

**Proposition 13.**

$$a_i(n) \equiv \begin{cases} r_i(n), & \text{if } n \text{ is even,} \\ \\ s_i(n), & \text{if } n \text{ is odd.} \end{cases} \pmod 2$$

*Proof.* The proposition follows from (45), (47) and Theorem 2. □

Finally, note that, from (36) and (37) the following homogeneous recursions for the coefficients of $P_n(x)$ follow.

**Theorem 14.** *For odd* $n \geq 3$ *and* $i \geq 0$,

$$(m-i)a_i(n) = na_i(n-1) + \sum_{j=0}^{i-1}(-1)^{i-j+1}\binom{m-j}{m-i-1}a_j(n).$$

*For even* $n \geq 4$ *and* $i \geq 0$,

$$(n - 2i - 1)a_i(n) = \frac{n}{2}a_i(n-1)+$$

$$2\sum_{j=0}^{i-1}(-1)^{i-j+1}\left(m\binom{m-j}{m-i} - \binom{m-j}{m-i-1}\right)a_j(n).$$

## 10. Arithmetic Proof of the Integrality $P_n(x)$ in Integer Points

From Theorem 2 we conclude that the polynomial, $P_n(x)$, takes integer values for integer $x = k$. Here we give an independent arithmetic proof of this fact using the explicit expression (3). It is well known (cf. [5], Section 8, Problem 87) that, if a polynomial $P(x)$ of degree $m$ takes integer values for $x = 0, 1, \ldots, m$, then it takes integer values for every integer $x$. Since, as we proved at the end of Section 6, $\deg P_n(k) = \lfloor\frac{n-1}{2}\rfloor$, we suppose that $0 \leq k \leq \lfloor\frac{n-1}{2}\rfloor$. Moreover, from the results of Section 3, $P_n(0)$ and $P_n(1)$ are integers. (In the case when $n + 1$ is an odd prime, $P_n(1) = (2^n - 1)(\frac{n}{2})!/(n+1)$ is integer, since $2^n - 1 \equiv 0 \pmod{n+1}$, while in the case when $n + 1$ is an odd composite number, no divisor exceeds $\frac{n+1}{3}$, therefore, $(\frac{n}{2})! \equiv 0 \pmod{n+1}$.) Thus we can suppose that

$$2 \leq k \leq \lfloor\frac{n-1}{2}\rfloor. \tag{48}$$

Suppose that $n$ is *even* (the case of odd $n$ is considered quite analogously). Let $p$ be a prime. Denote the maximal power of $p$ dividing $n$ by $[n]_p$. We say that, for integer $l$, $h$, the fraction $\frac{l}{h}$ is *p-integer*, if $[l]_p - [h]_p \geq 0$.

A) Firstly, we show that, for $n \geq 4$, $P_n(k)$ is 2-integer. Indeed, $2k + n - 1$ is odd, while $4k + 3n - 4$ is even. Therefore, by (26), using a trivial induction, we see that $P_n(k)$ is 2-integer.

Further we use the explicit formula (3) of Theorem 1.

B) Let $p$ be an odd prime divisor of $\binom{n+2k-2}{k-1}$ which does not coincide with any factor of the product $(n+2k-1)(n+2k-2)\cdots(n+k)$. Thus $p$ could divide one or several *composite* factors of this product. Therefore, the following condition holds

$$3 \leq p \leq \frac{n + 2k - 1}{3}. \tag{49}$$

Let us show that

$$a(n;k) = \frac{\binom{\frac{n}{2}+k-1}{k}}{\binom{n+2k-1}{k}} \left(\frac{n-2}{2}\right)! =$$

$$2^{-k} \frac{(n + 2k - 2)(n + 2k - 4) \cdots n}{(n + 2k - 1)(n + 2k - 2) \cdots (n + k)} \left(\frac{n-2}{2}\right)!$$

is $p$-integer and, consequently, $P_n(k)$ is $p$-integer.

