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ON RAMANUJAN CUBIC POLYNOMIALS

VLADIMIR SHEVELEV

Abstract. A polynomial x^3+px^2+qx+r with the condition $pr^{\frac{1}{3}}+3r^{\frac{2}{3}}+q=0$ we call a Ramanujan cubic polynomial (RCP). We study different interest properties of RCP, in particular, an important role of a parameter $\frac{pq}{r}$. We prove some new beautiful identities containing sums of 6 cubic radicals of values of trigonometrical functions as well.

Paper is deducated to the $120-th$ *anniversary of Srinivasa Ramanujan.*

1. INTRODUCTION

In his second notebook [4], S.Ramanujan proved the following theorem.

Theorem 1. *([4], p.325;[2]). Let* α,β *and* γ *denote the roots of the cubic equation*

$$x^3-ax^2+bx-1=0. \tag{1}$$

Then, for a suitable determination of roots,

$$\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}+\gamma^{\frac{1}{3}}=(a+6+3t)^{\frac{1}{3}}, \tag{2}$$

and

$$(\alpha\beta)^{\frac{1}{3}}+(\beta\gamma)^{\frac{1}{3}}+(\gamma\alpha)^{\frac{1}{3}}=(b+6+3t), \tag{3}$$

where

$$t^3-3(a+b+3)t-(ab+6(a+b)+9)=0. \tag{4}$$

A proof of Theorem 1 can be found in paper [2]. Evidently, the simplest condition for successful application of Theorem 1 is the condition

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$$a+b+3=0 \tag{5}$$

that implies

$$t=(ab+6(a+b)+9)^{\frac{1}{3}}=(ab-9)^{\frac{1}{3}}. \tag{6}$$

If, for real nonzero r, to consider

$$x_1=-r^{\frac{1}{3}}\alpha,\ \ x_2=-r^{\frac{1}{3}}\beta,\ \ x_3=-r^{\frac{1}{3}}\gamma \tag{7}$$

and denote

$$ar^{\frac{1}{3}}=p,\ \ br^{\frac{2}{3}}=q \tag{8}$$

then in the real case by (1)-(8) we obtain the following result.

Theorem 2. *Let $p,q,r,\in\mathbb{R},\ \ r\neq 0$ such that*

$$pr^{\frac{1}{3}}+3r^{\frac{2}{3}}+q=0, \tag{9}$$

and let the polynomial

$$x^3+px^2+qx+r \tag{10}$$

have real roots x_1,x_2,x_3. Then

$$x_1^{\frac{1}{3}}+x_2^{\frac{1}{3}}+x_3^{\frac{1}{3}}=(-p-6r^{\frac{1}{3}}+3(9r-pq)^{\frac{1}{3}})^{\frac{1}{3}}, \tag{11}$$

and

$$(x_1x_2)^{\frac{1}{3}}+(x_2x_3)^{\frac{1}{3}}+(x_3x_4)^{\frac{1}{3}}=(q+6r^{\frac{2}{3}}-3(9r^2-pqr)^{\frac{1}{3}})^{\frac{1}{3}}, \tag{12}$$

Notice that Theorem 2 was proved directly in [6]. Notice also, that (12) can be written in the form

$$x_1^{-\frac{1}{3}}+x_2^{-\frac{1}{3}}+x_3^{-\frac{1}{3}}=r^{-\frac{1}{3}}(-q-6r^{\frac{2}{3}}+3(9r^2-pqr)^{\frac{1}{3}})^{\frac{1}{3}}. \tag{13}$$

In connection with Theorem 2 we introduce the following definition.

Definition 1. *Let $p, q, r \in \mathbb{R}$, $r \neq 0$. The cubic polynomial (10) is called a Ramanujan polynomial (RCP) if it has real roots and the condition (9) is satisfied.*

In this paper we study various properties of the RCP and present some new identities.

2. SOME EXAMPLES

Example 1. The polynomial $x^3 - 3x^2 - 6x + 8$ is an RCP with roots $1, -2, 4$. Thus, by (11) we have

$$1 - 2^{\frac{1}{3}} + 4^{\frac{1}{3}} = (-9 + 9 \cdot 2^{\frac{1}{3}})^{\frac{1}{3}}$$

and therefore

$$\left(\frac{1}{9}\right)^{\frac{1}{3}} - \left(\frac{2}{9}\right)^{\frac{1}{3}} + \left(\frac{4}{9}\right)^{\frac{1}{3}} = (2^{\frac{1}{3}} - 1)^{\frac{1}{3}}. \tag{14}$$

It is a Ramanujan‘s original identity (see[5],p.331).

