Distance Distributions in Ensembles of Irregular Low-Density Parity-Check Codes

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***Abstract*—We derive asymptotic expressions for the average distance distributions in ensembles of irregular low-density parity-check (LDPC) codes. The ensembles are defined by matrices with given profiles of column and row sums.**

***Index Terms*—Distance distributions, low-density parity-check (LDPC) codes.**

I. Introduction

The low-density parity-check (LDPC) codes are famous because of their performance in the vicinity of the Shannon limit under iterative decoding of modest complexity. However, this phenomenon is still far from being completely understood. One of the important parameters helping in analysis of codes' performance under maximum-likelihood decoding is their distance distribution. Such estimates are important in analyzing achievable limits of performance of LDPC codes, and optimization of their parameters.

The problem of estimation distance distribution of regular codes (i.e., codes defined by parity-check matrices with fixed column and row sums) was addressed in many papers, starting with the original Gallager's work [2], see for most general results [5]. Use of such estimates to bounding performance of LDPC codes was discussed in [2], [7], [9], [13].

Recently, it was found that irregular LDPC codes (i.e., codes defined by parity-check matrices with several possible values of column and row sums) perform better than the regular codes under iterative decoding [6], [11], [12], [14].

In this paper, we solve the problem of estimation of the average distance distribution in ensembles of irregular LDPC codes. This question was addressed independently in [4], where an implicit expression for such distributions was given. Here we give an explicit formula describing the average distance distributions.

The paper is organized as follows. We start by defining the considered ensemble of irregular codes, and define the average distance distribution in this ensemble. Then we count the number of matrices belonging to the ensemble of special shape, yielding that the defined codes contain the word having ones on the first w positions and zeros on the remaining positions. Subsequently, we show how the previous result can be used for the general problem. Optimization of parameters in the derived expression accomplishes the proof. We conclude with a discussion on the number of solutions to a system of equations appearing in the optimization problem, and provide some evidence for a conjecture that the system possesses at most three solutions.

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II. Ensemble of Irregular Codes

We consider the following ensemble $\mathcal{C}_{m,n}$ of irregular codes associated with ensemble $\mathcal{H}_{m,n}$ of parity-check matrices. The codes are defined by $m \times n$ matrices from $\mathcal{H}_{m,n}$, and thus have rate R at least $1 - \frac{m}{n}$. Let h, g, and $r_1, r_2, \dots, r_g, s_1, s_2, \dots, s_h$ be nonnegative integers independent of n

$$\eta_1, \eta_2, \dots, \eta_h \in (0,1], \qquad \sum_{j=1}^{h} \eta_j = 1 \tag{1}$$

$$\nu_1, \nu_2, \dots, \nu_g \in (0,1], \qquad \sum_{i=1}^{g} \nu_i = 1. \tag{2}$$

Moreover, assume that the numbers $\nu_i m$ for $i = 1, \dots, g$ and $\eta_j n$ for $j = 1, \dots, h$ are integral. Let the following intervals of integer numbers be defined:

$$I_i = [1 + \sum_{\ell=1}^{i-1} \nu_\ell m, \sum_{\ell=1}^{i} \nu_\ell m], \qquad i = 1, \dots, g \tag{3}$$

$$J_j = [1 + \sum_{\ell=1}^{j-1} \eta_\ell n, \sum_{\ell=1}^{j} \eta_\ell n], \qquad j = 1, \dots, h. \tag{4}$$

An $m \times n$ matrix $H = (h_{i,j})$ belongs to the ensemble $\mathcal{H}_{m,n}$ if for every $i \in I_\ell$

$$\sum_{j=1}^{n} h_{i,j} = r_\ell \tag{5}$$

and for every $j \in J_\ell$

$$\sum_{i=1}^{m} h_{i,j} = s_\ell. \tag{6}$$

In other words, we partition the rows of the matrix into g strips, each of the size $\nu_i m$. Also, we partition all the columns into h strips, each of size $\eta_j n$. A matrix belongs to the defined ensemble if the row sums of the rows belonging to the ℓth (horizontal) strip are r_ℓ, $\ell = 1, \dots, g$, and the column sums of the columns in the ℓth (vertical) strip are s_ℓ, $\ell = 1, \dots, h$.

Given the parity-check matrix H, a word $\boldsymbol{v}$ belongs to the code if $H\boldsymbol{v}^T = \mathbf{0}$, i.e., if all the row sums of the submatrix of H consisting of the columns corresponding to the nonzero places in $\boldsymbol{v}$ are even.

Let θ be a number in the interval $(0, 1)$. Given a matrix $H \in \mathcal{H}_{m,n}$, we may find the number of submatrices consisting of θn columns of H having even row sums. Clearly, it is equal to the number of codewords of weight θn in the code defined by the matrix H. Let us define a class of matrices $\mathcal{H}_{m,n,\theta} \subset \mathcal{H}_{m,n}$ consisting of all matrices such that the sum of the first θn entries in each row is even.

Given a specific matrix, it seems to be a generally intractable problem for modestly large sizes of the matrix. However, the problem becomes simpler if we aim at determining the average of this number in the ensemble $\mathcal{C}_{m,n}$. Let us define

$$b_\theta := \frac{1}{n} \ln B_{\theta n} \tag{7}$$

where

$$B_{\theta n} = \frac{1}{|\mathcal{C}_{m,n}|} \sum_{C \in \mathcal{C}_{m,n}} |\boldsymbol{c} : \boldsymbol{c} \in C, \mathrm{wt}(\boldsymbol{c}) = \theta n| \tag{8}$$

and $\mathrm{wt}\,(\boldsymbol{c})$ is the Hamming weight of vector $\boldsymbol{c}$. Let

$$H(x) = -x \ln x - (1-x)\ln(1-x) \tag{9}$$

be the natural entropy function.

The main result of the paper is as follows.

Theorem 1: Let $\theta \in (0,1)$, $r_1, r_2, \dots, r_g$, $s_1, s_2, \dots, s_h$, be positive integers, $\gamma = \frac{m}{n}$, $\nu_1, \nu_2, \dots, \nu_g, \eta_1, \eta_2, \dots, \eta_h$, be numbers from $(0,1)$, such that

$$\sum_{i=1}^{g} \nu_i = 1, \quad \sum_{j=1}^{h} \eta_h = 1$$

and let

$$s = \sum_{i=1}^{h} \eta_i s_i,$$

$$\tilde{\theta} = 1 - \frac{\gamma}{s} \sum_{i=1}^{g} \nu_i r_i \frac{\rho_{r_i - 1}(t)}{\rho_{r_i}(t)} \tag{10}$$

where

$$\rho_j(t) = (1+t)^j + (1-t)^j.$$

Let

$$x_i = x_i(t, \xi) = \frac{\eta_i}{1 + \zeta^{-s_i} \xi^{-1}} \tag{11}$$

where

$$\zeta = \frac{\tilde{\theta}}{t(1-\tilde{\theta})}. \tag{12}$$

Let a finite discrete set $T = \{t_1, \dots\}$ be the set of positive first components of solutions (t_i, ζ_i) to the system

$$\sum_{i=1}^{h} x_i = \theta \tag{13}$$

$$\sum_{i=1}^{h} x_i s_i = \tilde{\theta} s. \tag{14}$$

Then if

$$\tilde{\theta} \le 1 - \frac{\sum_{i=1}^{g} \pi(r_i)}{\sum_{i=1}^{g} r_i} \tag{15}$$

where $\pi(a)$ is the parity function, $\pi(a) = 0$ if a is even, and $\pi(a) = 1$ otherwise, then

$$b_\theta = \max_{t \in T} \Bigg\{ -sH(\tilde{\theta}) + \gamma \sum_{i=1}^{g} \nu_i \ln \rho_{r_i}(t) - \gamma \ln 2 - s\tilde{\theta} \ln t + \sum_{i=1}^{h} \eta_i H\left(\frac{x_i}{\eta_i}\right) \Bigg\}. \tag{16}$$

If (15) does not hold then

$$b(\theta) = -\infty. \qquad \square$$

Note that the system (13), (14) is of polynomial type. There is numerical evidence that the number of positive solutions to the system, i.e., the size of T, does not exceed 3. We conjecture it always to be true. In the generalized regular case (all s_i's are equal) we have only one solution.

III. Proof of Theorem 1

Let us sketch the main steps of the proof of Theorem 1. We start in Section III-A from considering a subclass of $\mathcal{H}_{m,n}$ denoted $\mathcal{H}_{m,n,\theta}$ in the case when all the horizontal blocks have equal sizes. Using a technical lemma in Section III-B, we find the proportion P of the matrices from $\mathcal{H}_{m,n,\theta}$ within the class $\mathcal{H}_{m,n}$. The expression for P is given in the form of a finite sum which is logarithmically equivalent to its maximum summand. Furthermore, in Sections III-C–III-E, we compute the asymptotics for P (the main obstacle is overcome in Section III-D, where we solve a system of equations in respect to parameters corresponding to the maximal summand). In Section III-F, we establish two lemmas allowing generalization of the obtained result to the case of arbitrary sizes of horizontal blocks. In Section III-G, we find the asymptotics for P in the general case. In Section III-H, we prove the uniqueness of the positive solution to an equation appearing in the proof. Further, in Section III-I, we study $\lim_{n\to\infty} \frac{1}{n} \ln P$ as a function in a special parameter $\tilde{\theta}$ depending on θ. In Section III-J, we maximize the sum of two components of the distribution, namely, P and C depending on partition of θ within the vertical blocks. This accomplishes the proof of Theorem 1.

A. Counting Matrices of Given Profile

We start with a lemma allowing estimation of the number of matrices with given row and column profiles.

Lemma 1: Let $\mathcal{M}_n$ be an ensemble of $m \times n$ binary matrices

$$m = \gamma n, \quad 0 < \gamma \le 1 \tag{17}$$

$$|\Lambda^{\text{left}}| \overset{\ln}{\sim} \frac{(n(s_1\eta_1+s_2\eta_2+\cdots+s_{q-1}\eta_{q-1}+s_q(\theta-\eta_1-\eta_2-\cdots-\eta_{q-1})))!}{\prod_{i=1}^{g}\left(2!^{m_2^{(i)}}4!^{m_4^{(i)}}\ldots(r_i-\pi(r_i))!^{m^{(i)}_{r_i-\pi(r_i)}}\right)}\left(s_1!^{\eta_1 n}\ldots s_{q-1}!^{\eta_{q-1}n}s_q!^{(\theta-\eta_1-\cdots-\eta_{q-1})n}\right)^{-1} \quad (26)$$

$$|\Lambda^{\text{right}}| \overset{\ln}{\sim} \frac{(n((\eta_1+\cdots+\eta_q-\theta)s_q+s_{q+1}\eta_{q+1}+\cdots+s_h\eta_h))!}{\prod_{i=1}^{g}\left((r_i)!^{m_0^{(i)}}(r_i-2)!^{m_2^{(i)}}\ldots(2+\pi(r_i))!^{m^{(i)}_{r_i-\pi(r_i)-2}}\right)}\left(s_q!^{(\eta_1+\cdots+\eta_q-\theta)n}s_{q+1}!^{\eta_{q+1}n}\ldots s_h!^{\eta_h n}\right)^{-1} \quad (27)$$

$$|\mathcal{H}_{m,n}| \overset{\ln}{\sim} \frac{(n(\eta_1 s_1+\eta_2 s_2+\cdots+\eta_h s_h))!}{s_1!^{\eta_1 n}\ldots s_h!^{\eta_h n}r_1!^{m/g}\ldots r_g!^{m/g}} \quad (28)$$

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such that their row sums $r^{(i)}$, $i=1,\ldots,m$, and column sums $s^{(j)}$, $j=1,\ldots,n$, satisfy

1) *All* $r^{(i)}$'s and $s^{(j)}$'s are bounded, i.e., there exists a positive constant c bounding from above all the row and column sums.
2) All column sums are strictly positive. Then

$$|\mathcal{M}_n| \overset{\ln}{\sim} \frac{\left(\sum_{i=1}^{m} r^{(i)}\right)!}{\left(\prod_{i=1}^{m} r^{(i)}!\right)\left(\prod_{j=1}^{n} s^{(j)}!\right)} \quad (18)$$

where $a_n \overset{\ln}{\sim} b_n$ means that

$$\lim_{n\to\infty}\frac{\ln a_n}{\ln b_n}=1.$$

Proof: According to a modification [5] for rectangular matrices of a theorem due to O'Neil [10]

$$|\mathcal{M}_n| \sim \frac{\left(\sum_{i=1}^{m} r^{(i)}\right)!}{\left(\prod_{i=1}^{m} r^{(i)}\right)!\left(\prod_{j=1}^{n} s^{(j)}\right)!} \times\exp\left(\frac{-1}{2\left(\sum_{i=1}^{m} r^{(i)}\right)^2}\left(\sum_{i=1}^{m} r^{(i)}(r^{(i)}-1)\right)\left(\sum_{j=1}^{n} s^{(j)}(s^{(j)}-1)\right)\right). \quad (19)$$

By (17) and the conditions of the lemma

$$\sum_{i=1}^{m} r^{(i)} = \sum_{j=1}^{n} s^{(j)} \geq n \quad (20)$$

and

$$\frac{1}{2\left(\sum_{i=1}^{m} r^{(i)}\right)^2}\left(\sum_{i=1}^{m} r^{(i)}(r^{(i)}-1)\right)\left(\sum_{j=1}^{n} s^{(j)}(s^{(j)}-1)\right) \leq \frac{c^4\gamma n^2}{2n^2}=c_1.$$

Thus, the exponent in (19) is between two constants, e^{-c_1} and 1. It is left to prove that $|\mathcal{M}_n|$ tends to infinity with n. This easily follows since by (17) and (20)

$$\frac{\left(\sum_{i=1}^{m} r^{(i)}\right)!}{\left(\prod_{i=1}^{m} r^{(i)}!\right)\left(\prod_{j=1}^{n} s^{(j)}!\right)} \geq \frac{n!}{c!^{(1+\gamma)n}} \to \infty. \qquad \square$$

B. Counting Matrices From $\mathcal{H}_{m,n,\theta}$ for the Case of Horizontal Blocks of Equal Sizes

We deal with the ensemble $\mathcal{H}_{m,n}$ defined in the previous section. Consider first the case $\nu_1=\nu_2=\cdots=\nu_g=1/g$. The following balance identity holds:

$$\frac{m}{g}\sum_{i=1}^{g} r_i = n\sum_{j=1}^{h}\eta_j s_j. \quad (21)$$

Let for a matrix $A\in\mathcal{H}_{m,n}$ the following extra condition holds: *the sum of the first* $w=\theta n$, $0<\theta<1$, *entries in each row is even*. In this case, we write that

$$A\in\mathcal{H}_{m,n,\theta}.$$

The matrix A is comprised of g horizontal blocks and h vertical blocks, such that the row sums in each horizontal block and column sums in each vertical block are the same for all the rows and for all the columns. Given θ we may determine q such that

$$\theta\in[\eta_1+\eta_2+\cdots+\eta_{q-1},\eta_1+\eta_2+\cdots+\eta_q) \quad (22)$$

(in case $q=1$ we assume the left sum to be equal to 0). Suppose that among the rows of the ith horizontal block there are $m_v^{(i)}$ rows such that the sum of their first w entries equals v, $v=0,2,\ldots,r_i-\pi(r_i)$, π is the parity function

$$\pi(a)=\begin{cases}0, & a \text{ is even}\\ 1, & \text{otherwise.}\end{cases}$$

The matrix A is thus partitioned into two parts, A^{left} and A^{right}, containing correspondingly the first w and the last $n-w$ columns of A. Let
$\Lambda^{\text{left}}=\Lambda^{\text{left}}\left(m_0^{(1)},\ldots,m_0^{(i)},\ldots,m^{(i)}_{r_i-\pi(r_i)},\ldots,m^{(g)}_{r_g-\pi(r_g)}\right)$
be the ensemble of all possible matrices A^{left}. Analogously, we define ensemble Λ^{right}. The probability that an arbitrary matrix picked from the ensemble $\mathcal{H}_{m,n}$ belongs to $\mathcal{H}_{m,n,\theta}$ is

$$P=\frac{\sum\left(\prod_{i=1}^{g}\binom{m/g}{m_0^{(i)},m_2^{(i)},\ldots,m^{(i)}_{r_i-\pi(r_i)}}|\Lambda^{\text{left}}||\Lambda^{\text{right}}|\right)}{|\mathcal{H}_{m,n}|} \quad (23)$$

where the summation is over all $m_v^{(i)}$, $v=0,2,\ldots,r_i-\pi(r_i)$, $i=1,2,\ldots,g$, satisfying the following conditions:

$$\sum_{v=0,2,\ldots,r_i-\pi(r_i)} m_v^{(i)}=\frac{m}{g}, \qquad i=1,2,\ldots,g \quad (24)$$

$$\sum_{i=1}^{g}\sum_{v=0,2,\ldots,r_i-\pi(r_i)} v m_v^{(i)} = n(s_1\eta_1+s_2\eta_2+\cdots+s_{q-1}\eta_{q-1} +s_q(\theta-\eta_1-\eta_2-\cdots-\eta_{q-1})). \quad (25)$$