Let $k \geq 3$ be even. Then, after a simplification, we have

$$2^k a(n;k) = \frac{(n + k - 2)(n + k - 4) \cdots n}{(n + 2k - 1)(n + 2k - 3) \cdots (n + k + 1)} \left(\frac{n-2}{2}\right)!,$$

or

$$2^{\frac{k}{2}} a(n;k) = \frac{\left(\frac{n+k-2}{2}\right)!}{(n + 2k - 1)(n + 2k - 3) \cdots (n + k + 1)} \tag{50}$$

We distinguish several cases.

**Case a).** For $t \geq 2$, let $p^t$ divide at least one factor of the denominator. Then $p \leq (n+2k-1)^{\frac{1}{t}}$. Let us show that $p \leq \frac{n+k-2}{2t}$. We should show that $n+2k-1 \leq (\frac{n+k-2}{2t})^t$, or, since, by (48), $k \leq \frac{n-2}{2}$, it is sufficient to show that $\frac{3}{2}(n + k - 2) \leq (\frac{n+k-2}{2t})^t$, or $(2t)^{\frac{t}{t-1}} \leq (\frac{2}{3})^{\frac{1}{t-1}}(n + k - 2)$. Since $(\frac{2}{3})^{\frac{1}{t-1}} \geq \frac{2}{3}$, it is sufficient to prove that $(2t)^{\frac{t}{t-1}} \leq \frac{2}{3}(n + k - 2)$. Note that $e^t < p^t \leq n + 2k - 2$, $t \leq \ln(n + 2k - 2)$. Therefore we find $(2t)^{\frac{t}{t-1}} \leq (2\ln(n + 2k - 2))^2$. Furthermore, note that, if $n \geq 152$, then $\ln^2 n < \frac{n}{6}$. Thus $(2t)^{\frac{t}{t-1}} \leq \frac{2}{3}(n + k - 2)$. It is left to add that up to $n = 161$ we verified that the polynomials $P_n(k)$ have integer coefficients and, consequently, is integer-valued.

**Case b).** Let $p$ divide only one factor of the denominator. Then, in view of (48) and (49), $p \leq \frac{n+2k-1}{3} \leq \frac{n+k-2}{2}$ and, by (50), $a(n;k)$ is $p$-integer.

**Case c).** Let $p$ divide exactly $l$ factors of the denominator. Then

$$p \leq \frac{(n + 2k - 1) - (n + k + 1)}{l} = \frac{k - 2}{l},$$

and, since, by (48), $n \geq 2k+2$, we conclude that $\frac{n+k-2}{2} \geq \frac{3k}{2} \geq k-2 \geq lp$. Hence, by (50), $a(n;k)$ is $p$-integer.

It is left to notice that the case of odd $k$ is considered quite analogously.

C) Suppose that, as in B), $k \geq 2$ is even. Let $p$ be an odd prime divisor of $\binom{n+2k-1}{k}$ which coincides with some factor of the product $(n+2k-1)(n+2k-3)\cdots(n+k+1)$. In this case the fraction (50) is not integer. Thus in order to prove that $P_n(k)$ is $p$-integer, we should prove that $T_n(k)$ (18) is $p$-integer. By the condition, $p$ has form

$$p = n + 2k - 1 - 2r, \ \ 0 \leq r \leq \frac{k-2}{2}. \tag{51}$$

According to (18) and (51), we should prove that

$$\sum_{j=0}^{n-1} 2^j \binom{n+2k-j-2}{k-1} =$$

$$\sum_{j=0}^{n-1} 2^j \binom{p+2r-1-j}{k-1} \equiv 0 \pmod{p}, \tag{52}$$

or

$$A(n,r,k) :=$$

$$\sum_{j=0}^{n-1} 2^j (j - (2r-1))(j+1 - (2r-1)) \cdots (j+k-2-(2r-1)) \equiv 0 \pmod{p}.$$