Example 2. Ramanujan ([4], p.326, [1])offers the following identities:

$$\left(\cos\frac{2\pi}{7}\right)^{\frac{1}{3}} + \left(\cos\frac{4\pi}{7}\right)^{\frac{1}{3}} + \left(\cos\frac{8\pi}{7}\right)^{\frac{1}{3}} = \left(\frac{5 - 3 \cdot 7^{\frac{1}{3}}}{2}\right)^{\frac{1}{3}} \tag{15}$$

$$\left(\cos\frac{2\pi}{9}\right)^{\frac{1}{3}} + \left(\cos\frac{4\pi}{9}\right)^{\frac{1}{3}} + \left(\cos\frac{8\pi}{9}\right)^{\frac{1}{3}} = \left(\frac{3 \cdot 9^{\frac{1}{3}} - 6}{2}\right)^{\frac{1}{3}} \tag{16}$$

Notice that (sce, e.g.,[6])

$$\left(x - \cos\frac{2\pi}{7}\right)\left(x - \cos\frac{4\pi}{7}\right)\left(x - \cos\frac{8\pi}{7}\right) = x^3 + \frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{8} \tag{17}$$

$$\left(x - \cos\frac{2\pi}{9}\right)\left(x - \cos\frac{4\pi}{9}\right)\left(x - \cos\frac{8\pi}{9}\right) = x^3 - \frac{3}{4}x + \frac{1}{8} \tag{18}$$

and both polynomials are RCP_s. Thus, by (11) we obtain(15) and (16). Besides, using (13), (17) and (18) we find

$$\left(\sec\frac{2\pi}{7}\right)^{\frac{1}{3}}+\left(\sec\frac{4\pi}{7}\right)^{\frac{1}{3}}+\left(\sec\frac{8\pi}{7}\right)^{\frac{1}{3}}=\left(8-6\cdot 7^{\frac{1}{3}}\right)^{\frac{1}{3}} \tag{19}$$

$$\left(\sec\frac{2\pi}{9}\right)^{\frac{1}{3}}+\left(\sec\frac{4\pi}{9}\right)^{\frac{1}{3}}+\left(\sec\frac{8\pi}{9}\right)^{\frac{1}{3}}=\left(6(9^{\frac{1}{3}}-1)\right)^{\frac{1}{3}} \tag{20}$$

Example 3. Quite recently R.Witula and D.Slota [7] found, in particular, the following decomposition

$$\begin{aligned} &x^3+105x^2+588x-343= \\ &=(x-2\sin^6\frac{2\pi}{7}\cos\frac{4\pi}{7})(x-2\sin^6\frac{4\pi}{7}\cos\frac{8\pi}{7})(x-2\sin^6\frac{8\pi}{7}\cos\frac{2\pi}{7}) \end{aligned} \tag{21}$$

This polynomial is an RCP. Therefore by (11) and (13) we obtain the following identities (the first of them is presented in [7])

$$\begin{aligned} &\sin^2\frac{2\pi}{7}\left(2\cos\frac{4\pi}{7}\right)^{\frac{1}{3}}+\sin^2\frac{4\pi}{7}\left(2\cos\frac{8\pi}{7}\right)^{\frac{1}{3}}+ \\ &+\sin^2\frac{8\pi}{7}\left(2\cos\frac{2\pi}{7}\right)^{\frac{1}{3}}=-\frac{1}{4}\left(63(1+7^{\frac{1}{3}})\right)^{\frac{1}{3}}; \end{aligned} \tag{22}$$

$$\begin{aligned} &\csc^2\frac{2\pi}{7}\left(2\sec\frac{4\pi}{7}\right)^{\frac{1}{3}}+\csc^2\frac{4\pi}{7}\left(2\sec\frac{8\pi}{7}\right)^{\frac{1}{3}}+ \\ &+\csc^2\frac{8\pi}{7}\left(2\sec\frac{2\pi}{7}\right)^{\frac{1}{3}}=7\left(441(2-7^{\frac{1}{3}})\right)^{\frac{1}{3}} \end{aligned} \tag{23}$$

3. SOME PROPERTIES OF RCP_s

Theorem 3. *If x^3+px^2+qx+r is an RCP with roots x_1, x_2, x_3 then*

1) For any $a\in\mathbb{R},\ \ a\neq 0$, the polynomial

$$x^3+apx^2+a^2qx+a^3r$$

is also an RCP with roots ax_1, ax_2, ax_3.