By (18), we have (26)-(28) at the top of this page, and, finally, by (23), (24), and (26)–(28) we get the expression (29) at the

$$P \stackrel{\ln}{\sim} \left(\begin{matrix} n(\eta_1 s_1 + \cdots + \eta_h s_h) \\ n(\eta_1 s_1 + \cdots + \eta_{q-1} s_{q-1} + s_q(\theta - \eta_1 - \cdots - \eta_{q-1})) \end{matrix} \right)^{-1}$$
$$\times \sum \prod_{i=1}^{g} \frac{\binom{m/g}{m_0^{(i)}, m_2^{(i)}, \ldots, m_{r_i-\pi(r_i)}^{(i)}} r_i!^{m_0^{(i)}+m_2^{(i)}+\cdots+m_{r_i-\pi(r_i)}^{(i)}}}{(0!(r_i)!)^{m_0^{(i)}}(2!(r_i-2)!)^{m_2^{(i)}} \ldots ((r_i-\pi(r_i))!(\pi(r_i))!)^{m_{r_i-\pi(r_i)}^{(i)}}}$$
$$= \left(\begin{matrix} n(\eta_1 s_1 + \cdots + \eta_h s_h) \\ n(\eta_1 s_1 + \cdots + \eta_{q-1} s_{q-1} + s_q(\theta - \eta_1 - \cdots - \eta_{q-1})) \end{matrix} \right)^{-1}$$
$$\times \sum \prod_{i=1}^{g} \binom{m/g}{m_0^{(i)}, m_2^{(i)}, \ldots, m_{r_i-\pi(r_i)}^{(i)}} \binom{r_i}{2}^{m_2^{(i)}} \binom{r_i}{4}^{m_4^{(i)}} \cdots$$
$$\cdots \times \binom{r_i}{r_i - \pi(r_i) - 2}^{m_{r_i-\pi(r_i)-2}^{(i)}} \binom{r_i}{r_i - \pi(r_i)}^{m_{r_i-\pi(r_i)}^{(i)}} \quad (29)$$

top of the page, where the summation is over all numbers $m_v^{(i)}$ satisfying (24) and (25).

Set

$$s := \eta_1 s_1 + \cdots + \eta_h s_h \quad (30)$$
$$\tilde{\theta} := \frac{\eta_1 s_1 + \cdots + \eta_{q-1} s_{q-1} + s_q(\theta - \eta_1 - \cdots - \eta_{q-1})}{s}. \quad (31)$$

Evidently

$$0 < \tilde{\theta} < 1. \quad (32)$$

Furthermore,

$$P \stackrel{\ln}{\sim} \binom{ns}{n\tilde{\theta}s}^{-1} \Sigma_1 \quad (33)$$

where Σ_1 is the sum from (29).

From (24) and (25), or, in the new notations (30) and (31), the conditions become

$$\sum_{v=0,2,\ldots,r_i-\pi(r_i)} m_v^{(i)} = \frac{m}{g}, \qquad i = 1, 2, \ldots, g \quad (34)$$
$$\sum_{i=1}^{g} \sum_{v=2,4,\ldots,r_i-\pi(r_i)} v m_v^{(i)} = ns\tilde{\theta}. \quad (35)$$

Notice that for $P > 0$, the following upper bound is valid for $\tilde{\theta}$ (and, therefore, by (31), for θ):

$$\tilde{\theta} \le 1 - \frac{\sum_{i=1}^{g} \pi(r_i)}{\sum_{i=1}^{g} r_i}. \quad (36)$$

Indeed, by (34),

$$m_{r_i-\pi(r_i)}^{(i)} = \frac{m}{g} - \sum_{v=0,2,\ldots,r_i-\pi(r_i)-2} m_v^{(i)}.$$

Thus, (35) yields the expression at the bottom of the page and the equality

$$\sum_{i=1}^{g} m_0^{(i)}(r_i - \pi(r_i)) = -\sum_{i=1}^{g} \sum_{v=2,4,\ldots,r_i-\pi(r_i)-2} ((r_i - \pi(r_i)) - v) m_v^{(i)} + \frac{m}{g} \sum_{i=1}^{g} (r_i - \pi(r_i)) - ns\tilde{\theta}$$

follows. Thus, only if all r_i's are even there is no essential restriction on θ.

Since

$$\sum_{i=1}^{g} m_0^{(i)}(r_i - \pi(r_i)) \ge 0$$

then, necessarily

$$\frac{m}{g} \sum_{i=1}^{g} (r_i - \pi(r_i)) \ge ns\tilde{\theta}$$

or

$$\tilde{\theta} \le \frac{\gamma}{sg} \sum_{i=1}^{g} (r_i - \pi(r_i)). \quad (37)$$

However, by (30) and (21)

$$s = \frac{\gamma}{g} \sum_{i=1}^{g} r_i \quad (38)$$

and, finally, (38) and (37) yield (36).

Denote

$$M_{n,\theta} = \max \left\{ \prod_{i=1}^{g} \binom{m/g}{m_0^{(i)}, m_2^{(i)}, \ldots, m_{r_i-\pi(r_i)}^{(i)}} \times \binom{r_i}{2}^{m_2^{(i)}} \binom{r_i}{4}^{m_4^{(i)}} \cdots \binom{r_i}{r_i - \pi(r_i)}^{m_{r_i-\pi(r_i)}^{(i)}} \right\} \quad (39)$$

$$\sum_{i=1}^{g} \left(\sum_{v=2,4,\ldots,r_i-\pi(r_i)-2} v m_v^{(i)} + (r_i - \pi(r_i)) \left(\frac{m}{g} - \sum_{v=0,2,\ldots,r_i-\pi(r_i)-2} m_v^{(i)} \right) \right) = ns\tilde{\theta}$$

where the maximum is taken over all $m_v^{(i)}$ satisfying (34) and (35).

Since $\sum_1$ in (33) contains a number of summands that is polynomial in n, we have

$$P \overset{\ln}{\sim} \exp(-nsH(\tilde{\theta}))M_{n,\theta}. \tag{40}$$

Moreover, the asymptotics for $M_{n,\theta}$ is calculated under conditions (34), (35).

C. The Basic System of Equations for Computing the Asymptotics of $M_{n,\theta}$

By (34) and (35)

$$m_0^{(i)} = \frac{m}{g} - m_2^{(i)} - \cdots - m_{r_i-\pi(r_i)}^{(i)}, \qquad i = 1, 2, \ldots, g-1 \tag{41}$$

$$m_0^{(g)} + m_{r_g-\pi(r_g)}^{(g)} = \frac{m}{g} - m_2^{(g)} - \cdots - m_{r_g-\pi(r_g)-2}^{(g)} \tag{42}$$

$$\begin{aligned}(r_g - \pi(r_g))m_{r_g-\pi(r_g)}^{(g)} &= ns\tilde{\theta} - \left(\sum_{j=1}^{g-1} \sum_{v=2,4,\ldots,r_j-\pi(r_j)} vm_v^{(j)}\right) - 2m_2^{(g)} \\ &\quad - \cdots - (r_g - \pi(r_g) - 2)m_{r_g-\pi(r_g)-2}^{(g)}.\end{aligned} \tag{43}$$

Notice that by (34), $r_g > \pi(r_g)$. Thus, we assume in what follows that $r_g \neq 1$. Therefore, by (43)

$$\begin{aligned}m_{r_g-\pi(r_g)}^{(g)} = \kappa_g &:= \frac{ns\tilde{\theta}}{r_g - \pi(r_g)} - \frac{1}{r_g - \pi(r_g)}\left(\sum_{j=1}^{g-1} \sum_{v=2,4,\ldots,r_j-\pi(r_j)} vm_v^{(j)}\right) \\ &\quad - \frac{2}{r_g - \pi(r_g)}m_2^{(g)} - \cdots - \frac{r_g - \pi(r_g) - 2}{r_g - \pi(r_g)}m_{r_g-\pi(r_g)-2}^{(g)}\end{aligned} \tag{44}$$

$$\begin{aligned}m_0^{(g)} = \tilde{\kappa}_0^{(g)} &:= \frac{m}{g} - \frac{ns\tilde{\theta}}{r_g - \pi(r_g)} + \frac{1}{r_g - \pi(r_g)}\left(\sum_{j=1}^{g-1} \sum_{v=2,4,\ldots,r_j-\pi(r_j)} vm_v^{(j)}\right) \\ &\quad + \left(\frac{2}{r_g - \pi(r_g)} - 1\right)m_2^{(g)} + \cdots + \left(\frac{r_g - \pi(r_g) - 2}{r_g - \pi(r_g)} - 1\right)m_{r_g-\pi(r_g)-2}^{(g)}\end{aligned} \tag{45}$$

$$m_0^{(i)} = \kappa_0^{(i)} := \frac{m}{g} - m_2^{(i)} - \cdots - m_{r_i-\pi(r_i)}^{(i)}, \quad i = 1, \ldots, g-1. \tag{46}$$

By (39) and (34) we get (47) at the bottom of the page. Equating to zero the partial derivatives in $m_{2v}^{(i)}$, $v = 1, 2, \ldots, (r_i - \pi(r_i))/2$, $i = 1, 2, \ldots, g$, of the sum from (47) (derivatives in all but $m_{r_g-\pi(r_g)}^{(g)}$). From (44) and (46) we find the following derivatives:

1) in $m_{2v}^{(i)}$, $i = 1, \ldots, g-1$, $v = 1, 2, \ldots, (r_i - \pi(r_i))/2$

$$\begin{aligned}\ln \kappa_0^{(i)} - \ln m_{2v}^{(i)} + \frac{2v}{r_g - \pi(r_g)} \ln \kappa_g - \frac{2v}{r_g - \pi(r_g)} \ln \tilde{\kappa}_0^{(g)} \\ + \ln \binom{r_i}{2v} - \frac{2v}{r_g - \pi(r_g)} \ln \binom{r_g}{r_g - \pi(r_g)} = 0;\end{aligned} \tag{48}$$

2) in $m_{2v}^{(g)}$, $v = 1, 2, \ldots, (r_g - \pi(r_g))/2$

$$\begin{aligned}-\ln m_{2v}^{(g)} + \frac{2v}{r_g - \pi(r_g)} \ln \kappa_g + \left(1 - \frac{2v}{r_g - \pi(r_g)}\right) \ln \tilde{\kappa}_0^{(g)} \\ + \ln \binom{r_g}{2v} - \frac{2v}{r_g - \pi(r_g)} \ln \binom{r_g}{r_g - \pi(r_g)} = 0.\end{aligned} \tag{49}$$

This is the sought system for computing $M_{n,\theta}$.

D. Solution of the Basic System

We prove in this subsection the following result.

Theorem 2: The system (48)–(49) has the unique solution given by

$$m_{2v}^{(i)} = \frac{2\gamma n}{g} \frac{\binom{r_i}{2v} t^{2v}}{(1+t)^{r_i} + (1-t)^{r_i}} \tag{50}$$

where $v = 0, 1, \ldots, \frac{r_i - \pi(r_i)}{2}$, $i = 1, 2, \ldots, g$, and t is the unique positive root of

$$\sum_{i=1}^{g} r_i \frac{(1+t)^{r_i-1} + (1-t)^{r_i-1}}{(1+t)^{r_i} + (1-t)^{r_i}} = (1 - \tilde{\theta}) \sum_{i=1}^{g} r_i. \tag{51}$$

Proof: Plugging $v = 1$ into (48), multiplying it by v and subtracting from (48) for arbitrary $v \geq 2$, we get

$$(1 - v) \ln \kappa_0^{(i)} + \ln \frac{(m_2^{(i)})^v \binom{r_i}{2v}}{m_{2v}^{(i)} \binom{r_i}{2}^v} = 0 \tag{52}$$

or

$$\kappa_0^{(i)} = \frac{(m_2^{(i)})^{\frac{v}{v-1}} \binom{r_i}{2v}^{\frac{1}{v-1}}}{(m_{2v}^{(i)})^{\frac{1}{v-1}} \binom{r_i}{2}^{\frac{v}{v-1}}} \tag{53}$$

where $v = 2, 3, \ldots, \frac{r_i - \pi(r_i)}{2}$, $i = 1, 2, \ldots, g-1$.

Analogously, from (49) we get

$$\tilde{\kappa}_0^{g} = \frac{(m_2^{(g)})^{\frac{v}{v-1}} \binom{r_g}{2v}^{\frac{1}{v-1}}}{(m_{2v}^{(g)})^{\frac{1}{v-1}} \binom{r_g}{2}^{\frac{v}{v-1}}} \tag{54}$$

where $v = 2, 3, \ldots, \frac{r_g - \pi(r_g)}{2}$.

$$\begin{aligned}\ln M_{n,\theta} \sim \max \Bigg\{ \sum_{i=1}^{g} \Bigg(\frac{m}{g} \ln \frac{m}{g} - m_0^{(i)} \ln m_0^{(i)} - m_2^{(i)} \ln m_2^{(i)} - \cdots - m_{r_i-\pi(r_i)}^{(i)} \ln m_{r_i-\pi(r_i)}^{(i)} + \\ + m_2^{(i)} \ln \binom{r_i}{2} + m_4^{(i)} \ln \binom{r_i}{4} + \cdots + m_{r_i-\pi(r_i)}^{(i)} \ln \binom{r_i}{r_i - \pi(r_i)} \Bigg) \Bigg\}.\end{aligned} \tag{47}$$

Notice that (44) and (45) yield

$$\kappa_g + \tilde{\kappa}_0^{(g)} = \frac{m}{g} - m_2^{(g)} - m_4^{(g)} - \cdots - m_{r_g-2-\pi(r_g)}^{(g)}. \quad (55)$$

Moreover, by (46) we can rewrite (48) and (49) (for $v = 1$) in the following way:

$$\frac{m}{gm_2^{(i)}} - 1 - \mu_4^{(i)} - \cdots - \mu_{r_i-\pi(r_i)}^{(i)} = \frac{\left(\binom{r_g}{r_g-\pi(r_g)}\frac{\tilde{\kappa}_0^{(g)}}{\kappa_g}\right)^{\frac{2}{r_g-\pi(r_g)}}}{\binom{r_i}{2}} \quad (56)$$

where

$$\mu_{2v}^{(i)} = \frac{m_{2v}^{(i)}}{m_2^{(i)}} \quad (57)$$

where $v = 1, 2, \ldots, \frac{r_i-\pi(r_i)}{2}$, $i = 1, 2, \ldots, g$.

Assume as well

$$\lambda_{2v}^{(i)} = \frac{m_{2v}^{(g)}}{m_2^{(i)}} \quad (58)$$

where $v = 1, 2, \ldots, \frac{r_i-\pi(r_i)}{2}$, $i = 1, 2, \ldots, g$.