Note that, since $n - 2r = p - 2k + 1$, we have

$$\sum_{j=0}^{n-1} x^{j+k-2-(2r-1)} =$$

$$(x^{n+k-2r-1} - x^{k-2r-1})(x-1)^{-1} = (x^{p-k} - x^{k-2r-1})(x-1)^{-1}.$$

Therefore,

$$A(n,r,k) = 2^{2r} \sum_{j=0}^{n-1} (x^{j+k-2-(2r-1)})^{(k-1)} \mid_{x=2} =$$

$$2^{2r}((x^{p-k} - x^{k-2r-1})(x-1)^{-1})^{(k-1)} \mid_{x=2}.$$

Thus we should prove that

$$((x^{p-k} - x^{k-2r-1})(x-1)^{-1})^{(k-1)} \mid_{x=2} \equiv 0 \pmod{p},$$

or, using the Leibnitz formula,

$$\sum_{j=0}^{k-1} (-1)^{k-j-1} \binom{k-1}{j} (k-j-1)!(p-k)(p-k-1)\cdots(p-k-j+1)2^{p-k-j} \equiv$$

$$\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{k-1}{j}(k-j-1)!(k-2r-1)(k-2r-2)\cdots(k-2r-j)2^{k-2r-j-1} \pmod{p}.$$

Since $2^{p-1} \equiv 1 \pmod{p}$, we should prove the identity

$$\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{k-1}{k-j-1}(k-j-1)!(p-k)(p-k-1)\cdots(p-k-j+1)\mid_{p=0} 2^{-k-j+1} =$$

$$\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{k-1}{k-j-1}(k-j-1)!(k-2r-1)(k-2r-2)\cdots(k-2r-j)2^{k-2r-j-1},$$

or, after simple transformations, the identity

$$\sum_{j=0}^{k-1}\binom{k+j-1}{j}2^{-j} = 2^{2k-2r-2}\sum_{j=0}^{k-1}(-1)^j\binom{k-2r-1}{j}2^{-j}. \qquad (53)$$

It is known (([7], Ch.1, problem 7), that

$$\sum_{i=0}^{n}\binom{2n-i}{n}2^{i-n} = 2^n.$$

Putting $n - i = j$, we have

$$\sum_{j=0}^{n}\binom{n+j}{n}2^{-j} = \sum_{j=0}^{n}\binom{n+j}{j}2^{-j} = 2^n.$$

Therefore, the left hand side of (53) is $2^{k-1}$ and it is left to prove that

$$\sum_{j=0}^{k-1}(-1)^j\binom{k-2r-1}{j}2^{k-j} = 2^{2r+1}.$$

We have

$$\sum_{j=0}^{k-1}(-1)^j\binom{k-2r-1}{j}2^{k-j} = \sum_{j=0}^{k-2r-1}(-1)^j\binom{k-2r-1}{j}2^{k-j} =$$

$$2^{2r+1}\sum_{j=0}^{k-2r-1}(-1)^j\binom{k-2r-1}{j}2^{k-2r-1-j} = 2^{2r+1}(2-1)^{k-2r-1} = 2^{2r+1}$$

and we are done. The case of odd $k \geq 3$ is considered quite analogously. So, formulas (50) and (51) take the form

$$2^{\frac{k-1}{2}}a(n;k) = \frac{(\frac{n+k-1}{2})!}{(n+2k-1)(n+2k-3)\cdots(n+k)},$$

$$p = n + 2k - 2r - 1, \;\; 0 \leq r \leq \frac{k-1}{2},$$

and, for odd $k$, the proof reduces to the same congruence (52).