2) *The polynomial*

$$x^3+qx^2+(pr)x+r^2$$

is also an RCP with roots

$$\left\{\frac{r}{x_1}, \frac{r}{x_2}, \frac{r}{x_3}\right\}.$$

3) *The numbers*

$$\left\{\frac{r^{\frac{2}{3}}}{x_1}, \frac{r^{\frac{2}{3}}}{x_2}, \frac{r^{\frac{2}{3}}}{x_3}\right\}$$

are a permutation of the numbers

$$\left\{r^{\frac{1}{3}} - x_1,\ r^{\frac{1}{3}} - x_2,\ r^{\frac{1}{3}} - x_3\right\}.$$

4) $\frac{pq}{r} \leq \frac{9}{4}$.

*Proof.*1)-2) Straightforward.

3) Let

$$(x-(x_1-r^{\frac{1}{3}}))(x-(x_2-r^{\frac{1}{3}}))(x-(x_3-r^{\frac{1}{3}})) = x^3+p_1x^2+q_1x+r_1$$

After some simple calculations we have

$$\begin{gathered} p_1 = p + 3r^{\frac{1}{3}} \\ q_1 = q + 2pr^{\frac{1}{3}} + 3r^{\frac{2}{3}} \\ r_1 = r + qr^{\frac{1}{3}} + pr^{\frac{2}{3}} + r. \end{gathered}$$

Using (9) we find

$$p_1 = -\frac{q}{r^{\frac{1}{3}}},\ q_1 = pr^{\frac{1}{3}},\ r_1 = -r \tag{24}$$

Now we see that (9) is satisfied for numbers (24):

$$p_1 r_1^{\frac{1}{3}} + 3r_1^{\frac{2}{3}} + q_1 = 0.$$

i.e. the polynomial $x^3+p_1x^2+q_1x+r_1$ is an RCP with the roots $x_1 - r^{\frac{1}{3}},\ x_2 - r^{\frac{1}{3}},\ x_3 - r^{\frac{1}{3}}$.

According to 1) for $a = -r^{\frac{1}{3}}$ we notice that polynomial $x^3+qx^2+(pr)x+r^2$ has the roots

$$\left\{r^{\frac{1}{3}}(r^{\frac{1}{3}} - x_1),\ r^{\frac{1}{3}}(r^{\frac{1}{3}} - x_2),\ r^{\frac{1}{3}}(r^{\frac{1}{3}} - x_3)\right\}.$$

On the other hand, by 2) it has the roots $\left\{\frac{r}{x_1},\ \frac{r}{x_2},\ \frac{r}{x_3}\right\}$ and 3) follows.

4) Notice that, the condition (9) yields

$$\frac{pq}{r} = -\left(\frac{p}{r^{\frac{1}{3}}}\right)^2 - 3\left(\frac{p}{r^{\frac{1}{3}}}\right) \tag{25}$$

Thus,

$$\frac{9}{4} - \frac{pq}{r} = \frac{9}{4} + 3\frac{p}{r^{\frac{1}{3}}} + \left(\frac{p}{r^{\frac{1}{3}}}\right)^2 = \left(\frac{3}{2} + \frac{p}{r^{\frac{1}{3}}}\right)^2 \geq 0.\blacksquare$$

Example 4. By 1) of Theorem 1 together with the RCP (17) we have also the RCP $x^3 + x^2 - 2x - 1$ with the roots $x_1 = 2\cos\frac{2\pi}{7}$, $x_2 = 2\cos\frac{4\pi}{7}$, $x_3 = 2\cos\frac{8\pi}{7}$.