Substitute $v = 1$ into (48), multiply by v, and then subtract the result from (49). We get

$$-\ln m_{2v}^{(g)} - v\ln\kappa_0^{(i)} + v\ln m_2^{(i)} + \ln\tilde{\kappa}_0^{(g)} + \ln\binom{r_i}{2v} - v\ln\binom{r_i}{2} = 0. \quad (59)$$

From (59) and (54) for $v \geq 2$ we obtain

$$\frac{\kappa_0^{(i)}\binom{r_i}{2}}{m_2^{(i)}} = \left(\frac{\binom{r_g}{2v}m_2^{(g)}}{\binom{r_g}{2}m_{2v}^{(g)}}\right)^{\frac{1}{v-1}}. \quad (60)$$

Comparing (60) with (53) we conclude

$$\frac{\binom{r_g}{2v}m_2^{(g)}}{\binom{r_g}{2}m_{2v}^{(g)}} = \frac{\binom{r_i}{2v}m_2^{(i)}}{\binom{r_i}{2}m_{2v}^{(i)}}. \quad (61)$$

In the previously introduced notation we have

$$\frac{\binom{r_i}{2}}{\binom{r_i}{2v}}\mu_{2v}^{(i)} = \frac{\binom{r_g}{2}}{\binom{r_g}{2v}}\frac{\lambda_{2v}^{(i)}}{\lambda_2^{(i)}}. \quad (62)$$

Divide (53) by $m_2^{(i)}$. By (46) and (57) we have

$$1 + \mu_4^{(i)} + \mu_6^{(i)} + \cdots + \mu_{r_i-\pi(r_i)}^{(i)} + \frac{\binom{r_i}{2v}^{\frac{1}{v-1}}}{\binom{r_i}{2}^{\frac{v}{v-1}}(\mu_{2v}^{(i)})^{\frac{1}{v-1}}} = \frac{m}{gm_2^{(i)}}. \quad (63)$$

Notice that by (54)

$$\tilde{\kappa}_0^{(g)} = \frac{\binom{r_g}{2v}^{\frac{1}{v-1}}(m_2^{(g)})^{\frac{v}{v-1}}}{\binom{r_g}{2}^{\frac{v}{v-1}}(m_{2v}^{(g)})^{\frac{1}{v-1}}}, \qquad v = 2, 3, \ldots, \frac{r_g-\pi(r_g)}{2}. \quad (64)$$

Thus, the right-hand side of (64) is a constant independent of v.

Dividing (46) by $m_2^{(i)}$, by (57) from (56) we find

$$\binom{r_i}{2}\frac{\kappa_0^{(i)}}{m_2^{(i)}} = \left(\binom{r_g}{r_g-\pi(r_g)}\frac{\tilde{\kappa}_0}{\kappa_g}\right)^{\frac{2}{r_g-\pi(r_g)}}. \quad (65)$$

However, by (53)

$$\binom{r_i}{2}\frac{\kappa_0^{(i)}}{m_2^{(i)}} = \left(\frac{m_2^{(i)}\binom{r_i}{2v}}{m_{2v}^{(i)}\binom{r_i}{2}}\right)^{\frac{1}{v-1}}. \quad (66)$$

Now, by (66) and (65) we have

$$\binom{r_g}{r_g-\pi(r_g)}^{-1}\frac{\kappa_g}{\tilde{\kappa}_0^{(g)}} = \left(\mu_{2v}^{(i)}\frac{\binom{r_i}{2}}{\binom{r_i}{2v}}\right)^{\frac{r_g-\pi(r_g)}{2(v-1)}}. \quad (67)$$

Therefore, by (64) and (67) we get

$$\binom{r_g}{r_g-\pi(r_g)}^{-1}\kappa_g = \frac{\binom{r_g}{2v}^{\frac{1}{v-1}}(m_2^{(g)})^{\frac{v}{v-1}}}{\binom{r_g}{2}^{\frac{v}{v-1}}(m_{2v}^{(g)})^{\frac{1}{v-1}}}\left(\mu_{2v}^{(i)}\frac{\binom{r_i}{2}}{\binom{r_i}{2v}}\right)^{\frac{r_g-\pi(r_g)}{2(v-1)}}. \quad (68)$$

From (55) and (57) it follows that

$$\lambda_2^{(i)} + \lambda_4^{(i)} + \cdots + \lambda_{r_g-\pi(r_g)-2}^{(i)} + \frac{\tilde{\kappa}_0^{(g)} + \kappa_g}{m_2^{(i)}} = \frac{m}{gm_2^{(i)}}. \quad (69)$$

On the other hand, by (64) and (68) and taking into account (57) and (58) we find

$$\begin{aligned}\frac{\tilde{\kappa}_0^{(g)}+\kappa_g}{m_2^{(i)}} &= \frac{1}{m_2^{(i)}}\left(\frac{\binom{r_g}{2v}(m_2^{(g)})^v}{\binom{r_g}{2}^v m_{2v}^{(g)}}\right)^{\frac{1}{v-1}} \times\left(1+\binom{r_g}{r_g-\pi(r_g)}\left(\mu_{2v}^{(i)}\frac{\binom{r_i}{2}}{\binom{r_i}{2v}}\right)^{\frac{r_g-\pi(r_g)}{2(v-1)}}\right)\\ &= \left(\frac{\binom{r_g}{2v}(\lambda_2^{(i)})^v}{\binom{r_g}{2}^v\lambda_{2v}^{(i)}}\right)^{\frac{1}{v-1}} \times\left(1+\binom{r_g}{r_g-\pi(r_g)}\left(\mu_{2v}^{(i)}\frac{\binom{r_i}{2}}{\binom{r_i}{2v}}\right)^{\frac{r_g-\pi(r_g)}{2(v-1)}}\right).\end{aligned} \quad (70)$$

Finally, from (69) and (70) we get

$$\begin{aligned}&\lambda_2^{(i)} + \lambda_4^{(i)} + \cdots + \lambda_{r_g-2-\pi(r_g)}^{(i)}\\ &+ \frac{\binom{r_g}{2v}^{\frac{1}{v-1}}(\lambda_2^{(i)})^{\frac{v}{v-1}}}{\binom{r_g}{2}^{\frac{v}{v-1}}(\lambda_{2v}^{(i)})^{\frac{1}{v-1}}}\left(1+\binom{r_g}{r_g-\pi(r_g)}\left(\mu_{2v}^{(i)}\frac{\binom{r_i}{2}}{\binom{r_i}{2v}}\right)^{\frac{r_g-\pi(r_g)}{2(v-1)}}\right)\\ &= \frac{m}{gm_2^{(i)}}, \qquad i = 1, 2, \ldots, g.\end{aligned} \quad (71)$$

Taking into account (67) set

$$F := \left(\binom{r_g}{r_g-\pi(r_g)}^{-1}\frac{\kappa_g}{\tilde{\kappa}_0^{(g)}}\right)^{\frac{2}{r_g-\pi(r_g)}} = \left(\mu_{2v}^{(i)}\frac{\binom{r_i}{2}}{\binom{r_i}{2v}}\right)^{\frac{1}{v-1}} \quad (72)$$

and noticing that this yields

$$\mu_{2v}^{(i)} = F^{v-1}\frac{\binom{r_i}{2v}}{\binom{r_i}{2}}, \qquad v = 1, 2, \ldots, \frac{r_i-\pi(r_i)}{2} \quad (73)$$

we find from (63) that

$$\frac{1}{F\binom{r_i}{2}}\left(1+\binom{r_i}{2}F+\binom{r_i}{4}F^2+\cdots+\binom{r_i}{r_i-\pi(r_i)}F^{\frac{r_i-\pi(r_i)}{2}}\right) = \frac{m}{gm_2^{(i)}} \quad (74)$$

where $i = 1, 2, \ldots, g$.

Furthermore, by (64) and (58)

$$\frac{\tilde{\kappa}_0^{(g)}}{m_2^{(i)}} = \frac{\binom{r_g}{2v}^{\frac{1}{v-1}} (\lambda_2^{(i)})^{\frac{v}{v-1}}}{\binom{r_g}{2}^{\frac{v}{v-1}} (\lambda_{2v}^{(i)})^{\frac{1}{v-1}}}. \tag{75}$$

Denote

$$G = \frac{m_2^{(g)}}{\binom{r_g}{2}\tilde{\kappa}_0^{(g)}}. \tag{76}$$

By (75) and (58)

$$G = \left(\frac{\binom{r_g}{2}\lambda_{2v}^{(i)}}{\binom{r_g}{2v}\lambda_2^{(i)}}\right)^{\frac{1}{v-1}}. \tag{77}$$

Then, (72), (77), and (62) yield

$$G = F \tag{78}$$

and, therefore

$$\lambda_{2v}^{(i)} = \frac{\binom{r_g}{2v}}{\binom{r_g}{2}} F^{v-1} \lambda_2^{(i)}. \tag{79}$$

By (71) and (75)

$$\lambda_2^{(i)} + \lambda_4^{(i)} + \cdots + \lambda_{r_j-2-\pi(r_g)}^{(i)} + \frac{\tilde{\kappa}_0^{(g)}}{m_2^{(i)}} \left(1 + \binom{r_g}{r_g-\pi(r_g)} \left(\mu_{2v}^{(i)} \frac{\binom{r_i}{2}}{\binom{r_i}{2v}}\right)^{\frac{r_g-\pi(r_g)}{2(v-1)}}\right) = \frac{m}{g m_2^{(i)}}. \tag{80}$$

By (72)

$$\lambda_2^{(i)} + \lambda_4^{(i)} + \cdots + \lambda_{r_g-2-\pi(r_g)}^{(i)} + \frac{\tilde{\kappa}_0^{(g)}}{m_2^{(i)}} \left(1 + \binom{r_g}{r_g-\pi(r_g)} F^{\frac{r_g-\pi(r_g)}{2}}\right) = \frac{m}{g m_2^{(i)}}.$$

However, by (76) and (78)

$$F^{-1} = \frac{\tilde{\kappa}_0^{(g)}}{m_2^{(g)}} \binom{r_g}{2}$$

and

$$\frac{\tilde{\kappa}_0^{(g)}}{m_2^{(g)}} = \frac{1}{F\binom{r_g}{2}}.$$

Multiplying both sides of the last equation by

$$\lambda_2^{(i)} = \frac{m_2^{(g)}}{m_2^{(i)}}$$

we obtain

$$\frac{\tilde{\kappa}_0^{(g)}}{m_2^{(i)}} = \frac{\lambda_2^{(i)}}{F\binom{r_g}{2}}.$$

Thus, by (80) we arrive at the following identity:

$$\lambda_2^{(i)} \left(1 + \frac{1}{\binom{r_g}{2}F} + \frac{\binom{r_g}{r_g-\pi(r_g)}}{\binom{r_g}{2}} F^{\frac{r_g-\pi(r_g)}{2}-1}\right) + \lambda_4^{(i)} + \lambda_6^{(i)} + \cdots + \lambda_{r_g-2-\pi(r_g)}^{(i)} = \frac{m}{g m_2^{(i)}}. \tag{81}$$

Comparing (81) with (74), and using (79), we conclude with (82) at the bottom of the page. From an easy identity

$$1 + \binom{k}{2}x + \binom{k}{4}x^2 + \cdots + \binom{k}{k-\pi(k)} x^{\frac{k-\pi(k)}{2}} = \frac{(1+\sqrt{x})^k + (1-\sqrt{x})^k}{2} \tag{83}$$

and denoting

$$t := \sqrt{F} \tag{84}$$

we get

$$\lambda_2^{(i)} = \frac{\binom{r_g}{2}}{\binom{r_i}{2}} \frac{(1+t)^{r_i} + (1-t)^{r_i}}{(1+t)^{r_g} + (1-t)^{r_g}}. \tag{85}$$

To derive an equation for t let us return to the expressions for κ_g, (44) and (68). By (72) and (75)

$$\frac{\kappa_g}{m_{2i}} = \left(\frac{\binom{r_g}{2v}(\lambda_2^{(i)})^v}{\binom{r_g}{2}^v \lambda_{2v}^{(i)}}\right)^{\frac{1}{v-1}} \binom{r_g}{r_g-\pi(r_g)} F^{\frac{r_g-\pi(r_g)}{2}} = \frac{\lambda_2^{(i)}}{\binom{r_g}{2}} \left(\frac{\binom{r_g}{2v}\lambda_2^{(i)}}{\binom{r_g}{2}\lambda_{2v}^{(i)}}\right)^{\frac{1}{v-1}} \binom{r_g}{r_g-\pi(r_g)} F^{\frac{r_g-\pi(r_g)}{2}} \tag{86}$$

(by (77))

$$= \frac{\lambda_2^{(i)}}{\binom{r_g}{2}} \binom{r_g}{r_g-\pi(r_g)} G^{-1} F^{\frac{r_g-\pi(r_g)}{2}}$$

(by (78))

$$= \frac{\lambda_2^{(i)}}{\binom{r_g}{2}} \binom{r_g}{r_g-\pi(r_g)} F^{\frac{r_g-2-\pi(r_g)}{2}}.$$

Therefore, from (44) multiplied by $\frac{r_g-\pi(r_g)}{2m_2^{(i)}}$, (57), (58), and (86) we have

$$\lambda_2^{(i)} + 2\lambda_4^{(i)} + 3\lambda_6^{(i)} + \cdots + \frac{r_g-\pi(r_g)-2}{2}\lambda_{r_g-\pi(r_g)-2}^{(i)} + 1 + 2\mu_4^{(i)} + 3\mu_6^{(i)} + \cdots + \frac{r_i-\pi(r_i)}{2}\mu_{r_i-\pi(r_i)}^{(i)} + \frac{\binom{r_g}{r_g-\pi(r_g)}}{r_g+\pi(r_g)-1}\lambda_2^{(i)} F^{\frac{r_g-\pi(r_g)-2}{2}} = \frac{ns\tilde{\theta}}{2m_2^{(i)}} - \frac{1}{2m_2^{(i)}} \sum_{j=1, j\neq i}^{g-1} \sum_{v=2,4,\ldots,r_j-\pi(r_j)} v m_v^{(j)} \tag{87}$$

$$\lambda_2^{(i)} = \frac{\binom{r_g}{2}}{\binom{r_i}{2}} \frac{1 + \binom{r_i}{2}F + \binom{r_i}{4}F^2 + \cdots + \binom{r_i}{r_i-\pi(r_i)} F^{\frac{r_i-\pi(r_i)}{2}}}{1 + \binom{r_g}{2}F + \binom{r_g}{4}F^2 + \cdots + \binom{r_g}{r_g-2-\pi(r_g)} F^{\frac{r_g-\pi(r_i)}{2}-1} + \binom{r_g}{r_g-\pi(r_g)} F^{\frac{r_g-\pi(r_g)}{2}}}. \tag{82}$$

or, taking into account (73) and (79), after multiplication by $\binom{r_i}{2}\binom{r_j}{2}$ we obtain

$$\binom{r_g}{2}\left(\binom{r_i}{2}+2\binom{r_i}{4}F+3\binom{r_i}{6}F^2+\cdots+\frac{r_i-\pi(r_i)}{2}\binom{r_i}{r_i-\pi(r_i)}F^{\frac{r_i-\pi(r_i)-2}{2}}\right)+\binom{r_i}{2}\lambda_2^{(i)}\left(\binom{r_g}{2}+2\binom{r_g}{4}F+3\binom{r_g}{6}F^2+\cdots+\frac{r_g-\pi(r_g)-2}{2}\binom{r_g}{r_g-\pi(r_g)-2}F^{\frac{r_g-\pi(r_g)-4}{2}}+\frac{r_g-\pi(r_g)}{2}\binom{r_g}{r_g-\pi(r_g)}F^{\frac{r_g-\pi(r_g)-2}{2}}\right)=\frac{\binom{r_g}{2}\binom{r_i}{2}}{2m_2^{(i)}}\left(ns\tilde{\theta}-\sum_{j=1,j\neq i}^{g-1}\sum_{v=2,4,\ldots,r_j-\pi(r_j)}vm_v^{(j)}\right). \quad (88)$$

Notice, that by (83)

$$\binom{k}{2}+2\binom{k}{4}x+\cdots+\frac{k-\pi(k)}{2}\binom{k}{k-\pi(k)}x^{\frac{k-\pi(k)-2}{2}}=\left(1+\binom{k}{2}x+\binom{k}{4}x^2+\cdots+\binom{k}{k-\pi(k)}x^{\frac{k-\pi(k)}{2}}\right)'=\frac{k}{4\sqrt{x}}\left((1+\sqrt{x})^{k-1}-(1-\sqrt{x})^{k-1}\right). \quad (89)$$

Assuming in (88)$\sqrt{F}=t$, by (89) we have

$$r_i\binom{r_g}{2}\frac{1}{4t}\left((1+t)^{r_i-1}-(1-t)^{r_i-1}\right)+\lambda_2^{(i)}r_g\binom{r_g}{2}\frac{1}{4t}\left((1+t)^{r_g-1}-(1-t)^{r_g-1}\right)=\frac{\binom{r_g}{2}\binom{r_i}{2}}{2m_2^{(i)}}\left(ns\tilde{\theta}-\sum_{j=1,j\neq i}^{g-1}\sum_{v=2,4,\ldots,r_j-\pi(r_j)}vm_v^{(j)}\right). \quad (90)$$

Moreover, taking into account (79), after multiplication of (81) by $\binom{r_g}{2}F$ we get

$$\lambda_2^{(i)}\left(1+\binom{r_g}{2}F+\binom{r_g}{4}F^2+\cdots+\binom{r_g}{r_g-\pi(r_g)}F^{\frac{r_g-\pi(r_g)}{2}}\right)=\frac{\gamma n}{gm_2^{(i)}}\binom{r_g}{2}F \quad (91)$$

or, by (83), and having in mind $\sqrt{F}=t$, we derive

$$\frac{\lambda_2^{(i)}}{2}\left((1+t)^{r_g}+(1-t)^{r_g}\right)=\frac{\gamma n}{gm_2^{(i)}}\binom{r_g}{2}t^2. \quad (92)$$

Dividing (90) by (92) we have

$$\frac{r_i\binom{r_g}{2}\frac{1}{2t}\left((1+t)^{r_i-1}-(1-t)^{r_i-1}\right)}{\lambda_2^{(i)}\left((1+t)^{r_g}+(1-t)^{r_g}\right)}+\frac{r_g\binom{r_i}{2}\frac{1}{2t}\left((1+t)^{r_i-1}-(1-t)^{r_i-1}\right)}{(1+t)^{r_g}+(1-t)^{r_g}}=\frac{g}{2t^2}\binom{r_i}{2}\left(\frac{s\tilde{\theta}}{\gamma}-\frac{1}{\gamma n}\sum_{j=1,j\neq i}^{g-1}\sum_{v=2,4,\ldots,r_j-\pi(r_j)}vm_v^{(j)}\right). \quad (93)$$

Let us use (85) for $\lambda_2^{(i)}$. After multiplication of (93) by $\frac{2t^2}{\binom{r_i}{2}}$ we get

$$r_it\frac{(1+t)^{r_i-1}-(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}}+r_gt\frac{(1+t)^{r_g-1}-(1-t)^{r_g-1}}{(1+t)^{r_g}+(1-t)^{r_g}}=g\left(\frac{s\tilde{\theta}}{\gamma}-\frac{1}{\gamma n}\sum_{j=1,j\neq i}^{g-1}\sum_{v=2,4,\ldots,r_j-\pi(r_j)}vm_v^{(j)}\right) \quad (94)$$

where $i=1,\ldots,g-1$.