## 11. Representation of $P_n(x)$ in Basis $\{\binom{x}{i}\}$

The structure of the explicit formula (3) allows us to conjecture that the coefficients of $P_n(x)$ in basis $\{\binom{x}{i}\}$ have simpler properties. A process of expansion of a polynomial $P(x)$ in the binomial basis is indicated in [5] in a solution of Problem 85: "Functions $1,\ x,\ x^2,\ldots,x^n$ one can consecutively express in the form of linear combinations with the constant coefficients of $1,\ \frac{x}{1},\ \frac{x(x-1)}{2},\ldots,\frac{x(x-1)\cdots(x-n+1)}{n!}$." Therefore,

$$P(x) = b_0 \binom{x}{m} + b_1 \binom{x}{m-1} + \cdots + b_{m-1}\binom{x}{1} + b_m,$$

where $b_0,\ b_1,\ldots,b_m$ are defined from the equations

$$P(0) = b_m,$$

$$P(1) = b_m + \binom{1}{1}b_{m-1},$$

$$P(2) = b_m + \binom{2}{1}b_{m-1} + \binom{2}{2}b_{m-2},$$

$$\vdots$$

$$P(m) = b_m + \binom{m}{1}b_{m-1} + \cdots + \binom{m}{m}b_0.$$

This process can be simplified in the following way. In the identity

$$n^x = (1 + (n-1))^x =$$

$$1 + (n-1)\binom{x}{1} + (n-1)^2\binom{x}{2} + \cdots + (n-1)^x\binom{x}{x} =$$

$$n^0 + (n-n^0)\binom{x}{1} + (n-n^0)^2\binom{x}{2} + \cdots + (n-n^0)^x\binom{x}{x}$$

we can evidently replace powers $n^j,\ j = 0,\ldots,x$, by the arbitrary numbers $a_j,\ j = 0,\ldots,x$. Thus we have a general identity

$$a_x = a_0 + (a_1 - a_0)\binom{x}{1} + (a_2 - 2a_1 + a_0)\binom{x}{2} + (a_3 - 3a_2 + 3a_1 - a_0)\binom{x}{3} + \cdots +$$

$$\left(a_x - \binom{x}{1}a_{x-1} + \binom{x}{2}a_{x-2} - \cdots + (-1)^x\binom{x}{x}a_0\right)\binom{x}{x}.$$

Essentially, we quickly obtained a special case of the so-called "Newton's forward difference formula" (cf. [10]). Here, put $a_j = P(j),\ j = 0,\ldots,m$, and, firstly,

consider values $0 \leq x \leq m$. Since $\binom{x}{l} = 0$ for $l > m$, we obtain the required representation under the condition $0 \leq x \leq m$ :

$$P(x) = P(0) + (P(1) - P(0))\binom{x}{1} +$$

$$(P(2) - 2P(1) + P(0))\binom{x}{2} + \cdots + (P(m) - \binom{m}{1}P(m-1) +$$

$$\binom{m}{2}P(m-2) - \cdots + (-1)^m \binom{m}{m}P(0))\binom{x}{m}. \qquad (54)$$

It is left to note that, since a polynomial of degree $m$ is fully defined by its values in $m + 1$ points $0, 1, \ldots, m$, then (54) is the required representation for all $x$.

So, for the polynomials $\{P_n(x)\}$, we have

$$P_1 = 1,$$

$$P_2 = 1,$$

$$P_3 = 3\binom{x}{1} + 4,$$

$$P_4 = 2\binom{x}{1} + 4,$$

$$P_5 = 10\binom{x}{2} + 30\binom{x}{1} + 32,$$

$$P_6 = 6\binom{x}{2} + 22\binom{x}{1} + 32,$$

$$P_7 = 42\binom{x}{3} + 196\binom{x}{2} + 378\binom{x}{1} + 384,$$

$$P_8 = 24\binom{x}{3} + 128\binom{x}{2} + 296\binom{x}{1} + 384,$$

$$P_9 = 216\binom{x}{4} + 1368\binom{x}{3} + 3816\binom{x}{2} + 6120\binom{x}{1} + 6144,$$

$$P_{10} = 120\binom{x}{4} + 840\binom{x}{3} + 2664\binom{x}{2} + 5016\binom{x}{1} + 6144,$$

$$P_{11} = 1320\binom{x}{5} + 10560\binom{x}{4} + 38544\binom{x}{3} + 84480\binom{x}{2} + 122760\binom{x}{1} + 122880,$$

$$P_{12} = 760\binom{x}{5} + 6240\binom{x}{4} + 25152\binom{x}{3} + 62112\binom{x}{2} + 103920\binom{x}{1} + 122880.$$