Example 5. For the RCP from Example 4 we have $\frac{r^{\frac{2}{3}}}{x_1} = \frac{1}{2}\sec\frac{2\pi}{7} > 0$ while $r^{\frac{1}{3}} - x_1 = -1 - 2\cos\frac{2\pi}{7} < 0$. Therefore, the permutation in 3) of Theorem 1 is not identical. We verity *approximately* by a pocket calculator the precise equalities

$$\frac{1}{x_1} = 1 + x_3,\ \frac{1}{x_2} = 1 + x_1,\ \frac{1}{x_3} = 1 + x_2.$$

Example 6. Analogously, by (18) we have the RCP $x^3 - 3x + 1$ with the roots $x_1 = 2\cos\frac{2\pi}{9}$, $x_2 = 2\cos\frac{4\pi}{9}$, $x_3 = 2\cos\frac{8\pi}{9}$. Here $\frac{r^{\frac{2}{3}}}{x_1} = \frac{1}{2}\csc\frac{2\pi}{9} > 0$ while $r^{\frac{1}{3}} - x_1 = 1 - 2\cos\frac{2\pi}{9} < 0$. Therefore, the permutation in

3) of Theorem 1 again is not identical. We verity as above that

$$\frac{1}{x_1} = 1 - x_2,\ \frac{1}{x_2} = 1 - x_3,\ \frac{1}{x_3} = 1 - x_1.$$

4. A NEW FORMULA AND IDENTITIES

Theorem 4. *In the conditions of Theorem 3 the following formula holds*

$$\left(\frac{x_1}{x_2}\right)^{\frac{1}{3}} + \left(\frac{x_2}{x_1}\right)^{\frac{1}{3}} + \left(\frac{x_1}{x_3}\right)^{\frac{1}{3}} + \left(\frac{x_3}{x_1}\right)^{\frac{1}{3}} + \left(\frac{x_2}{x_3}\right)^{\frac{1}{3}} + \left(\frac{x_3}{x_2}\right)^{\frac{1}{3}} = \left(\frac{pq}{r} - 9\right)^{\frac{1}{3}} \tag{26}$$

*Proof.*Denote X the left hand side of (26). Notice that, in [6] it was shown that (9) yields the following identity for the roots of RCP (10):

$$X = \left(\frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{x_1}{x_3} + \frac{x_3}{x_1} + \frac{x_2}{x_3} + \frac{x_3}{x_2} - 6\right)^{\frac{1}{3}}. \tag{27}$$

Thus, we have

$$X = (-\frac{1}{r}(x_1^2x_2 + x_2^2x_1 + x_1^2x_3 + x_3^2x_1 + x_2^2x_3 + x_3^2x_2) - 6)^{\frac{1}{3}} =$$

$$= (-\frac{1}{3r}((x_1+x_2+x_3)^3 - (x_1^3+x_2^3+x_3^3) - 6x_1x_2x_3) - 6)^{\frac{1}{3}}. \tag{28}$$

As is well known (see, e.g. [3])

$$x_1^3 + x_2^3 + x_3^3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3, \tag{29}$$

where $\sigma_i = \sigma_i(x_1, x_2, x_3), \quad i = 1, 2, 3,$ are the elementary symmetric polynomials. For RCP (10) we have

$$\sigma_1 = -p, \quad \sigma_2 = q, \quad \sigma_3 = -r. \tag{30}$$

Now by (28)-(30) we complete the proof:

$$X = \left(-\frac{1}{3r}(-p^3 + p^3 - 3pq + 3r + 6r) - 6\right)^{\frac{1}{3}} = \left(\frac{pq}{r} - 9\right)^{\frac{1}{3}}. \blacksquare$$

Example 7. Using the RCP (17),(18) by Theorem 4 we find the following identities:

$$\left(\frac{\cos\frac{2\pi}{7}}{\cos\frac{4\pi}{7}}\right)^{\frac{1}{3}} + \left(\frac{\cos\frac{4\pi}{7}}{\cos\frac{2\pi}{7}}\right)^{\frac{1}{3}} + \left(\frac{\cos\frac{2\pi}{7}}{\cos\frac{8\pi}{7}}\right)^{\frac{1}{3}} + \left(\frac{\cos\frac{8\pi}{7}}{\cos\frac{2\pi}{7}}\right)^{\frac{1}{3}} +$$