By (62), (79) for $F=t^2$, and (57) we find

$$m_{2v}^{(i)}=t^{2v-2}\frac{\binom{r_i}{2v}}{\binom{r_i}{2}}m_2^{(i)}. \quad (95)$$

Substitute it into the right-hand side of (94). Solving it for the double sum we have

$$\frac{g}{\gamma n}\sum_{j=1,j\neq i}^{g-1}m_2^{(j)}\left(\sum_{v=1}^{\frac{r_j-\pi(r_j)}{2}}2vt^{2v-2}\frac{\binom{r_j}{2v}}{\binom{r_j}{2}}\right)=\frac{gs\tilde{\theta}}{\gamma}-r_it\frac{(1+t)^{r_i-1}-(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}}-r_gt\frac{(1+t)^{r_g-1}-(1-t)^{r_g-1}}{(1+t)^{r_g}+(1-t)^{r_g}} \quad (96)$$

where $i=1,2,\ldots,g-1$.

Notice that due to (89) (when $x=t^2$) the left-hand side of (96) is

$$\frac{g}{\gamma nt}\sum_{j=1,j\neq i}^{g-1}\frac{m_2^{(j)}}{r_j-1}\left((1+t)^{r_j-1}-(1-t)^{r_j-1}\right). \quad (97)$$

Multiply (96) by $\frac{\gamma nt}{g}$, and set

$$x_j:=\frac{m_2^{(j)}}{r_j-1}\left((1+t)^{r_j-1}-(1-t)^{r_j-1}\right) \quad (98)$$

$$d(t):=tn\left(s\tilde{\theta}-\frac{r_g\gamma}{g}t\frac{(1+t)^{r_g-1}-(1-t)^{r_g-1}}{(1+t)^{r_g}+(1-t)^{r_g}}\right) \quad (99)$$

$$b_i(t):=\frac{r_i\gamma}{g}t^2n\frac{(1+t)^{r_i-1}-(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}}. \quad (100)$$

Then (96) can be rewritten in the following way:

$$\sum_{j=1,j\neq i}^{g-1}x_j=d(t)-b_i(t), \qquad i=1,\ldots,g-1. \quad (101)$$

Let us now return to (87). Moving

$$1+2\mu_4^{(i)}+\cdots+\frac{r_i-\pi(r_i)}{2}\mu_{r_i-\pi(r_i)}^{(i)}$$

into the double sum (under assumption $F=t^2$) we have

$$\lambda_2^{(i)}+2\lambda_4^{(i)}+3\lambda_6^{(i)}+\cdots+\frac{r_g-\pi(r_g)-2}{2}\lambda_{r_g-\pi(r_g)-2}^{(i)}+\frac{\binom{r_g}{r_g-\pi(r_g)}}{r_g+\pi(r_g)-1}\lambda_2^{(i)}t^{r_g-\pi(r_g)-2}=\frac{ns\tilde{\theta}}{2m_2^{(i)}}-\frac{1}{2m_2^{(i)}}\sum_{j=1}^{g-1}\sum_{v=2,4,\ldots,r_j-\pi(r_j)}vm_v^{(j)}. \quad (102)$$

Furthermore, in the left-hand side of (90) the first summand vanishes. Thus, dividing both parts by $\binom{r_i}{2}$ we obtain

$$\lambda_2^{(i)} r_g \frac{1}{4t}\left((1+t)^{r_g-1}-(1-t)^{r_g-1}\right) = \frac{\binom{r_g}{2}}{2m_2^{(i)}}\left(ns\tilde{\theta}-\sum_{j=1}^{g-1}\sum_{v=2,4,\dots,r_j-\pi(r_j)} vm_v^{(j)}\right). \quad (103)$$

Multiply (103) by $2t^2$, and divide the result by (92). We get

$$r_g t\frac{(1+t)^{r_g-1}-(1-t)^{r_g-1}}{(1+t)^{r_g}+(1-t)^{r_g}} = g\left(\frac{s\tilde{\theta}}{\gamma}-\frac{1}{\gamma n}\sum_{j=1}^{g-1}\sum_{v=2,4,\dots,r_j-\pi(r_j)} vm_v^{(j)}\right). \quad (104)$$

Solving (104) for the double sum and taking into account (95) and (97) we have

$$\frac{g}{\gamma nt}\sum_{j=1}^{g-1}\frac{m_2^{(j)}}{r_j-1}\left((1+t)^{r_j-1}-(1-t)^{r_j-1}\right) = \frac{gs\tilde{\theta}}{\gamma}-r_g t\frac{(1+t)^{r_g-1}-(1-t)^{r_g-1}}{(1+t)^{r_g}+(1-t)^{r_g}}. \quad (105)$$

Multiplying by $\frac{\gamma nt}{g}$ and using notation from (98) and (99) we find

$$\sum_{j=1}^{g-1} x_j = d(t). \quad (106)$$

Comparing it with (101) we conclude that

$$x_i = b_i(t) \quad (107)$$

and thus from (106) it follows that

$$d(t)-\sum_{i=1}^{g-1} b_i(t) = 0. \quad (108)$$

Substituting $d_i(t)$ and $b_i(t)$ by their expressions (99) and (100) we derive the sought equation in t

$$\sum_{i=1}^{g} r_i t\frac{(1+t)^{r_i-1}-(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}} = \frac{s\tilde{\theta}g}{\gamma}. \quad (109)$$

This equation can be written in an alternative form. From (109) it follows that

$$r_1+r_2+\cdots+r_g-\sum_{i=1}^{g} r_i t\frac{(1+t)^{r_i-1}-(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}} = r_1+r_2\cdots+r_g-\frac{gs\tilde{\theta}}{\gamma}$$

or

$$\sum_{i=1}^{g} r_i\left(1-t\frac{(1+t)^{r_i-1}-(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}}\right) = \sum_{i=1}^{g} r_i-\frac{gs\tilde{\theta}}{\gamma}.$$

Thus,

$$\sum_{i=1}^{g} r_i\frac{(1+t)^{r_i-1}+(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}} = \sum_{i=1}^{g} r_i-\frac{gs\tilde{\theta}}{\gamma}. \quad (110)$$

By (30) and balance (21) it follows that

$$s = \frac{\gamma}{g}\sum_{i=1}^{g} r_i. \quad (111)$$

Therefore, we have

$$\sum_{i=1}^{g} r_i\frac{(1+t)^{r_i-1}+(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}} = (1-\tilde{\theta})\sum_{i=1}^{g} r_i. \quad (112)$$

Let us continue the proof of Theorem 2. By (107) and (100)

$$x_i = \frac{r_i\gamma}{g}t^2 n\frac{(1+t)^{r_i-1}-(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}}. \quad (113)$$

From (98) and (113) we get

$$m_2^{(i)} = \frac{x_i(r_i-1)}{(1+t)^{r_i-1}-(1-t)^{r_i-1}} = \frac{2\gamma n}{g}\frac{\binom{r_i}{2}t^2}{(1+t)^{r_i}+(1-t)^{r_i}} \quad (114)$$

where $i=1,2,\dots,g-1$. Therefore, by (95),

$$m_{2v}^{(i)} = \frac{2\gamma n}{g}\frac{\binom{r_i}{2v}t^{2v}}{(1+t)^{r_i}+(1-t)^{r_i}} \quad (115)$$

where $v=1,2,\dots,\frac{r_i-\pi(r_i)}{2}$, $i=1,2,\dots,g-1$. Moreover, by (58) and (92) it follows that

$$m_2^{(j)} = \frac{2\gamma n}{g}\frac{\binom{r_g}{2}t^2}{(1+t)^{r_g}+(1-t)^{r_g}}. \quad (116)$$

Then from (79) and (84) after multiplying by $m_2^{(i)}$ and taking into account (58) we obtain

$$m_{2v}^{(g)} = \frac{\binom{r_j}{2v}}{\binom{r_g}{2}}t^{2v-2}m_2^{(g)}. \quad (117)$$

By (116) and (117)

$$m_{2v}^{(g)} = \frac{2\gamma n}{g}\frac{\binom{r_g}{2v}t^{2v}}{(1+t)^{r_g}+(1-t)^{r_g}} \quad (118)$$

where $v=1,2,\dots,\frac{r_v-\pi(r_v)-2}{2}$.

What is left is to find $m_{r_g}^{(g)}, m_0^{(i)}, i=1,2,\dots,g-1$, and $m_0^{(g)}$.

By (86) with (84) and (58)

$$m_{r_g}^{(g)} = \kappa_g = \frac{m_2^{(g)}}{\binom{r_g}{2}}\binom{r_g}{r_g-\pi(r_g)}t^{r_g-\pi(r_g)-2}. \quad (119)$$

By (119) and (116)

$$m_{r_g}^{(g)} = \frac{2\gamma n}{g}\frac{\binom{r_g}{r_g-\pi(r_g)}t^{r_g-\pi(r_g)}}{(1+t)^{r_g}+(1-t)^{r_g}}. \quad (120)$$

Furthermore, from (34) and (115) for $i=1,2,\dots,g-1$

$$m_0^{(i)} = \frac{n\gamma}{g}-\frac{2n\gamma}{g}\sum_{v=1}^{\frac{r_g-\pi(r_g)}{2}}\frac{\binom{r_g}{2v}t^{2v}}{(1+t)^{r_i}+(1-t)^{r_i}} = \frac{n\gamma}{g}\left(1-\frac{2}{(1+t)^{r_i}+(1-t)^{r_g}}\sum_{v=1}^{\frac{r_v-\pi(r_v)}{2}}\binom{r_i}{2v}t^{2v}\right).$$

By (83), when $x=t^2$ and $k=r_i$

$$\sum_{i=1}^{\frac{r_i-\pi(r_i)}{2}}\binom{r_i}{2v}t^{2v} = \frac{(1+t)^{r_i}-(1-t)^{r_i}}{2}-1$$

and therefore,

$$m_0^{(i)} = \frac{2\gamma n}{g}\frac{1}{(1+t)^{r_i}+(1-t)^{r_i}} \quad (121)$$

where $i = 1, 2, \dots, g-1$. Analogously, from (34), (118), and (120) we have

$$m_0^{(g)} = \frac{2\gamma n}{g} \frac{1}{(1+t)^{r_g} + (1-t)^{r_g}}. \tag{122}$$

Comparing (115), (118), (120)–(122) we conclude that a uniform expression is valid

$$m_{2v}^{(i)} = \frac{2\gamma n}{g} \frac{\binom{r_i}{2v} t^{2v}}{(1+t)^{r_i} + (1-t)^{r_i}} \tag{123}$$

where $v = 0, 1, \dots, \frac{r_i - \pi(r_i)}{2}$, $i = 1, 2, \dots, g$.

Theorem 2 is proved up to uniqueness of the positive root to (51). This will be done in Section III-H for a more general case. □

E. Calculation of Asymptotics for $M_{n,\theta}$ and P

Substituting $m_{2v}^{(i)}$ from (123) into (47) we have (124) at the bottom of the page. By (83)

$$\sum_{v=0}^{\frac{r_i-\pi(r_i)}{2}} \binom{r_i}{2v} t^{2v} = \frac{(1+t)^{r_i} + (1-t)^{r_i}}{2}. \tag{125}$$

By (89)

$$\sum_{v=0}^{\frac{r_i-\pi(r_i)}{2}} 2v \binom{r_i}{2v} t^{2v} = r_i t \frac{(1+t)^{r_i-1} - (1-t)^{r_i-1}}{2}. \tag{126}$$

By (124)–(126)

$$\ln M_{n,\theta} = -\frac{\gamma n}{g} \left(t \ln t \sum_{i=1}^{g} r_i \frac{(1+t)^{r_i-1} - (1-t)^{r_i-1}}{(1+t)^{r_i} + (1-t)^{r_i}} + \sum_{i=1}^{g} \ln \frac{2}{(1+t)^{r_i} + (1-t)^{r_i}} \right).$$

Using (109) we obtain

$$\ln M_{n,\theta} \sim \frac{\gamma n}{g} \left(-\frac{g s \tilde{\theta}}{\gamma} \ln t + \sum_{i=1}^{g} \ln \frac{(1+t)^{r_i} + (1-t)^{r_i}}{2} \right) \tag{127}$$

or, taking into account (38)

$$\ln M_{n,\theta} \sim \frac{\gamma n}{g} \left(-\tilde{\theta} \ln t \sum_{i=1}^{g} r_i + \sum_{i=1}^{g} \ln \frac{(1+t)^{r_i} + (1-t)^{r_i}}{2} \right). \tag{128}$$

Then, by (40) and (38)

$$\ln P \sim \frac{\gamma n}{g} \sum_{i=1}^{g} \left(-H(\tilde{\theta}) r_i - \tilde{\theta} r_i \ln t + \ln \frac{(1+t)^{r_i} + (1-t)^{r_i}}{2} \right). \tag{129}$$

Thus, we have the following result.

Theorem 3: If

$$\tilde{\theta} \le 1 - \frac{\sum_{i=1}^{g} \pi(r_i)}{\sum_{i=1}^{g} r_i} \tag{130}$$

then

$$\mathcal{P}(\tilde{\theta}) = \lim_{n\to\infty} \frac{1}{n} \ln P = \frac{\gamma}{g} \sum_{i=1}^{g} \left(-H(\tilde{\theta}) r_i - \tilde{\theta} r_i \ln t + \ln \frac{(1+t)^{r_i} + (1-t)^{r_i}}{2} \right) \tag{131}$$

where t is the unique positive root of (112), otherwise

$$\mathcal{P}(\tilde{\theta}) = -\infty.$$ □

F. Lemmas of Equivalence and Gluing

To scrutinize the more general case we will need the following lemmas.

Lemma 2: The maximum of the product (39) will not alter if some of r_i's are equal to each other.

Proof: It is easy to check using Stirling approximation. We omit the details. □

The lemma shows that we are allowed to glue blocks with equal row sums into one common block, and *vice versa*, split a block to several blocks having the same row sums.