## 12. On Coefficients of $P_n(x)$ in Basis $\{\binom{x}{i}\}$

Let

$$P_n(x) = b_0(n)\binom{x}{m} + b_1(n)\binom{x}{m-1} + \cdots + b_{m-1}(n)\binom{x}{1} + b_m(n),$$

where $m = \lfloor \frac{n-1}{2} \rfloor$.

Since, for integer $k$, we have the explicit formula for $P_n(k)$, (3), then, according to (54), we have the following explicit formula for $b_i(n)$, $i = 0, \ldots, m$ :

$$b_i(n) = \sum_{k=0}^{m-i} (-1)^{m-i-k} \binom{m-i}{k} P_n(k). \tag{55}$$

Let

$$P_n(x) = \sum_{j=0}^{m} a_j(n) x^{m-j}.$$

Then

$$b_i(n) = \sum_{j=0}^{m} a_j(n) \sum_{k=0}^{m-i} (-1)^{m-i-k} k^{m-j} \binom{m-i}{k}. \tag{56}$$

Since the $l$-th difference of $f(x)$ is (cf. [1], formula 25.1.1)

$$\Delta^l f(x) = \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} f(x+k),$$

one can write (56) in the form

$$b_i(n) = \sum_{j=0}^{m} a_j(n) \Delta^{m-i} x^{m-j} \mid_{x=0}.$$

Here the summands corresponding to $j > i$, evidently, equal 0. Therefore, we have

$$b_i(n) = \sum_{j=0}^{i} a_j(n) \Delta^{m-i} x^{m-j} \mid_{x=0}. \tag{57}$$

**Theorem 15.** *For $n \geq 1$, we have*

$$b_0(n) = \begin{cases} n(\frac{n-1}{2})!, & \text{if } n \text{ is odd,} \\ \\ (\frac{n}{2})!, & \text{if } n \text{ is even;} \end{cases} \tag{58}$$

$$b_1(n) = \begin{cases} \frac{1}{6}n(5n-7)(\frac{n-1}{2})!, & \text{if } n \text{ is odd,} \\ \\ \frac{1}{6}(5n-8)(\frac{n}{2})!, & \text{if } n \text{ is even.} \end{cases} \tag{59}$$

*In general, for a fixed $i$, $b_i(n) = (m-i)!Y_i(n)$, if $n$ is odd, and $b_i(n) = (m-i)!Z_i(n)$, if $n$ is even, where $Y_i$ and $Z_i$ are polynomials in $n$ of degree $2i+1$.*

*Proof.* Note that the Stirling number of the second kind $S(n,m)$ is connected with the $m$-th difference of $\Delta^m x^n \mid_{x=0}$ in the following way (see [1], formulas 24.1.4):

$$S(n,m)m! = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k^n = \Delta^m x^n \mid_{x=0} . \tag{60}$$

In particular, since $S(m,m) = 1$, $S(m+1,m) = \binom{m+1}{2}$, we have

$$\Delta^m x^m \mid_{x=0} = m!$$

and

$$\Delta^m x^{m+1} \mid_{x=0} = \frac{m}{2}(m+1)! .$$

Therefore, by (57),

$$b_0(n) = m!a_0(n),$$

$$b_1(n) = \frac{m-1}{2}m!a_0(n) + (m-1)!a_1(n),$$

and, by (40) and (41) (where $m = \lfloor\frac{n-1}{2}\rfloor$), we find formulas (58) and (59).

Further, we need the following lemma.