$$+ \left(\frac{\cos\frac{4\pi}{7}}{\cos\frac{8\pi}{7}}\right)^{\frac{1}{3}} + \left(\frac{\cos\frac{8\pi}{7}}{\cos\frac{4\pi}{7}}\right)^{\frac{1}{3}} = -7^{\frac{1}{3}}; \tag{31}$$

$$\left(\frac{\cos\frac{2\pi}{9}}{\cos\frac{4\pi}{9}}\right)^{\frac{1}{3}} + \left(\frac{\cos\frac{4\pi}{9}}{\cos\frac{2\pi}{9}}\right)^{\frac{1}{3}} + \left(\frac{\cos\frac{2\pi}{9}}{\cos\frac{8\pi}{9}}\right)^{\frac{1}{3}} + \left(\frac{\cos\frac{8\pi}{9}}{\cos\frac{2\pi}{9}}\right)^{\frac{1}{3}} +$$

$$+ \left(\frac{\cos\frac{4\pi}{9}}{\cos\frac{8\pi}{9}}\right)^{\frac{1}{3}} + \left(\frac{\cos\frac{8\pi}{9}}{\cos\frac{4\pi}{9}}\right)^{\frac{1}{3}} = -9^{\frac{1}{3}}. \tag{32}$$

Example 8. Using RCP(21) we find

$$\left(\frac{\sin\frac{2\pi}{7}}{\sin\frac{4\pi}{7}}\right)^2 \left(\frac{\cos\frac{4\pi}{7}}{\cos\frac{8\pi}{7}}\right)^{\frac{1}{3}} + \left(\frac{\sin\frac{4\pi}{7}}{\sin\frac{2\pi}{7}}\right)^2 \left(\frac{\cos\frac{8\pi}{7}}{\cos\frac{4\pi}{7}}\right)^{\frac{1}{3}} +$$

$$+\left(\frac{\sin\frac{2\pi}{7}}{\sin\frac{8\pi}{7}}\right)^2\left(\frac{\cos\frac{4\pi}{7}}{\cos\frac{2\pi}{7}}\right)^{\frac{1}{3}}+\left(\frac{\sin\frac{8\pi}{7}}{\sin\frac{2\pi}{7}}\right)^2\left(\frac{\cos\frac{2\pi}{7}}{\cos\frac{4\pi}{7}}\right)^{\frac{1}{3}}+$$

$$+\left(\frac{\sin\frac{4\pi}{7}}{\cos\frac{8\pi}{7}}\right)^2\left(\frac{\cos\frac{8\pi}{7}}{\cos\frac{2\pi}{7}}\right)^{\frac{1}{3}}+\left(\frac{\sin\frac{8\pi}{7}}{\sin\frac{4\pi}{7}}\right)^2\left(\frac{\cos\frac{2\pi}{7}}{\cos\frac{8\pi}{7}}\right)^{\frac{1}{3}}=-3\cdot 7^{\frac{1}{3}}. \tag{33}$$

5. ON RCP WITH THE SAME VALUE OF $\frac{pq}{r}$

Theorem 5. *If for two RCP_s*

$$y^3+p_1y^2+q_1y+r_1,\quad z^3+p_2z^2+q_2z+r_2$$

the following condition holds

$$\frac{p_1q_1}{r_1}=\frac{p_2q_2}{r_2}$$

then for its roots $y_1,y_2,y_3;\ \ z_1,z_2,z_3$ the numbers

$$\left\{\frac{y_1}{y_2},\ \frac{y_2}{y_1},\ \frac{y_1}{y_3},\ \frac{y_3}{y_1},\ \frac{y_2}{y_3},\ \frac{y_3}{y_2}\right\}$$

are a permutation of the numbers

$$\left\{\frac{z_1}{z_2},\ \frac{z_2}{z_1},\ \frac{z_1}{z_3},\ \frac{z_3}{z_1},\ \frac{z_2}{z_3},\ \frac{z_3}{z_2}\right\}$$

*Proof.*Evidently, it is sufficient to prove that if x_1,x_2,x_3 are roots of RCP (10) then the numbers

$$\xi_1=\frac{x_1}{x_2}+(\frac{x_1}{x_2})^{-1},\ \ \xi_2=\frac{x_1}{x_3}+(\frac{x_1}{x_3})^{-1},\ \ \xi_3=\frac{x_2}{x_3}+(\frac{x_2}{x_3})^{-1} \tag{34}$$

depend on $\frac{pq}{r}$ only.