Lemma 3: Let in a matrix from $\mathcal{H}_{m,n,\theta}$ some of the blocks have length $m_j = o(m)$, $j \in \{1, \dots, g\}$. Then $\ln \max M_{n,\theta}$ (see (39)) asymptotically does not depend on m_j's.

$$\begin{aligned}
\ln M_{n,\theta} &\sim \sum_{i=1}^{g} \left\{ \frac{n\gamma}{g} \ln \frac{n\gamma}{g} - \sum_{v=0}^{\frac{r_i-\pi(r_i)}{2}} m_{2v}^{(i)} \ln \frac{m_{2v}^{(i)}}{\binom{r_i}{2v}} \right\} \\
&= \gamma n \ln n + \gamma n \ln \frac{\gamma}{g} - \sum_{i=1}^{g} \frac{2\gamma n}{g((1+t)^{r_i} + (1-t)^{r_i})} \sum_{v=0}^{\frac{r_i-\pi(r_i)}{2}} \binom{r_i}{2v} t^{2v} \ln \left(\frac{2\gamma n}{g} \frac{t^{2v}}{(1+t)^{r_i} + (1-t)^{r_i}} \right) \\
&= \frac{2\gamma n}{g} \left(\frac{g}{2} \ln n + \frac{g}{2} \ln \frac{\gamma}{g} - \sum_{i=1}^{g} \frac{1}{(1+t)^{r_i} + (1-t)^{r_i}} \right. \\
&\quad \left. \cdot \left(\left(\ln \frac{2\gamma n}{g((1+t)^{r_i} + (1-t)^{r_i})} \right) \cdot \sum_{v=0}^{\frac{r_i-\pi(r_i)}{2}} \binom{r_i}{2v} t^{2v} + (\ln t) \sum_{v=0}^{\frac{r_i-\pi(r_i)}{2}} 2v \binom{r_i}{2v} t^{2v} \right) \right).
\end{aligned} \tag{124}$$

Proof: It is enough to prove the lemma for two blocks, one having length $m_1 \sim m$, and the second one being of length $m_2 = o(m)$. Moreover, assume that all the column sums equal s. Then the product (39) assumes form

$$\Pi := \binom{m_1}{m_0^{(1)}, m_2^{(1)}, \dots, m_{r_1-\pi(r_1)}^{(1)}} \times \binom{m_2}{m_0^{(2)}, m_2^{(2)}, \dots, m_{r_2-\pi(r_2)}^{(2)}} \times \binom{r_1}{2}^{m_2^{(1)}} \binom{r_1}{4}^{m_4^{(1)}} \cdots \binom{r_1}{r_1-\pi(r_1)}^{m_{r_1-\pi(r_1)}^{(1)}} \times \binom{r_2}{2}^{m_2^{(2)}} \binom{r_2}{4}^{m_4^{(2)}} \cdots \binom{r_2}{r_2-\pi(r_2)}^{m_{r_2-\pi(r_2)}^{(2)}} \tag{132}$$

under conditions

$$\begin{aligned} &m_0^{(1)} + m_2^{(1)} + \cdots + m_{r_1-\pi(r_1)}^{(1)} = m_1 \\ &m_0^{(2)} + m_2^{(2)} + \cdots + m_{r_2-\pi(r_2)}^{(2)} = m_2 \\ &2m_2^{(1)} + 4m_4^{(1)} + \cdots + (r_1 - \pi(r_1))m_{r_1-\pi(r_1)}^{(1)} \\ &\quad + 2m_2^{(2)} + 4m_4^{(2)} + \cdots + (r_2 - \pi(r_2))m_{r_2-\pi(r_2)}^{(2)} = s\theta n. \end{aligned} \tag{133}$$

Thus,

$$\begin{aligned} &\binom{r_2}{2}^{m_2^{(2)}} \binom{r_2}{4}^{m_4^{(2)}} \cdots \binom{r_2}{r_2-\pi(r_2)}^{m_{r_2-\pi(r_2)}^{(2)}} \\ &\le \binom{r_2}{\frac{r_2-\pi(r_2)}{2}}^{m_0^{(2)}+m_2^{(2)}+\cdots+m_{r_2-\pi(r_2)}^{(2)}} = \binom{r_2}{\frac{r_2-\pi(r_2)}{2}}^{m_2} \\ &= \binom{r_2}{\frac{r_2-\pi(r_2)}{2}}^{o(m)}. \end{aligned} \tag{134}$$

Moreover, by Stirling

$$\binom{m_2}{m_0^{(2)}, m_2^{(2)}, \dots, m_{r_2-\pi(r_2)}^{(2)}} \overset{\ln}{\approx} \frac{m_2^{m_2}}{\left(m_0^{(2)}\right)^{m_0^{(2)}} \left(m_2^{(2)}\right)^{m_2^{(2)}} \cdots \left(m_{r_2-\pi(r_2)}^{(2)}\right)^{m_{r_2-\pi(r_2)}^{(2)}}}. \tag{135}$$

Therefore, by (133) and (134)

$$\ln \binom{m_2}{m_0^{(2)}, m_2^{(2)}, \dots, m_{r_2-\pi(r_2)}^{(2)}} \binom{r_2}{2}^{m_2^{(2)}} \binom{r_2}{4}^{m_4^{(2)}} \cdots \binom{r_2}{r_2-\pi(r_2)}^{m_{r_2-\pi(r_2)}^{(2)}} = o(m \ln m)$$

while

$$\ln \Pi \sim Cm \ln m$$

for a constant $C > 0$.

Thus,

$$\ln \Pi \sim \binom{m_1}{m_0^{(1)}, m_2^{(1)}, \dots, m_{r_1-\pi(r_1)}} \binom{r_1}{2}^{m_2^{(1)}} \binom{r_1}{4}^{m_4^{(1)}} \cdots \binom{r_1}{r_1-\pi(r_1)}^{m_{r_1-\pi(r_1)}^{(1)}}. \qquad \square$$

G. Asymptotics of P in the General Case of Arbitrary Sizes of Horizontal Blocks

Now we are ready to accomplish the proof. Consider a more general (than the previous) case when there are g horizontal blocks of (perhaps different) lengths $m_1, m_2, \dots, m_g$

$$m_1 + m_2 + \cdots + m_g = m. \tag{136}$$

By Lemma 3, we may assume that all m_j's are proportional to m, and

$$\begin{aligned} m_1 = \nu_1 m, m_2 = \nu_2 m, \dots m_g = \nu_g \\ m\nu_1 + \nu_2 + \cdots + \nu_g = 1. \end{aligned} \tag{137}$$

Using the ”gluing lemma” (Lemma 2) and (112) we arrive at the following equation in t:

$$\sum_{i=1}^{g} \nu_i r_i \frac{(1+t)^{r_i-1} + (1-t)^{r_i-1}}{(1+t)^{r_i} + (1-t)^{r_i}} = (1-\tilde{\theta}) \sum_{i=1}^{g} \nu_i r_i \tag{138}$$

and (111) reduces to

$$s = \gamma \sum_{i=1}^{g} \nu_i r_i. \tag{139}$$

Thus, Theorem 3 reduces to the following.

Theorem 4: If

$$\tilde{\theta} \le 1 - \frac{\sum_{i=1}^{g} \pi(r_i)}{\sum_{i=1}^{g} r_i} \tag{140}$$

then

$$\begin{aligned} \mathcal{P}(\tilde{\theta}) &= \lim_{n\to\infty} \frac{1}{n} \ln P \qquad (141) \\ &= \gamma \sum_{i=1}^{g} \left(-H(\tilde{\theta})\nu_i r_i - \tilde{\theta}(\ln t)\nu_i r_i + \nu_i \ln \left(\frac{(1+t)^{r_i} + (1-t)^{r_i}}{2} \right) \right) \\ &= -sH(\tilde{\theta}) - s\tilde{\theta}(\ln t) + \gamma \sum_{i=1}^{g} \nu_i \ln \left(\frac{(1+t)^{r_i} + (1-t)^{r_i}}{2} \right) \end{aligned}$$

where t is the unique positive root of (138), and $\tilde{\theta}$ is defined in (31); otherwise

$$\mathcal{P}(\tilde{\theta}) = -\infty. \qquad \square$$

H. The Uniqueness of the Positive Solution to (138)

We prove now that (138) always possesses a positive solution, and, moreover, this solution turns out to be unique. Indeed, set, for $t > 0$

$$z = \frac{1-t}{1+t}, \quad t = \frac{1-z}{1+z}. \tag{142}$$

Then

$$\frac{(1+t)^{r_i-1}+(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}} = \frac{1}{1+t}\frac{1+z^{r_i-1}}{1+z^{r_i}} = \frac{z+1}{2}\frac{1+z^{r_i-1}}{1+z^{r_i}} = \frac{1}{2}\left(1+z\frac{1+z^{r_i-2}}{1+z^{r_i}}\right)$$

and (138) reduces to

$$U(z) := z\sum_{i=1}^{g}\nu_i r_i \frac{1+z^{r_i-2}}{1+z^{r_i}} = (1-2\tilde{\theta})\sum_{i=1}^{g}\nu_i r_i. \quad (143)$$

Let

$$\psi_i(z) = \frac{z+z^{r_i-1}}{1+z^{r_i}} \quad (144)$$

and notice that

$$\psi_i'(z) = (1+z^{r_i})^{-2}(1-(r_i-1)z^{r_i}+(r_i-1)z^{r_i-2}-z^{2r_i-2}). \quad (145)$$

Since, by (142), $z^2 < 1$, then

$$z^{r_i} < z^{r_i-2}$$

and by (145)

$$\psi_i'(z) > 0. \quad (146)$$

Then $U(z)$ is monotonously increasing from

$$U(-1) = \lim_{z\to -1+0} U(z) = -\sum_{i=1}^{g}\nu_i r_i \lim_{z\to -1}\frac{1+z^{r_i-2}}{1+z^{r_i}}$$

and up to $U(1) = \sum_{i=1}^{g}\nu_i r_i$. However,

$$\lim_{z\to -1}\frac{1+z^{r_i-2}}{1+z^{r_i}} = \begin{cases} 1, & \text{if } r_i \text{ is even} \\ \frac{r_i-2}{r_i}, & \text{otherwise.}\end{cases}$$

Thus,

$$U(-1) = -\sum_{i=1}^{g}\nu_i r_i + 2\sum_{i=1}^{g}\nu_i \pi(r_i).$$

Since

$$\tilde{\theta} \in \left[0,\, 1-\frac{\sum_{i=1}^{g}\nu_i\pi(r_i)}{\sum_{i=1}^{g}\nu_i r_i}\right]$$

the right-hand side of (143) belongs to

$$\left[-\sum_{i=1}^{g}\nu_i r_i + 2\sum_{i=1}^{g}\nu_i\pi(r_i), \sum_{i=1}^{g}\nu_i r_i\right].$$

This means that (143) always has a unique solution $z = z_0$ in the interval $[-1, 1]$, and our claim follows. Moreover, if $z_0 \in [-1, 0]$ then $t_0 \in [1, \infty]$, and if $z_0 \in [0, 1]$ then $t_0 \in [0, 1]$.

Notice also that from (143) and the fact that $U(z)$ monotonously grows it follows that $\tilde{\theta}(z)$ monotonously decreases, and, therefore, by (142) $\tilde{\theta}(t)$ is monotonously increasing.

I. Study of $\mathcal{P}(\theta) := \lim_{n\to\infty}\frac{1}{n}\ln P$

In this subsection we establish the following result.

Theorem 5:

1) For $0 < \tilde{\theta} < 1/2$, $\mathcal{P}(\tilde{\theta})$ is monotonously decreasing.

2) For even $\min r_i$, $\mathcal{P}(\tilde{\theta})$ possesses minimum at $\tilde{\theta} = 1/2$, which is the absolute minimum if all r_i's are even, and is a local one otherwise.

3) For odd $\min r_i$, $\tilde{\theta} = 1/2$ is not an extremal point, and

$$\min_{\tilde{\theta}}\mathcal{P}(\tilde{\theta}) = -\gamma\left(\sum_{i=1}^{g}\nu_i r_i\right) H\left(1-\frac{\sum_{i=1}^{g}\nu_i\pi(r_i)}{\sum_{i=1}^{g}\nu_i r_i}\right) \quad (147)$$

4) The same expression is valid for 2) when not all the sums r_i are even. Then $\mathcal{P}(\tilde{\theta})$ has at least one local maximum for $\tilde{\theta} > 1/2$.

5) For $t > 1$, all the stationary points of the function $\mathcal{P}(\tilde{\theta}(t))$ are roots of the equation

$$\sum_{i=1}^{g}\nu_i r_i\frac{(t-1)^{r_i}}{(-1)^{r_i}(t+1)^{r_i}+(t-1)^{r_i}} = 0. \quad (148)$$

Proof:

1) From the above arguments it follows that the function $t(\tilde{\theta})$ is invertible. By (141) we have (149) at the top of the following page. Notice that by (38), (109) becomes

$$\sum_{i=1}^{g}\nu_i r_i\frac{(1+t)^{r_i-1}-(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}} = \frac{\tilde{\theta}}{t}\sum_{i=1}^{g}\nu_i r_i. \quad (150)$$

From (149) and (150) we find

$$\Gamma(\tilde{\theta}) = \gamma\ln\frac{\tilde{\theta}}{t(1-\tilde{\theta})}\sum_{i=1}^{g}\nu_i r_i = s\ln\frac{\tilde{\theta}}{t(1-\tilde{\theta})}. \quad (151)$$

Thus, the stationary points of $\mathcal{P}(\tilde{\theta})$ satisfy

$$\frac{\tilde{\theta}(t)}{1-\tilde{\theta}(t)} = t \quad (152)$$

or

$$\tilde{\theta}(t) = \frac{t}{1+t}. \quad (153)$$

For these points by (150) we have

$$\sum_{i=1}^{g}\nu_i r_i\frac{(1+t)^{r_i-1}-(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}} = \frac{1}{1+t}\sum_{i=1}^{g}\nu_i r_i. \quad (154)$$

The equation has root $t = 1$, which, by (152) corresponds to $\tilde{\theta} = \frac{1}{2}$. Let us show that to the left from the point $t = 1$ (or, which is the same, $\tilde{\theta} \in (0, \frac{1}{2})$) there are no stationary points of $\mathcal{P}(\tilde{\theta})$. Indeed, by (143), for $\tilde{\theta} < \frac{1}{2}$, $z > 0$, and, therefore, by (142), $0 < t < 1$. Thus,

$$\frac{(1+t)^{r_i-1}-(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}} < \frac{1}{1+t}$$

and (154) is not valid. By (150)

$$\frac{\tilde{\theta}}{t} < \frac{1}{1+t}, \qquad \frac{1}{1-\tilde{\theta}} < t+1$$

and therefore,

$$\frac{\tilde{\theta}}{t(1-\tilde{\theta})} < 1.$$

$$\begin{aligned}\Gamma(\tilde{\theta}) &:= \frac{d}{d\tilde{\theta}}\mathcal{P}(\tilde{\theta}) \\ &= \gamma \sum_{i=1}^{g} \nu_i \frac{d}{d\tilde{\theta}}\left(-r_i H(\tilde{\theta}) - \tilde{\theta} r_i \ln t + \ln\left((1+t)^{r_i} + (1-t)^{r_i}\right)\right) \\ &= \gamma \sum_{i=1}^{g} \nu_i r_i \left(\ln\frac{\tilde{\theta}}{1-\tilde{\theta}} - \ln t - \frac{1}{\tilde{\theta}'(t)}\left(\frac{\tilde{\theta}}{t} - \frac{(1+t)^{r_i-1} - (1-t)^{r_i-1}}{(1+t)^{r_i} + (1-t)^{r_i}}\right)\right).\end{aligned} \tag{149}$$

By (151)

$$\mathcal{P}'(\tilde{\theta}) < 0, \qquad \tilde{\theta} \in (0, \frac{1}{2}).$$

Thus, $\mathcal{P}(\tilde{\theta})$ monotonously decreases down to the value $\mathcal{P}(\frac{1}{2})$. By (141)

$$\mathcal{P}(\frac{1}{2}) = -\gamma \ln 2 \tag{155}$$

2) and 3) Assume now that $\tilde{\theta} > \frac{1}{2}$. Then, by (143), $z < 0$, and $t \in (1, \infty)$. Then

$$\frac{(1+t)^{r_i-1} - (1-t)^{r_i-1}}{(1+t)^{r_i} + (1-t)^{r_i}} > \frac{1}{1+t}, \qquad r_i \equiv 0 \bmod 2 \tag{156}$$

$$\frac{(1+t)^{r_i-1} - (1-t)^{r_i-1}}{(1+t)^{r_i} + (1-t)^{r_i}} < \frac{1}{1+t}, \qquad r_i \equiv 1 \bmod 2. \tag{157}$$

By (150), we have

$$\begin{aligned}\frac{\tilde{\theta}}{t}\sum_{i=1}^{g}\nu_i r_i &= \sum_{i=1}^{g}\nu_i r_i \frac{(1+t)^{r_i-1} - (1-t)^{r_i-1}}{(1+t)^{r_i} + (1-t)^{r_i}} \\ &= \sum_{i=1}^{g}\nu_i r_i \left(\frac{(t+1)^{r_i-1} + (-1)^{r_i}(t-1)^{r_i-1}}{(t+1)^{r_i} + (-1)^{r_i}(t-1)^{r_i}} - \frac{(t+1)^{r_i-1}}{(t+1)^{r_i}}\right) + \frac{1}{t+1}\sum_{i=1}^{g}\nu_i r_i \\ &= \frac{2}{t+1}\left(\sum_{i\le g: r_i \text{even}} \frac{\nu_i r_i (t-1)^{r_i-1}}{(t+1)^{r_i} + (t-1)^{r_i}} - \sum_{i \le g: r_i \text{odd}} \frac{\nu_i r_i (t-1)^{r_i-1}}{(t+1)^{r_i} - (t-1)^{r_i}}\right) \\ &\quad + \frac{1}{t+1}\sum_{i=1}^{g}\nu_i r_i.\end{aligned} \tag{158}$$

With the help of (158) we will study $\mathcal{P}(\tilde{\theta})$ in a (small) right vicinity of $\tilde{\theta} = \frac{1}{2}$ (or, in a right vicinity of $t = 1$).