**Lemma 16.** $S(n+k,n)$ *is a polynomial in $n$ of degree $2k$.*

*Proof.* For $k \geq 1$, let

$$Q_k(n) = S(n+k,n).$$

Note that, since $S(n,n) = 1$, we have $Q_0(n) = 1$. Further, since $S(n,0) = \delta_{n,0}$, for $k \geq 1$, $Q_k(0) = 0$. From the main recursion for $S(n,m)$ which is $S(n,m) = mS(n-1,m) + S(n-1,m-1)$, we have

$$Q_k(n) - Q_k(n-1) = nQ_{k-1}(n).$$

Also, in view of $Q_k(0) = 0$, we find the recursion

$$Q_0(n) = 1, \quad Q_k(n) = \sum_{i=1}^{n} iQ_{k-1}(i). \tag{61}$$

Using a simple induction, from (61) we obtain the lemma. $\square$

**Remark 17.** The first few polynomials $\{Q_k(n)\}$ are:

$$Q_0 = 1,$$

$$Q_1 = \frac{1}{2}n(n+1),$$

$$Q_2 = \frac{1}{24}n(n+1)(n+2)(3n+1),$$

$$Q_3 = \frac{1}{48}n^2(n+1)^2(n+2)(n+3),$$

$$Q_4 = \frac{1}{5760}n(n+1)(n+2)(n+3)(n+4)(15n^3+30n^2+5n-2).$$

It can be proven that the sequence of denominators coincides with $A053657$ [8], such that the denominator of $Q_k(n)$ is $\prod p^{\sum_{j\geq 0}\lfloor \frac{k}{(p-1)p^j}\rfloor}$, where the product is over all primes.

Note that from (57) and (60) we find

$$b_i(n) = (m-i)!\sum_{j=0}^{i} a_j(n)S(m-j,m-i), \ \ m = \lfloor\frac{n-1}{2}\rfloor.$$

Since, by Lemma 10, $S(m-j,m-i)$ is a polynomial in $n$ of degree $2((m-j)-(m-i)) = 2(i-j)$, while, by Theorem 12, $a_j(n)$ is a polynomial of degree $2j+1$, it follows that $a_j(n)S(m-j,m-i)$ is a polynomial of degree $2i+1$. Thus $\sum_{j=0}^{i} a_j(n)S(m-j,m-i)$ is a polynomial of degree $2i+1$. This completes the proof. $\quad\square$

The first polynomials $Y_i(n), \ Z_i(n)$ are

$$Y_0 = n,$$

$$Y_1 = \frac{1}{12}(n-1)n(5n-7),$$

$$Y_2 = \frac{1}{480}(n-3)(n-1)n(43n^2-168n+149),$$

$$Y_3 = \frac{1}{13440}(n-5)(n-3)(n-1)n(177n^3-1319n^2+3063n-2161);$$

$$Z_0 = \frac{n}{2},$$

$$Z_1 = \frac{1}{24}(n-2)n(5n-8),$$

$$Z_2 = \frac{1}{960}(n-4)(n-2)n(43n^2-182n+184),$$

$$Z_3 = \frac{1}{26880}(n-6)(n-4)(n-2)n(3n-8)(59n^2-306n+352)).$$

Finally, we prove the following attractive result.

**Theorem 18.** 1) *For odd $n$, $b_j(n)/n, j = 0, \ldots, m-1$, are integer. Moreover, for $n \geq 3$,*

$$b_i(n) = n(b_i(n-1) + b_{i-1}(n-1)), \quad i = 1, \ldots, m-1.$$

2) *For even $n \geq 4$,*

$$2b_i(n) = b_i(n-1) + b_{i-1}(n-1) + m!\binom{m}{i}, \quad i = 1, \ldots, m-1. \qquad (62)$$