For elementary symmetric polynomials of ξ_1,ξ_2,ξ_3 we have

$$\xi_1+\xi_2+\xi_3=-\frac{1}{r}(x_1^2x_2+x_2^2x_1+x_1^2x_3+x_3^2x_1+x_2^2x_3+x_3^2x_1) \tag{35}$$

and as above ((28)-(29)) we find

$$\xi_1+\xi_2+\xi_3=\frac{pq}{r}-3. \tag{36}$$

$$\xi_1\xi_2+\xi_1\xi_3+\xi_2\xi_3=\frac{1}{r^2}(-r(x_1^3+x_2^3+x_3^3)+(x_1x_2)^3+(x_1x_3)^3+(x_2x_3)^3-$$
$$-r(x_1x_2^2+x_2x_1^2+x_1^2x_3+x_3^2x_1+x_2^2x_3+x_3^2x_2)).$$

Using (35)-(36) and taking into account that by (29),(30)

$$x_1^3 + x_2^3 + x_3^3 = -p^3 + 3pq - 3r,$$
$$(x_1x_2)^3 + (x_1x_3)^3 + (x_2x_3)^3 = -(p*)^3 + 3p^*q^* - 3r^*,$$

where

$$p^* = -(x_1x_2 + x_1x_3 + x_2x_3) = -q,$$
$$q^* = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 = x_1x_2x_3(x_1 + x_2 + x_3) = pr,$$
$$r^* = -(x_1x_2x_3)^2 = -r^2,$$

such that

$$(x_1x_2)^3 + (x_1x_3)^3 + (x_2x_3)^3 = q^3 - 3pqr + 3r^2,$$

we have

$$\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3 = 4\frac{pq}{r} - 24. \tag{37}$$

It is interesting that, *only* for proof that $\xi_1\xi_2\xi_3$ depends on $\frac{pq}{r}$, we use condition (9). Indeed, by (34) we have

$$\xi_1\xi_2\xi_3 = \frac{1}{r^2}(x_1^2 + x_2^2 + x_3^2)((x_1x_2)^2 + (x_1x_3)^2 + (x_2x_3)^2) - 1.$$

Taking into account that

$$x_1^2 + x_2^2 + x_3^2 = p^2 - 2q$$

$$(x_1x_2)^2 + (x_1x_3)^2 + (x_2x_3)^2 = (p^*)^2 - 2q^* = q^2 - 2pr$$

we find

$$\xi_1\xi_2\xi_3 = \left(\frac{pq}{r}\right)^2 + 4\left(\frac{pq}{r}\right) - 1 - 2\frac{p^3r + q^3}{r^2}. \tag{38}$$

Since by (9)

$$pr^{\frac{1}{3}} + q = -3r^{\frac{2}{3}}$$

then we have

$$\frac{p^3r+q^3}{r^2} = \frac{1}{r^2}(pr^{\frac{1}{3}}+q)(p^2r^{\frac{2}{3}}-pqr^{\frac{1}{3}}+q^2) =$$
$$= -\frac{3}{r^{\frac{4}{3}}}(pr^{\frac{1}{3}}+q)^2-3pqr^{\frac{1}{3}}) =$$
$$= -\frac{3}{r^{\frac{4}{3}}}(9r^{\frac{4}{3}}-3pqr^{\frac{1}{3}}).$$

Now according to (38) we find

(39) $$\xi_1\xi_2\xi_3 = \left(\frac{pq}{r}-7\right)^2+4.$$

Thus, by (36), (37) and (39) the numbers ξ_1,ξ_2,ξ_3 (34) are the roots of the following polynomial

(40) $$\xi^3-\left(\frac{pq}{r}-3\right)\xi^2+4\left(\frac{pq}{r}-6\right)\xi-\left(\left(\frac{pq}{r}-7\right)^2+4\right)$$

and, consequently, they depend on $\frac{pq}{r}$ only.■

Notice that, using (25) we see that the numbers $\frac{x_1}{x_2},\frac{x_2}{x_1},\frac{x_1}{x_3},\frac{x_3}{x_1},\frac{x_2}{x_3},\frac{x_3}{x_2}$ depend even on $\frac{p}{r^{\frac{1}{3}}}$ only.