Set $t = 1 + \Delta t, \Delta t > 0$, and notice that

$$\frac{(\Delta t)^{r_i-1}}{(2+\Delta t)^{r_i} \pm (\Delta t)^{r_i}} \sim \frac{(\Delta t)^{r_i-1}}{2^{r_i}} \qquad (\Delta t \to 0). \tag{159}$$

Therefore, the positivity of the difference in (158) depends on the positivity of

$$\Upsilon = \sum_{i=1}^{g}(-1)^{r_i}\nu_i r_i \left(\frac{\Delta t}{2}\right)^{r_i-1}.$$

However, assuming that all r_i's are distinct (see the ”gluing lemma”), we have

$$\Upsilon \sim \nu_{\hat{i}} r_{\hat{i}} (-1)^{r_{\hat{i}}} \left(\frac{\Delta t}{2}\right)^{r_{\hat{i}}-1} \tag{160}$$

where $\hat{i}$ is the index of the minimum r_i

$$r_{\hat{i}} = \min_{1\le i\le g} r_i.$$

From (160) it follows that if $r_{\hat{i}}$ is even then for t belonging to a small enough right vicinity of $t_0 = 1$, by (150) and (158)

$$\frac{\tilde{\theta}}{t} > \frac{1}{1+t}$$

and thus

$$\frac{1}{1-\tilde{\theta}} > t+1, \qquad \frac{\tilde{\theta}}{t(1-\tilde{\theta})} > 1.$$

However, if $r_{\hat{i}}$ is odd, for the same values of t

$$\frac{\tilde{\theta}}{t} < \frac{1}{1+t}, \qquad \frac{\tilde{\theta}}{t(1-\tilde{\theta})} < 1.$$

In the first case, $\mathcal{P}(\tilde{\theta})$ attains minimum at $\tilde{\theta} = \frac{1}{2}$, and the minimum is

$$\min_{\tilde{\theta}} \mathcal{P}(\tilde{\theta}) = -\gamma \ln 2. \tag{161}$$

In the second case, $\mathcal{P}(\tilde{\theta})$ continues decreasing in at least small vicinity of $\tilde{\theta} = \frac{1}{2}$.

Let us show, moreover, that

$$\lim_{t\to\infty}\tilde{\theta}(t) = 1 - \frac{\sum_{i=1}^{g}\nu_i \pi(r_i)}{\sum_{i=1}^{g}\nu_i r_i} \tag{162}$$

i.e., the equality in (36) corresponds to the limit case of $t = \infty$.

Indeed, for an odd r and t tending to infinity

$$\frac{(1+t)^{r-1} - (1-t)^{r-1}}{(1+t)^r + (1-t)^r} \sim \frac{(r-1)}{r}\frac{1}{t}. \tag{163}$$

For even r

$$\frac{(1+t)^{r-1} - (1-t)^{r-1}}{(1+t)^r + (1-t)^r} \sim \frac{1}{t}. \tag{164}$$

Using these relations along with (150), we find

$$\frac{\tilde{\theta}(t)}{t}\sum_{i=1}^{g}\nu_i r_i \sim \frac{1}{t}\sum_{i=1}^{g}\nu_i(r_i - \pi(r_i))$$

and (162) follows.

It follows from (162) that if $r_{\hat{i}}$ is odd then

$$\min_{\tilde{\theta}} \mathcal{P}(\tilde{\theta}) = \mathcal{P}\left(1 - \frac{\sum_{i=1}^{g} \nu_i \pi(r_i)}{\sum_{i=1}^{g} \nu_i r_i}\right). \tag{165}$$

Let us show that

$$\min_{\tilde{\theta}} \mathcal{P}(\tilde{\theta}) = -\gamma\left(\sum_{i=1}^{g} \nu_i r_i\right) H\left(1 - \frac{\sum_{i=1}^{g} \nu_i \pi(r_i)}{\sum_{i=1}^{g} \nu_i r_i}\right). \tag{166}$$

Notice that for growing t

$$\ln \frac{(1+t)^{r_i} + (1-t)^{r_i}}{2} \sim (r_i - \pi(r_i)) \ln t.$$

From (141), we find

$$\mathcal{P}\left(1 - \frac{\sum_{i=1}^{g} \nu_i \pi(r_i)}{\sum_{i=1}^{g} \nu_i r_i}\right) = -\gamma\left(\sum_{i=1}^{g} \nu_i r_i\right) H\left(1 - \frac{\sum_{i=1}^{g} \nu_i \pi(r_i)}{\sum_{i=1}^{g} \nu_i r_i}\right) + \left(\sum_{i=1}^{g} \nu_i r_i\right) \lim_{t\to\infty}\left(\left(1 - \frac{\sum_{i=1}^{g} \nu_i \pi(r_i)}{\sum_{i=1}^{g} \nu_i r_i}\right) - \tilde{\theta}(t)\right) \ln t. \tag{167}$$

By L'Hopital's rule

$$\lim_{t\to\infty}\left(\left(1 - \frac{\sum_{i=1}^{g} \nu_i \pi(r_i)}{\sum_{i=1}^{g} \nu_i r_i}\right) - \tilde{\theta}(t)\right) \ln t = \lim_{t\to\infty}\left(\tilde{\theta}'(t) t \ln^2 t\right). \tag{168}$$

By (138)

$$\tilde{\theta}'(t) = \left(\sum_{i=1}^{g} \nu_i r_i\right)^{-1} \times \sum_{i=1}^{g} \nu_i r_i \frac{(1+t)^{2r_i-2} - (1-t)^{2r_i-2} + 4(r_i-1)t(1-t^2)^{r_i-2}}{\left((1+t)^{r_i} + (1-t)^{r_i}\right)^2} = O\left(\frac{1}{t^3}\right). \tag{169}$$

Finally, (167) and (169) yield (166).

4) Using again (158) we will study $\mathcal{P}(\tilde{\theta})$ in a small (left) vicinity of

$$\tilde{\theta} = 1 - \frac{\sum_{i=1}^{g} \nu_i \pi(r_i)}{\sum_{i=1}^{g} \nu_i r_i}$$

or, as it was earlier shown, for t big enough. We have for $t \to \infty$,

$$\sum_{i\le g: r_i \text{ even}} \frac{\nu_i r_i (t-1)^{r_i-1}}{(t+1)^{r_i} + (t-1)^{r_i}} - \sum_{i\le g: r_i \text{ odd}} \frac{\nu_i r_i (t-1)^{r_i-1}}{(t+1)^{r_i} - (t-1)^{r_i}} \sim \sum_{i\le g: r_i \text{ even}} \frac{\nu_i r_i}{2t} - \sum_{i\le g: r_i \text{ odd}} \frac{\nu_i}{2}. \tag{170}$$

For t big enough, the expression (170) becomes negative if among the numbers $r_1, r_2, \ldots, r_g$ there is at least one odd number. If here $\min_{1\le i\le g} r_i$ is even, then for $t > 1$ the increasing of $\mathcal{P}(\tilde{\theta})$ will be necessary changed by decreasing. Therefore, $\mathcal{P}(\tilde{\theta})$ has at least one local maximum point for $\tilde{\theta} > \frac{1}{2}$.

5) From (152), (150), and (138) when $t > 1$ we have

$$t = \frac{\tilde{\theta}}{1-\tilde{\theta}} = t\left(\sum_{i=1}^{g} \nu_i r_i \frac{(1+t)^{r_i-1} - (1-t)^{r_i-1}}{(1+t)^{r_i} + (1-t)^{r_i}}\right) \times \left(\sum_{i=1}^{g} \nu_i r_i \frac{(1+t)^{r_i-1} + (1-t)^{r_i-1}}{(1+t)^{r_i} + (1-t)^{r_i}}\right)^{-1}$$

and (148) follows. □

Remark 1: It is easy to see that if a is a positive constant, then

$$\lim_{x\to+\infty} xH\left(1 - \frac{a}{x}\right) = +\infty$$

and thus (166) yields that if there is at least one odd r_i and at least one r_j (indepenedently of its parity) tends to $+\infty$, then $\min \mathcal{P}(\tilde{\theta}) \to -\infty$. □

J. Computation of the Spectrum

Let us change now the definition of θ. We assume that we are choosing the θn columns in the following way. In the ith vertical block (of size $\eta_i n$) we pick the first $\theta_i n$ columns, where $\theta_i \le \eta_i$. Here $i = 1, 2, \ldots, h$. Clearly

$$\theta_1 + \theta_2 + \cdots + \theta_h = \theta, \qquad \theta_i \le \eta_i. \tag{171}$$

Set temporarily

$$\tilde{\eta}_i = \theta_i, \qquad i = 1, \ldots, h \tag{172}$$

$$\tilde{\eta}_{h+i} = \eta_i - \theta_i, \qquad i = 1, \ldots, h \tag{173}$$

$$\tilde{s}_i = s_i, \qquad i = 1, \ldots, h \tag{174}$$

$$\tilde{s}_{h+i} = s_i, \qquad i = 1, \ldots, h. \tag{175}$$

It is clear that

$$\sum_{i=1}^{2h} \tilde{\eta}_i = 1, \qquad \sum_{i=1}^{h} \tilde{\eta}_i = \theta. \tag{176}$$

For this new system, let us calculate $\tilde{\theta}$. Using (22), (31), for $q = h$, and (171)–(176), we obtain

$$\tilde{\theta} = \frac{\sum_{i=1}^{h} \tilde{\eta}_i s_i - s_h(\tilde{\eta}_1 + \tilde{\eta}_2 + \cdots + \tilde{\eta}_h) + \theta s_h}{\sum_{i=1}^{2h} \tilde{\eta}_i \tilde{s}_i} = \frac{\sum_{i=1}^{h} \tilde{\eta}_i s_i}{\sum_{i=1}^{2h} \tilde{\eta}_i \tilde{s}_i}. \tag{177}$$

From (172) we get

$$\tilde{\theta} = \frac{\sum_{i=1}^{h} \theta_i s_i}{\sum_{i=1}^{h} \theta_i s_i + \sum_{i=1}^{h} (\eta_i - \theta_i) s_i} = \frac{\sum_{i=1}^{h} \theta_i s_i}{\sum_{i=1}^{g} \eta_i s_i} = \frac{\sum_{i=1}^{h} \theta_i s_i}{s}. \tag{178}$$

Furthermore

$$C(\theta_1, \theta_2, \ldots, \theta_h) := \lim_{n\to\infty} \frac{1}{n} \ln \prod_{i=1}^{h} \binom{\eta_i n}{\theta_i n} = \sum_{i=1}^{h} (\eta_i \ln \eta_i - \theta_i \ln \theta_i - (\eta_i - \theta_i) \ln(\eta_i - \theta_i)). \tag{179}$$

Our goal is to compute (see (7) and (8))

$$b_{\tilde\theta} = \ln \sum \exp\left(\mathcal{P}(\tilde\theta) + C(\theta_1,\dots,\theta_h)\right) \sim \max\left(\mathcal{P}(\tilde\theta) + C(\theta_1,\dots,\theta_h)\right) \quad (180)$$

where the sum and the maximum are taken over all $\theta_1,\dots,\theta_h$ satifying (171).

We will use the following technical trick. Let us introduce a δ-ensemble, δ is a small positive parameter, similar to the initial one. In the δ-ensemble, we consider the matrices having the same distribution of the row sums, but there is an extra zero column strip (indexed $h+1$) of breadth δn. Set

$$\eta_i^{(\delta)} = \eta_i(1-\delta), \quad i=1,\dots,h, \qquad \eta_{h+1}^{\delta} = \delta. \quad (181)$$

Then

$$\sum_{i=1}^{h+1} \eta_i^{(\delta)} = 1-\delta+\delta = 1.$$

Moreover, assume that in the ith vertical strip of breadth $\eta_i^{(\delta)} n$ all column sums are s_i, $i=1,\dots,h$, and

$$s_{h+1} = 0. \quad (182)$$

Let the numbers $\theta_i^{(\delta)}$, $i=1,2,\dots,h$, be small δ-perturbations of the numbers θ_i, so that

$$\lim_{\delta\to+0} \theta_i^{(\delta)} = \theta_i \quad (183)$$

$$\theta_1^{(\delta)} + \cdots + \theta_h^{(\delta)} \le \theta. \quad (184)$$

Set

$$\theta_{h+1}^{(\delta)} = \theta - \theta_1^{(\delta)} - \cdots - \theta_h^{(\delta)} \quad (185)$$

$$0 \le \theta_i^{(\delta)} \le \eta_i^{(\delta)}. \quad (186)$$

Thus, for $i=h+1$ it follows that

$$\lim_{n\to\infty} \theta_{h+1}^{(\delta)} = 0. \quad (187)$$

We also assume existence of the following limit:

$$\lim_{\delta\to+0} \frac{\theta_{h+1}^{(\delta)}}{\eta_{h+1}^{(\delta)} - \theta_{h+1}^{(\delta)}} = \xi. \quad (188)$$

Notice that from (178), (181), and (182) it follows that

$$\tilde\theta^{(\delta)} = \frac{\sum_{i=1}^{h} \theta_i^{(\delta)} s_i}{\sum_{i=1}^{h} \eta_i^{(\delta)} s_i}. \quad (189)$$

Moreover, analogously to (179) we have

$$C(\theta_1^{(\delta)}, \theta_2^{(\delta)}, \dots, \theta_{h+1}^{(\delta)}) = \lim_{n\to\infty} \frac{1}{n} \ln \prod_{i=1}^{h+1} \binom{\eta_i^{(\delta)} n}{\theta_i^{(\delta)} n} = \sum_{i=1}^{h+1} \left(\eta_i^{(\delta)} \ln \eta_i^{(\delta)} - \theta_i^{(\delta)} \ln \theta_i^{(\delta)} - (\eta_i^{(\delta)} - \theta_i^{(\delta)}) \ln(\eta_i^{(\delta)} - \theta_i^{(\delta)})\right). \quad (190)$$

Equating to zero the partial derivatives in $\theta_j^{(\delta)}$, $j=1,\dots,h$, of the sum $\mathcal{P}(\tilde\theta^{(\delta)}) + C(\theta_1^{(\delta)},\dots,\theta_{h+1}^{(\delta)})$, we find

$$\mathcal{P}'(\tilde\theta^{(\delta)}) \cdot \frac{s_j}{\sum_{i=1}^{h} \eta_i^{(\delta)} s_i} + \ln \frac{\eta_j^{(\delta)} - \theta_j^{(\delta)}}{\theta_j^{(\delta)}} + \ln \frac{\theta_{h+1}^{\delta}}{\eta_{h+1}^{(\delta)} - \theta_{h+1}^{(\delta)}} = 0. \quad (191)$$

Denote

$$\zeta_\delta := \exp\left(\frac{\mathcal{P}'(\tilde\theta^{(\delta)})}{\sum_{i=1}^{h} \eta_i^{(\delta)} s_i}\right), \qquad \xi_\delta := \frac{\theta_{h+1}^{(\delta)}}{\eta_{h+1}^{(\delta)} - \theta_{h+1}^{(\delta)}}. \quad (192)$$

Thus, (191) can be written as

$$\ln \frac{\theta_j^{(\delta)}}{\eta_j^{(\delta)} - \theta_j^{(\delta)}} = s_j \ln \zeta_\delta + \ln \xi_\delta,$$

and

$$\theta_j^{(\delta)} = \frac{\eta_j^{(\delta)}}{1 + \zeta_\delta^{-s_j} \xi_\delta^{-1}}, \qquad j=1,2,\dots,h. \quad (193)$$

Summing up (193) for $j=1,\dots,h$ we find

$$\sum_{j=1}^{h} \frac{\eta_j^{(\delta)}}{1 + \zeta_\delta^{-s_j} \xi_\delta^{-1}} = \theta - \theta_{h+1}^{(\delta)}. \quad (194)$$

Let us move to the limit when $\delta \to 0$. Then $\lim \theta_{h+1}^{(\delta)} = 0$, $\lim \tilde\theta^{(\delta)} = \tilde\theta$, and, by (188), $\lim \xi_\delta = \xi$. Therefore,

$$\sum_{j=1}^{h} \frac{\eta_j}{1 + \zeta^{-s_j} \xi^{-1}} = \theta \quad (195)$$

where, by (192)

$$\zeta := \exp\left(\frac{\mathcal{P}'(\tilde\theta)}{\sum_{i=1}^{h} \eta_i s_i}\right). \quad (196)$$

Thus, since given $\mathcal{P}(\tilde\theta)$ by (196) we know ζ, the (195) can be considered as an equation of degree h in one indeterminate ξ.