*Proof.* 1) According to (55), we should prove that for odd $n \geq 3$,

$$\sum_{k=0}^{m-i}(-1)^{m-i-k}\binom{m-i}{k}P_n(k) =$$

$$n\Big(\sum_{k=0}^{m-i}(-1)^{m-i-k}\binom{m-i}{k}P_{n-1}(k)+$$

$$\sum_{k=0}^{m-i-1}(-1)^{m-i-k-1}\binom{m-i-1}{k}P_{n-1}(k)\Big), \quad i = 1, 2, \ldots, m-1,$$

or, putting $m - i = t$,

$$\sum_{k=0}^{t}(-1)^k\binom{t}{k}P_n(k) = n\Big(\sum_{k=0}^{t}(-1)^k\binom{t}{k}P_{n-1}(k)-$$

$$\sum_{k=0}^{t}(-1)^k\binom{t-1}{k}P_{n-1}(k)\Big), \quad t = 1, 2, \ldots, m-1,$$

or, finally, for $t = 1, \ldots, \frac{n-3}{2}$,

$$\sum_{k=1}^{t}(-1)^{k-1}\Big(\binom{t}{k}P_n(k) - \binom{t-1}{k-1}nP_{n-1}(k)\Big) = P_n(0). \qquad (63)$$

To prove (63), note that, by (36), $nP_{n-1}(k) = P_n(k) - P_n(k-1)$. Hence,

$$\binom{t}{k}P_n(k) - \binom{t-1}{k-1}nP_{n-1}(k) =$$

$$P_n(k)\Big(\binom{t}{k} - \binom{t-1}{k-1}\Big) + \binom{t-1}{k-1}P_n(k-1) =$$

$$\binom{t-1}{k}P_n(k) + \binom{t-1}{k-1}P_n(k-1).$$

Thus the summands of (63) are

$$(-1)^{k-1}\Big(\binom{t}{k}P_n(k) - \binom{t-1}{k-1}nP_{n-1}(k)\Big) =$$

$$(-1)^{k-1}\binom{t-1}{k}P_n(k) - (-1)^{k-2}\binom{t-1}{k-1}P_n(k-1),$$

and the summing gives

$$\sum_{k=1}^{t}(-1)^{k-1}\left(\binom{t}{k}P_n(k) - \binom{t-1}{k-1}nP_{n-1}(k)\right) =$$

$$(-1)^{k-1}\binom{t-1}{k}P_n(k)\mid_{k=t} - (-1)^{k-2}\binom{t-1}{k-1}P_n(k-1)\mid_{k=1} = P_n(0).$$

2) Analogously, the proof of (62) reduces to the proof of the following equality for $t = 1, 2, \ldots, m-1$:

$$\sum_{k=0}^{t+1}(-1)^k\left(2\binom{t}{k}P_n(k) + \binom{t}{k-1}P_{n-1}(k)\right) = (-1)^t m!\binom{m}{t}. \qquad (64)$$

Note that, by (39),

$$(-1)^k\left(2\binom{t}{k}P_n(k) + \binom{t}{k-1}P_{n-1}(k)\right) =$$

$$(-1)^k\binom{t}{k}P_{n-1}(k+1) -$$

$$(-1)^{k-1}\binom{t}{k-1}P_{n-1}(k) + (-1)^k\binom{t}{k}\binom{k+m}{m}m! \qquad (65)$$

Since

$$\sum_{k=0}^{t+1}\left((-1)^k\binom{t}{k}P_{n-1}(k+1) - (-1)^{k-1}\binom{t}{k-1}P_{n-1}(k)\right) =$$

$$(-1)^k\binom{t}{k}P_{n-1}(k)\mid_{k=t+1} - (-1)^{k-1}\binom{t}{k-1}P_{n-1}(k)\mid_{k=0} = 0,$$

by (64) and (65), the proof reduces to the known combinatorial identity

$$\sum_{k=0}^{t}(-1)^k\binom{t}{k}\binom{k+m}{m} = (-1)^t\binom{m}{t}, \quad t = 1, \ldots, m-1$$

(see [7], Ch.1, formula (8) with $p = 0$ up to the notations).                                    □

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