Example 9. Together with decomposition (17) with $\frac{pq}{r}=2$ in [7] the following decomposition was found:

$$x^3+7x^2-98x-343 =$$
$$\left(x-128\cos\frac{2\pi}{7}\left(\sin\frac{2\pi}{7}\sin\frac{8\pi}{7}\right)^3\right)\left(x-128\cos\frac{4\pi}{7}\left(\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right)^3\right)\times$$
$$\times\left(x-128\cos\frac{8\pi}{7}\left(\sin\frac{4\pi}{7}\sin\frac{8\pi}{7}\right)^3\right).$$

It is an RCP with also $\frac{pq}{r}=2$. Hence, by Theorem 5 the numbers

$$\left\{\frac{\cos\frac{2\pi}{7}}{\cos\frac{4\pi}{7}}\left(\frac{\sin\frac{8\pi}{7}}{\sin\frac{4\pi}{7}}\right)^3,\ \frac{\cos\frac{4\pi}{7}}{\cos\frac{2\pi}{7}}\left(\frac{\sin\frac{4\pi}{7}}{\sin\frac{8\pi}{7}}\right)^3,\ \frac{\cos\frac{2\pi}{7}}{\cos\frac{8\pi}{7}}\left(\frac{\sin\frac{2\pi}{7}}{\sin\frac{4\pi}{7}}\right)^3,\right.$$
$$\left.\frac{\cos\frac{8\pi}{7}}{\cos\frac{2\pi}{7}}\left(\frac{\sin\frac{4\pi}{7}}{\sin\frac{2\pi}{7}}\right)^3,\ \frac{\cos\frac{4\pi}{7}}{\cos\frac{8\pi}{7}}\left(\frac{\sin\frac{2\pi}{7}}{\sin\frac{8\pi}{7}}\right)^3,\ \frac{\cos\frac{8\pi}{7}}{\cos\frac{4\pi}{7}}\left(\frac{\sin\frac{8\pi}{7}}{\sin\frac{2\pi}{7}}\right)^3\right\}$$

are a permutation of the numbers

$$\left\{\frac{\cos\frac{2\pi}{7}}{\cos\frac{4\pi}{7}},\quad \frac{\cos\frac{4\pi}{7}}{\cos\frac{2\pi}{7}},\quad \frac{\cos\frac{2\pi}{7}}{\cos\frac{8\pi}{7}},\quad \frac{\cos\frac{8\pi}{7}}{\cos\frac{2\pi}{7}},\quad \frac{\cos\frac{4\pi}{7}}{\cos\frac{8\pi}{7}},\quad \frac{\cos\frac{8\pi}{7}}{\cos\frac{4\pi}{7}}\right\}.$$

To get the corresponding equalities, it is sufficient to verify *approximately* by a pocket calculator that

$$\frac{\cos\frac{2\pi}{7}}{\cos\frac{4\pi}{7}}\left(\frac{\sin\frac{8\pi}{7}}{\sin\frac{4\pi}{7}}\right)^3=\frac{\cos\frac{4\pi}{7}}{\cos\frac{8\pi}{7}},\quad \frac{\cos\frac{2\pi}{7}}{\cos\frac{8\pi}{7}}\left(\frac{\sin\frac{2\pi}{7}}{\sin\frac{4\pi}{7}}\right)^3=\frac{\cos\frac{4\pi}{7}}{\cos\frac{2\pi}{7}},$$

$$\frac{\cos\frac{4\pi}{7}}{\cos\frac{8\pi}{7}}\left(\frac{\sin\frac{2\pi}{7}}{\sin\frac{8\pi}{7}}\right)^3=\frac{\cos\frac{8\pi}{7}}{\cos\frac{2\pi}{7}}, \tag{41}$$

etc.

Notice that, in [7] R.Witula and D.Slota found some other RCP_s with roots which depend on sines or cosines of arguments $\frac{2\pi}{7}, \frac{4\pi}{7}, \frac{8\pi}{7}$. However, it is interesting that, as we verified, for them $\frac{pq}{r}$ assumes *only* the values $2, -40, -180$.

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Departments of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel. e-mail:shevelev@bgu.ac.il