Consider the following function

$$f(x) = \sum_{j=1}^{h} \frac{\eta_j}{1 + a_j x}$$

where $a_j \ge 0$. When x grows from 0 to ∞ it monotonously decreases from 1 ($= \sum_{j=1}^{h} \eta_j$) down to 0. Therefore, for any $\theta \in (0,1)$, the equation

$$f(x) = \theta$$

has a unique positive solution. Thus, (195) possesses the only positive root $\xi = \xi_\theta$.

Moving to the limit $(\delta \to 0)$ in (193) we find the sought distribution of numbers θ_j

$$\theta_j = \frac{\eta_j}{1 + \zeta^{-s_j} \xi^{-1}}. \quad (197)$$

Our goal now is to confirm that the numbers θ_j from (197) deliver maximum to the distance distribution $\mathcal{P}(\tilde{\theta}) + C(\theta_1, \ldots, \theta_h)$. Notice, for instance, that in the case $s_1 = s_2 = \cdots = s_h = s$ we find from (195) that

$$\frac{1}{1+\zeta^{-s}\xi^{-1}} = \theta$$

and, therefore, from (197) it follows that

$$\theta_j = \theta\eta_j, \qquad j = 1, \ldots, h. \tag{198}$$

In this case, by (178), $\tilde{\theta} = \theta$, i.e., $\mathcal{P}(\tilde{\theta}) = \mathrm{const}$, and the maximum of the distance distribution corresponds to the maximum of $C(\theta_1, \ldots, \theta_h)$. It is easy to show that the values defined in (198) indeed provide the maximum of $C(\theta_1, \ldots, \theta_h)$ under condition $\theta_1 + \cdots + \theta_h = \theta$.

Let us consider now the general situation. Let first, $\tilde{\theta}$ be a point where $\mathcal{P}(\tilde{\theta})$ decreases. Then, by (196), $\zeta < 1$. Since the maximum of the second summand in the logarithm of the distance distribution $C(\theta_1, \ldots, \theta_h)$ corresponds to $\tilde{\theta} = \theta$, and $\mathcal{P}(\tilde{\theta})$ decreases, then $\theta_j, j = 1, \ldots, h$, from (193), will deliver the maximum of the distribution only if the corresponding value of $\tilde{\theta}$ is not greater than θ. By (178), (197), and (195), this means that the following inequality:

$$\sum_{i=1}^{h} \frac{\eta_i s_i}{1+\zeta^{-s_i}\xi^{-1}} \leq \sum_{i=1}^{h} \frac{\eta_i}{1+\zeta^{-s_i}\xi^{-1}} \cdot \sum_{i=1}^{h} \eta_i s_i \tag{199}$$

is valid in a small vicinity of θ. To prove this inequality we need the following lemma.

Lemma 4: Let a_i, b_i be arbitrary real numbers, $i = 1, \ldots, h$, $\eta_i \geq 0$, and $\eta_1 + \cdots + \eta_h = 1$. Then the following equality is valid:

$$\begin{aligned}
&\eta_1 a_1 b_1 + \eta_2 a_2 b_2 + \cdots + \eta_h a_h b_h \\
&\quad - (\eta_1 a_1 + \eta_2 a_2 + \cdots + \eta_h a_h)(\eta_1 b_1 + \eta_2 b_2 + \cdots + \eta_h b_h) \\
&= \eta_1\eta_2(a_1 - a_2)(b_1 - b_2) + \eta_1\eta_3(a_1 - a_3)(b_1 - b_3) \\
&\quad + \cdots + \eta_1\eta_h(a_1 - a_h)(b_1 - b_h) \\
&\quad + \eta_2\eta_3(a_2 - a_3)(b_2 - b_3) + \eta_2\eta_4(a_2 - a_4)(b_2 - b_4) \\
&\quad + \cdots + \eta_2\eta_h(a_2 - a_h)(b_2 - b_h) + \\
&\quad + \cdots + \eta_{h-1}\eta_h(a_{h-1} - a_h)(b_{h-1} - b_h).
\end{aligned} \tag{200}$$

Proof: First, compare the coefficients at $a_i b_i$, $i = 1, \ldots, h$. On the left-hand side, the coefficient is $\eta_i - \eta_i^2$. On the right-hand side, the coefficient is

$$\begin{aligned}
&(\eta_1\eta_i + \eta_2\eta_i + \cdots + \eta_{i-1}\eta_i) \\
&\quad + (\eta_i\eta_{i+1} + \eta_i\eta_{i+2} + \cdots + \eta_i\eta_h) \\
&\quad\quad = \eta_i(\eta_1 + \eta_2 + \cdots + \eta_h - \eta_i) = \eta_i(1 - \eta_i).
\end{aligned}$$

The coefficients of $a_i b_j$, $i \neq j$, are evidently equal $-\eta_i\eta_j$ (both in the left- and the right-hand sides). □

Set in (200)

$$a_i = \frac{1}{1+\zeta^{-s_i}\xi^{-1}}, \qquad b_i = s_i, \ \ i = 1, \ldots, h. \tag{201}$$

Since $\zeta < 1$, then

$$s_i \geq s_j \Leftrightarrow \frac{1}{1+\zeta^{-s_i}\xi^{-1}} \leq \frac{1}{1+\zeta^{-s_j}\xi^{-1}}. \tag{202}$$

From (202) it follows that all the summands in the right-hand side of (200) are nonpositive for the values defined by (201). This proves (199). Notice that the equality in (199) holds only if $s_1 = \cdots = s_h$.

Let now $\tilde{\theta}$ be a point where $\mathcal{P}(\tilde{\theta})$ increases. Then, the values of θ_j, $j = 1, \ldots, h$, defined by (193) deliver maximum to the distribution only when the corresponding value $\tilde{\theta}$ is not less than θ, that is, it happens when the inequality (199) holds in the opposite direction. Let us prove that this indeed holds when $\tilde{\theta}$ lies in a close enough vicinity of θ.

For the values (201), under condition $\zeta > 1$

$$s_i \geq s_j \Leftrightarrow \frac{1}{1+\zeta^{-s_i}\xi^{-1}} \geq \frac{1}{1+\zeta^{-s_j}\xi^{-1}}. \tag{203}$$

By (203), all the summands in the right-hand side of (200) are nonnegative for the values defined by (201). This proves the inequality opposite to (199).

Finally, in the stationary points of $\mathcal{P}(\tilde{\theta})$, $\zeta = 1$, and (199) becomes an equality which corresponds to

$$\tilde{\theta} = \theta.$$

According to (197)

$$\theta_i = \frac{\xi}{1+\xi}\eta_i, \qquad i = 1, \ldots, h \tag{204}$$

where, by (195) (it also follows from (204))

$$\frac{\xi}{1+\xi} = \theta.$$

This means that for the stationary points of $\mathcal{P}(\tilde{\theta})$ we have

$$\theta_i = \theta\eta_i, \qquad i = 1, \ldots, h.$$

For the derived values of θ_j (197), using (179) we find after simple transformations

$$C(\theta_1, \ldots, \theta_h) = \sum_{i=1}^{h} \frac{\eta_i}{1+\zeta^{-s_i}\xi^{-1}} \ln \frac{(1+\zeta^{-s_i}\xi^{-1})^{1+\zeta^{-s_i}\xi^{-1}}}{(\zeta^{-s_i}\xi^{-1})^{\zeta^{-s_i}\xi^{-1}}}. \tag{205}$$

Denote

$$x_i := \frac{\eta_i}{1+\zeta^{-s_i}\xi^{-1}}. \tag{206}$$

Then

$$C(\theta_1, \ldots, \theta_h) = \sum_{i=1}^{h} \eta_i H\left(\frac{x_i}{\eta_i}\right). \tag{207}$$

According to (195) and (206)

$$\sum_{i=1}^{h} x_i = \theta. \tag{208}$$

By (197) and (206) we get

$$\sum_{i=1}^{h} s_i x_i = \tilde{\theta} s \tag{209}$$

where $s = \sum_{i=1}^{h} \eta_i s_i$. Since

$$\zeta = \exp\left(\frac{\mathcal{P}'(\tilde{\theta})}{s}\right)$$

then, by (151)

$$\zeta = \frac{\tilde{\theta}}{t(1-\tilde{\theta})}. \tag{210}$$

By (208) and (209) we derive the following system of equations in t and ξ:

$$\sum_{i=1}^{h} x_i = \theta, \qquad \sum_{i=1}^{h} x_i s_i = \tilde{\theta} s \tag{211}$$

where ζ in (206) is defined by (210), and $\tilde{\theta}$ is defined by (138)–(139).

Now, by (180), (141), and (207) we obtain (16) from Theorem 1. □

This accomplishes the proof of Theorem 1. Moreover, we obtained the distribution of the numbers θ_i delivering maximum of the distance distribution component. It is by (197) and (206)

$$\theta_i = x_i, \qquad i = 1, \ldots, h. \tag{212}$$

IV. On the Solutions to the System (13)–(14)

We analyze here the number of solutions to (13) and (14). As we have mentioned, we conjecture that in the general case the number of the solutions never exceeds three. In this section, we provide evidence that the number of solutions can be more than one.

Assume that

$$s_h > s_{h-1} > \cdots > s_2, (1-\eta_1)s_2 \gg s_1, \ \eta_1 \geq \eta_2 \geq \cdots \geq \eta_h \tag{213}$$

$$\theta < \min\left(\frac{1}{2}, \eta_1\right). \tag{214}$$

By (214), θ belongs to the interval where $\mathcal{P}(\tilde{\theta})$ is decreasing. In this case, there exists a maximum in the left vicinity of θ. Since $\tilde{\theta} = \theta$ corresponds to the maximum of $C(\theta_1, \theta_2, \ldots, \theta_h)$, computed under condition $\sum_{i=1}^{h} \theta_i = \theta$, then the value of θ_i delivering the maximum to $C(\theta_1, \theta_2, \ldots, \theta_h)$ is $\theta_i = \eta_i \theta$. According to (197), the choice of $\tilde{\theta} \approx \theta$, corresponds to the following approximate equality:

$$\frac{1}{1+\zeta^{-s_i}\xi^{-1}} \approx \tilde{\theta}. \tag{215}$$

It means that the expression in (215) is almost independent of i. This is possible only if $\zeta \approx 1$, $\zeta < 1$, with high accuracy. Substituting $\zeta \approx 1$ into (13)

$$\sum_{i=1}^{h} x_i = \sum_{i=1}^{h} \frac{\eta_i}{1+\zeta^{-s_i}\xi^{-1}} = \theta \tag{216}$$

we find

$$\frac{1}{1+\xi^{-1}} \approx \theta, \qquad \xi^{-1} \approx \frac{1-\theta}{\theta}. \tag{217}$$

Now (14) is also satisfied

$$\sum_{i=1}^{h} \frac{s_i}{s} x_i = \sum_{i=1}^{h} \frac{s_i}{s} \frac{\eta_i}{1+\frac{1-\theta}{\theta}} = \theta \approx \tilde{\theta}. \tag{218}$$

Exact equality could be achieved by a small variation of ζ in the vicinity of 1 and ξ^{-1} around $\frac{1-\theta}{\theta}$.

Let us find now another solution to the system, essentially distant from the solution θ. Let $\zeta, 0 < \zeta < 1$ be small enough. Since when $s_2 \gg s_1$

$$\zeta^{-s_2} \gg \zeta^{-s_1} \tag{219}$$

by (213)

$$x_1 \gg x_2 > x_3 > \cdots > x_h. \tag{220}$$

Therefore, by (220) and (216)

$$x_1 = \frac{\eta_1}{1+\zeta^{-s_1}\xi^{-1}} \approx \theta \tag{221}$$

and

$$\zeta^{s_1}\xi \approx \frac{\theta}{\eta_1 - \theta}. \tag{222}$$

Furthermore, (14) yields

$$\sum_{i=1}^{h} \frac{s_i \eta_i}{s(1+\zeta^{-s_i}\xi^{-1})} = \tilde{\theta}. \tag{223}$$

Since, when ζ is small enough and $s_2 \gg s_1$, we have for $i \geq 2$, $\zeta^{-s_i}\xi^{-1} \gg 1$ and $s_i \zeta^{s_i} \ll s_1 \zeta^{s_1}$, then by (221) and (223)

$$\tilde{\theta} \approx \frac{s_1}{s} \frac{\eta_1}{1+\zeta^{-s_1}\xi^{-1}} \approx \frac{s_1 \theta}{s}. \tag{224}$$

Notice that for the stationary points of function $b_\theta^*(\tilde{\theta})$ from (16) under the maximum

$$s\tilde{\theta} = s_1 x_1 + \cdots + s_h x_h > s_1(x_1 + \cdots + x_h) = s_1 \theta. \tag{225}$$

This means that the stationary point (224) is situated in a small right vicinity of $\frac{s_1\theta}{s}$. The other stationary point which is close to θ does not belong to a small vicinity of $\frac{s_1\theta}{s}$. Indeed, by (213)

$$s \geq \eta_1 s_1 + (1-\eta_1)s_2 \gg s_1$$

and thus,

$$\frac{s_1 \theta}{s} \ll \theta. \tag{226}$$

We will illiustrate the above situation by an example.

Example: Let $s_1 = 2$, $s_2 = 28$ $(h = 2)$; $r_1 = 30$ $(g = 1)$; $\eta_1 = \eta_2 = 0.5$, $\theta = 0.1$, $\gamma = 0.5$. Thus, we consider a rate $1/2$ code. We have the following three solutions to (13) and (14), the first and the third correspond to our considerations. We provide accuracy of seven digits after the decimal point in the numbers.

1) $\tilde{\theta}_1 = 0.0958855$. This value is on the left of $\theta = 0.1$ and corresponds to a local maximum of $b^*_\theta(\tilde{\theta})$. The maximum equals -0.0207870. Then $\zeta = 0.9959481$, $t = 0.1064861$, $\xi = 0.1179562$ (the approximation (217) gives $\xi = 0.1111111$).
2) $\tilde{\theta}_2 = 0.0596296$. This value corresponds to the local minimum of $b^*_\theta(\tilde{\theta})$ equal -0.0239098. Here, $\zeta = 0.9580943$, $t = 0.0661842$, $\xi = 0.1871131$. This case corresponds to an intermediate value of ζ, which still is not far from $\zeta = 1$.
3) $\tilde{\theta}_3 = 0.0133336$. This value corresponds to the absolute maximum of $b^*_\theta(\theta)$, and thus provides the value of b_θ. The maximum equals $+0.0491889$. Here $\zeta = 0.5850010$, $t = 0.0231004$, $\xi = 0.7305134$.

Notice that in this example $s = 15$, and by (224)

$$\frac{s_1\theta}{s} = 0.0133333$$

i.e., $\tilde{\theta}_3$ differs from this value only in the seventh digit after the decimal point. In this case, $\zeta^2\xi = 0.2500008$, and the approximation (222) provides for this value 0.2500000.

V. On the Minimum Distance

In this section, we discuss the minimum distance of codes in the ensemble of irregular codes. To do this we have to study the derivative of the average distance distribution in the (right) vicinity of $\theta = 0$. Notice that in the case $h = 1$, i.e., when all the column sums are equal, we have essentially the regular case considered earlier in [5]. Thus, we assume from now on that $h \geq 2$. Let

$$2 \leq r_1 < r_2 < \cdots < r_g, \quad 2 \leq s_1 < s_2 > \cdots < s_h.$$

First, notice that by (13) we have that when $\theta \to 0$ then $x_i \to 0$, $i = 1, 2, \ldots, h$. Thus, by (14) it follows that $\tilde{\theta} \to 0$. By the earlier proved one-to-one correspondence between $\tilde{\theta}$ and t, related by (10), and $\tilde{\theta} \to 0$ it follows that $t \to 0$. On the contrary, when $t \to 0$, then $\tilde{\theta} \to 0$, and $\theta \to 0$. Therefore, $\theta \to 0$ is equivalent to all $t \in T$ tend to zero. In what follows, we need asymptotics of $\tilde{\theta}, \theta, \zeta, x_i$, etc., when $t \to 0$.

We start with $\tilde{\theta}$ as a function in t. Notice, that by (10) the function $\tilde{\theta}$ is even. Standard computation with binomial coefficients gives

$$\frac{(1+t)^{r_i-1}+(1-t)^{r_i-1}}{(1+t)^{r_i}+(1-t)^{r_i}} = 1-(r_i-1)t^2+\frac{1}{3}(r_i-1)(r_i^2+r_i-3)t^4+O(t^6). \quad (227)$$

Using (10) and the balance equation

$$\frac{\gamma}{s}\sum_{i=1}^{g}\nu_i r_i = 1 \quad (228)$$

we find

$$\tilde{\theta} = c_g t^2 - \tilde{c}_g t^4 + O(t^6) \quad (229)$$

where

$$c_g = \frac{\gamma}{s}\sum_{i=1}^{g}\nu_i r_i(r_i-1) \quad (230)$$

$$\tilde{c}_g = \frac{\gamma}{3s}\sum_{i=1}^{g}\nu_i r_i(r_i-1)(r_i^2+r_i-3). \quad (231)$$

It is easy to show that

$$\tilde{c}_g \geq c_g > 1. \quad (232)$$

From (229) it follows that

$$\tilde{\theta}'_t = 2c_g t - 4\tilde{c}_g t^3 + O(t^5). \quad (233)$$

Another corollary of (229) is that

$$\ln\tilde{\theta} = 2\ln t + \ln c_g - \frac{\tilde{c}_g}{c_g}t^2 + O(t^4). \quad (234)$$

Furthermore

$$\zeta = \frac{\tilde{\theta}}{t(1-\tilde{\theta})} = c_g t + (c_g^2 - \tilde{c}_g)t^3 + O(t^5). \quad (235)$$

By (14) we have

$$x_i \leq \frac{\tilde{\theta}s}{s_i} < \frac{s}{s_i}c_g t^2. \quad (236)$$

Let us elaborate on (236). We have by (11)

$$x_i = \frac{\eta_i\zeta^{s_i}\xi}{1+\zeta^{s_i}\xi}. \quad (237)$$

When $t \to 0$

$$\zeta^{s_i}\xi = o(1), \quad i = 1, 2, \ldots, h. \quad (238)$$

Indeed, if

$$\limsup_{t\to 0}\zeta^{s_i}\xi > 0$$

then (237) contradicts (236). Therefore, by (235) and (237)

$$t^{s_i}\xi = o(1), \qquad i = 1, 2, \ldots, h \quad (239)$$

$$x_i = \eta_i\zeta^{s_i}\xi - \eta_i\zeta^{2s_i}\xi^2 + O(\zeta^{3s_i}\xi^3) \quad (240)$$

and, since $s_i > s_1, i = 2, \ldots, h$, by (235) and (240)

$$x_i = o(x_1), \qquad i \geq 2 \quad (241)$$

$$x_i = o(x_2), \qquad i \geq 3. \quad (242)$$

Thus, by (240) and (235)

$$\sum_{i=1}^{h}x_i s_i = s_1x_1 + s_2x_2 + O(x_3) = \eta_1 s_1 c_g^{s_1}\xi t^{s_1} + \eta_1 s_1^2 c_g^{s_1-1}(c_g^2-\tilde{c}_g)\xi t^{s_1+2} - \eta_1 s_1 c_g^{2s_1}\xi^2 t^{2s_1} + \eta_2 s_2 c_g^{s_2}\xi t^{s_2} + o(\min(\xi t^{s_1+2}, \xi t^{s_2}, \xi^2 t^{2s_1})). \quad (243)$$

On the other hand

$$\sum_{i=1}^{h} x_i s_i = \tilde{\theta} s = s c_g t^2 - s\tilde{c}_g t^4 + O(t^6). \tag{244}$$

Comparing (243) and (244) we conclude that

$$\xi = \frac{s}{\eta_1 s_1 c_g^{s_1-1}} \frac{1}{t^{s_1-2}} + \begin{cases} O\left(\frac{1}{t^{s_1-4}}\right), & s_2 \ge s_1+2 \\ O\left(\frac{1}{t^{s_1-3}}\right), & s_2 = s_1+1. \end{cases} \tag{245}$$

In particular, if $s_1 = 2$,

$$\xi = \frac{s}{2\eta_1 c_g} + \begin{cases} O\left(t^2\right), & s_2 \ge 4 \\ O\left(t\right), & s_2 = 3. \end{cases} \tag{246}$$

In the case $s_1 = 2$ and $s_2 = 3$, we find a more accurate representation of ξ,

$$\xi = \frac{s}{2\eta_1 c_g} + At + Bt^2 + O(t^3). \tag{247}$$

Comparing (243) and (244), and taking into account (246) we find

$$A = -\frac{3\eta_2 s}{4\eta_1^2}, \qquad B = \frac{s}{2\eta_1}\left(1 + \frac{s-4\eta_1}{2\eta_1} c_g + \frac{9\eta_2^2}{4\eta_1^2} c_g^2\right). \tag{248}$$

In the case, $s_1 = 2$, $s_2 = 4$, we find a more accurate representation of ξ

$$\xi = \frac{s}{2\eta_1 c_g} + Ct^2 + O(t^3). \tag{249}$$

Comparing (243) and (244), and taking into account (246), we find

$$C = \frac{1}{\eta_1}\left(\frac{s\tilde{c}_g}{2c_g^2} + \frac{s^2}{4\eta_1} - \frac{s\eta_2 c_g}{\eta_1} - s\right). \tag{250}$$

Finally, when $s_2 \ge 5$

$$\xi = \frac{s}{2\eta_1 c_g} + Dt^2 + O(t^3) \tag{251}$$

where

$$D = \frac{1}{\eta_1}\left(\frac{s\tilde{c}_g}{2c_g^2} + \frac{s^2}{4\eta_1} - s\right). \tag{252}$$

Let b_θ^* be the expression under the maximum in (16). Then

$$\begin{aligned} \frac{db_\theta^*}{d\tilde{\theta}} &= s(\ln\tilde{\theta} - \ln(1-\tilde{\theta})) \\ &\quad + \gamma \sum_{i=1}^{g} r_i \nu_i \frac{(1+t)^{r_i-1} - (1-t)^{r_i-1}}{(1+t)^{r_i} + (1-t)^{r_i}} \frac{dt}{d\tilde{\theta}} \\ &\quad - s\ln t - \frac{s\tilde{\theta}}{t}\frac{dt}{d\tilde{\theta}} + \sum_{i=1}^{h} \ln\frac{\eta_i - x_i}{x_i}\frac{dx_i}{d\tilde{\theta}}. \end{aligned} \tag{253}$$

The last expression can be simplified: from (150) and (228) it follows that

$$\tilde{\theta} = \frac{t\gamma}{s}\sum_{i=1}^{g} \nu_i r_i \frac{(1+t)^{r_i-1} - (1-t)^{r_i-1}}{(1+t)^{r_i} + (1-t)^{r_i}}. \tag{254}$$

Taking into account (254) and (12), we find

$$\frac{db_\theta^*}{d\tilde{\theta}} = s\ln\zeta + \sum_{i=1}^{h}\ln\frac{\eta_i - x_i}{x_i}\frac{dx_i}{d\tilde{\theta}}. \tag{255}$$

Moreover, by (235)

$$\ln\zeta = \ln t + \ln c_g + \frac{c_g^2 - \tilde{c}_g}{c_g} t^2 + O(t^4). \tag{256}$$

We will consider several situations.

A. Case $s_1 \ge 3$

By (241), (13), and (14) we find

$$x_1 \sim \theta \sim \tilde{\theta}\frac{s}{s_1} \quad (t \to 0) \tag{257}$$

$$\frac{dx_1}{d\tilde{\theta}} = \frac{s}{s_1} + o(1). \tag{258}$$

This yields

$$\sum_{i=1}^{h}\ln\frac{\eta_i - x_i}{x_i}\frac{dx_i}{d\tilde{\theta}} = -\frac{s}{s_1}\ln x_1 + \frac{s}{s_1}\ln\eta_1 + o(1). \tag{259}$$

Using (255), (256), and (259) we conclude that

$$\frac{db_\theta^*}{d\tilde{\theta}} = s\ln t - \frac{2s}{s_1}\ln t + O(1). \tag{260}$$

By (257) and (260), we have that in the considered case $s_1 \ge 3$

$$\lim_{\theta\to 0}\frac{db_\theta^*}{d\theta} = -\infty. \tag{261}$$

This means that the average minimum distance in the considered ensemble is linear in n.

B. Case $s_1 = 2$

Let us consider now the case $s_1 = 2$. Notice that by (240) and (246)

$$x_i = O(t^5), \qquad i \ge 3 \tag{262}$$

a) $s_1 = 2$, $s_2 = 3$.

By (235), (240), (247), and (248)

$$x_1 = \frac{c_g s}{2}t^2 - \frac{3}{4}\frac{\eta_2}{\eta_1}sc_g^2 t^3 + O(t^4) \tag{263}$$

$$x_2 = \frac{\eta_2}{2\eta_1}sc_g^2 t^3 - \frac{3\eta_2^2 s}{4\eta_1^2}c_g^3 t^4 + O(t^5). \tag{264}$$

Therefore,

$$\theta = x_1 + x_2 + O(t^5) = \frac{c_g s}{2}t^2 - \frac{\eta_2}{4\eta_1}sc_g^2 t^3 + O(t^4). \tag{265}$$

Furthermore, by (233), (263), and (264)

$$\frac{dx_1}{d\tilde{\theta}} = \frac{s}{2} - \frac{9\eta_2}{8\eta_1}sc_g t + O(t^2) \tag{266}$$

$$\frac{dx_2}{d\tilde{\theta}} = \frac{3\eta_2}{4\eta_1}sc_g t + O(t^2) \tag{267}$$

$$\frac{dx_i}{d\tilde{\theta}} = o(t^3), \qquad i \ge 3. \tag{268}$$

Thus,

$$\begin{aligned} \sum_{i=1}^{h}\ln\frac{\eta_i - x_i}{x_i}\frac{dx_i}{d\tilde{\theta}} &= -s\ln t - \frac{s}{2}\ln\frac{c_g s}{2\eta_1} + \\ &\quad + \frac{3s\eta_2}{8\eta_1}c_g t\left(2 + \ln\frac{s}{2\eta_1 c_g}\right) + O(t^2\ln t). \end{aligned} \tag{269}$$

By (255), (259), and (269)

$$\frac{db^*_\theta}{d\tilde{\theta}} = \frac{s}{2}\ln\frac{2\eta_1 c_g}{s} + \frac{3st\eta_2 c_g}{8\eta_1}\left(2+\ln\frac{s}{2\eta_1 c_g}\right) + O(t^2\ln t). \tag{270}$$

Since, by (229) and (265)

$$\frac{d\tilde{\theta}}{d\theta} = \frac{2}{s} + \frac{3\eta_2}{2s\eta_1}c_g t + O(t^2) \tag{271}$$

then

$$\frac{db^*_\theta}{d\theta} = \ln\frac{2\eta_1 c_g}{s} + \frac{3}{2}\frac{\eta_2}{\eta_1}c_g t + O(t^2\ln t). \tag{272}$$

Therefore, if $2\eta_1 c_g < s$ then $\frac{db^*_\theta}{d\tilde{\theta}} < 0$, and there is a linear in n minimum distance.

Otherwise the derivative is positive.

b) $s_1 = 2$, $s_2 = 4$.

Analogously to the previous case

$$\frac{db^*_\theta}{d\theta} = \ln\frac{2\eta_1 c_g}{s} - \frac{9}{2}\frac{\eta_2}{\eta_1}c_g t\ln t + O(t). \tag{273}$$

It is easy to check that the condition for negativity of the derivative coincides with the previous case.

c) $s_1 = 2$, $s_2 \geq 5$.

This case differs from the previous one in constants D of (252) and C of (250). However, it affects only terms of small orders, and (273) holds true.

REFERENCES

[1] R. G. Gallager, *Information Theory and Reliable Communication*. New York: Wiley, 1968.

[2] ——, *Low Density Parity Check Codes*. Cambridge, MA: MIT Press, 1963.

[3] I. J. Good and J. F. Crook, "The enumeration of arrays and a generalization related to contingency tables," *Discr. Math.*, vol. 19, no. 1, pp. 23–45, 1977.

[4] C. Di, R. Urbanke, and T. Richardson, "Weight distributions: How deviant can you be?," in *Proc. Int. Symp. Information Theory (ISIT 2001)*, Washington, DC, 2001, p. 50.

[5] S. Litsyn and V. Shevelev, "On ensembles of low-density parity check codes: Asymptotic distance distributions," *IEEE Trans. Inform. Theory*, vol. 48, p. 887, Apr. 2002.

[6] M. Luby, M. Mitzenmacher, A. Shokrollahi, and D. Spielman, "Analysis of low density codes and improved designs using irregular graphs," in *Proc. 30th Annu. ACM Symp. Theory of Computing*, 1998, pp. 249–258.

[7] D. J. C. MacKay, "Good error-correcting codes based on very sparse matrices," *IEEE Trans. Inform. Theory*, vol. 45, pp. 399–431, Mar. 1999.

[8] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*. Amsterdam, The Netherlands: Elsevier, 1977.

[9] G. Miller and D. Burshtein, "Bounds on the maximum-likelihood decoding error probability of low-density parity-check codes," *IEEE Trans. Inform. Theory*, vol. 47, pp. 2696–2710, Nov. 2001.

[10] P. E. O'Neil, "Asymptotics and random matrices with row-sum and column-sum restrictions," *Bull. Amer. Math. Soc.*, vol. 75, pp. 1276–1282, 1969.

[11] T. Richardson and R. Urbanke, "The capacity of low-density parity-check codes under message-passing decoding," *IEEE Trans. Inform. Theory*, vol. 47, pp. 599–618, Feb. 2001.

[12] T. Richardson, A. Shokrollahi, and R. Urbanke, "Design of capacity-approaching irregular low-density parity-check codes," *IEEE Trans. Inform. Theory*, vol. 47, pp. 619–637, Feb. 2001.

[13] I. Sason and S. Shamai (Shitz), "Improved upper bounds on the ensemble performance of ML decoded low density parity check codes," *IEEE Commun. Lett.*, vol. 4, pp. 89–91, Mar. 2000.

[14] A. Shokrollahi, "Capacity achieving sequences," in *Codes, Systems, and Graphical Models*, B. Marcus and J. Rosenthal, Eds. New York: Springer-Verlag, 2000, IMA Series on Mathematics and its Applications, pp. 153–166.