TANGENT POWER SUMS AND THEIR APPLICATIONS

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Abstract
For integers m and p, we study the tangent power sums $\sum_{k=1}^{m} \tan^{2p} \frac{\pi k}{2m+1}$. We give recurrence, asymptotic and explicit formulas for these polynomials and indicate their connections with Newman's digit sums in base $2m$. In particular, for increasing m, we prove a monotonic strengthening of the Moser-Newman digit phenomenon for certain intervals.

1. Introduction

Everywhere below we suppose that $n \geq 1$ is an odd number and p is a positive integer. In the present paper we study tangent power sums of the form

$$\sigma(n,p) = \sum_{k=1}^{\frac{n-1}{2}} \tan^{2p} \frac{\pi k}{n}. \tag{1}$$

Shevelev [14] and Hassan [5] independently proved the following statements:

Theorem 1. *For every p, $\sigma(n,p)$ is an integer and a multiple of n.*

Theorem 2. *For fixed p, $\sigma(n,p)$ is a polynomial in n of degree $2p$ with the leading term*

$$\frac{2^{2p-1}(2^{2p}-1)}{(2p)!}|B_{2p}|n^{2p}, \tag{2}$$

where B_{2p} is a Bernoulli number.

Hassan [5] proved these results (see his Theorem 4.3 and formula 4.19) using a sampling theorem associated with second-order discrete eigenvalue problems.

Shevelev's proof [14] (see his Remarks 1 and 2) used some elementary arguments including the well-known Littlewood expression for the power sums of elementary polynomials in a determinant form [6].

In this paper we give another proof of these two theorems. In addition, we find several other representations, numerical results, and identities involving $\sigma(n,p)$. We give digit theory applications of $\sigma(n,p)$ in Section 5; and in Section 7, using the digit interpretation and a combinatorial idea, we find an explicit expression for $\sigma(n,p)$ (Theorem 7).

2. Proof of Theorem 1

Proof. Let $\omega = e^{\frac{2\pi i}{n}}$. Note that

$$\tan\frac{\pi k}{n} = i\frac{1-\omega^k}{1+\omega^k} = -i\frac{1-\omega^{-k}}{1+\omega^{-k}},\ \ \tan^2\frac{\pi k}{n} = \frac{1-\omega^{-k}}{1+\omega^k}\frac{1-\omega^k}{1+\omega^{-k}} \tag{3}$$

For the factors of $\tan^2\frac{\pi k}{n}$ we have

$$\frac{1-\omega^{-k}}{1+\omega^k} = \frac{(-\omega^k)^{n-1}-1}{(-\omega^k)-1} = \sum_{j=0}^{n-2}(-\omega^k)^j,\ \ \frac{1-\omega^k}{1+\omega^{-k}} = \sum_{j=0}^{n-2}(-\omega^{-k})^j. \tag{4}$$

Since $\tan\frac{\pi k}{n} = -\tan\frac{\pi(n-k)}{n}$, we have

$$2\sigma(n,p) = \sum_{k=1}^{n-1}\tan^{2p}\frac{\pi k}{n} \tag{5}$$

and by (3)–(5),

$$2\sigma(n,p) = \sum_{k=1}^{n-1}(\sum_{j=0}^{n-2}(-\omega^k)^j)^p(\sum_{j=0}^{n-2}(-\omega^{-k})^j)^p =$$

$$\sum_{k=1}^{n-1}(\prod_{l=0}^{p-1}\sum_{j=0}^{n-2}(-\omega^k)^j\prod_{l=0}^{p-1}\sum_{j=0}^{n-2}(-\omega^{-k})^j) =$$

$$= \sum_{k=1}^{n-1}(\prod_{t=0}^{2p-1}\sum_{j=0}^{n-2}(-\omega^{(-1)^t k})^j). \tag{6}$$

Furthermore, we note that

$$(n-1)^t \equiv (-1)^t \pmod{n}. \tag{7}$$

Indeed, (7) is evident for odd t. If t is even and $t = 2^h s$ with odd s, then

$$(n-1)^t - (-1)^t = ((n-1)^s)^{2^h} - ((-1)^s)^{2^h} =$$

$$((n-1)^s - (-1)^s)((n-1)^s + (-1)^s)((n-1)^{2s} +$$

$$(-1)^{2s}) \cdot \ldots \cdot ((n-1)^{2^{h-1}s} + (-1)^{2^{h-1}s}),$$

and, since $(n-1)^s + 1 \equiv 0 \pmod{n}$, we are done. Using (7), we can write (6) in the form (summing from $k = 0$, adding the zero summand)

$$2\sigma(n,p) = \sum_{k=0}^{n-1} \prod_{t=0}^{2p-1} (1 - \omega^{k(n-1)^t} + \omega^{2k(n-1)^t} - \ldots - \omega^{(n-2)k(n-1)^t}). \tag{8}$$

Considering $0, 1, 2, \ldots, n-2$ as digits in base $n-1$, after the multiplication of the factors of the product in (8) we obtain summands of the form $(-1)^{(r)}\omega^{kr}$, $r = 0, \ldots, (n-1)^{2p} - 1$, where $s(r)$ is the digit sum of r in base $n-1$. Thus we have

$$2\sigma(n,p) = \sum_{k=0}^{n-1} \sum_{r=0}^{(n-1)^{2p}-1} (-1)^{s(r)}\omega^{kr} = \sum_{r=0}^{(n-1)^{2p}-1} (-1)^{s(r)} \sum_{k=0}^{n-1} (\omega^k)^r. \tag{9}$$

However,

$$\sum_{k=0}^{n-1} (\omega^k)^r = \begin{cases} n, \text{if } r \equiv 0 \pmod{n} \\ 0, \text{otherwise.} \end{cases}$$

Therefore, by (9),

$$2\sigma(n,p) = n \sum_{r=0,\ n|r}^{(n-1)^{2p}-1} (-1)^{s(r)} \tag{10}$$

and, consequently, $2\sigma(n,p)$ is an integer multiple of n. It remains to show that the right-hand side of (10) is even. It is sufficient to show that the sum contains an even number of summands. The number of summands is

$$1 + \lfloor \frac{(n-1)^{2p}}{n} \rfloor = 1 + \frac{(n-1)^{2p} - 1}{n} =$$

$$1 + \sum_{l=0}^{2p-1} (-1)^l \binom{2p}{l} n^{2p-1-l} \equiv 1 + \sum_{l=0}^{2p-1} (-1)^l \binom{2p}{l} \pmod{2}.$$

But

$$1 + \sum_{l=0}^{2p-1} (-1)^l \binom{2p}{l} = 1 - (-1)^{2p} \binom{2p}{2p} = 0.$$

This completes the proof of the theorem. □

3. Proof of Theorem 2

Proof. We start with a construction similar to the one found in [16]. As is well-known,

$$\sin n\alpha = \sum_{i=0}^{\frac{n-1}{2}} (-1)^i \binom{n}{2i+1} \cos^{n-(2i+1)} \alpha \sin^{2i+1} \alpha,$$

or

$$\sin n\alpha = \tan \alpha \cos^n \alpha \sum_{i=0}^{\frac{n-1}{2}} (-1)^i \binom{n}{2i+1} \tan^{2i} \alpha.$$

Let $\alpha = \frac{k\pi}{n},\ k = 1, 2, \ldots, \frac{n-1}{2}$. Since $\tan \alpha \neq 0,\ \cos \alpha \neq 0$, then

$$0 = \sum_{i=0}^{\frac{n-1}{2}} (-1)^i \binom{n}{2i+1} \tan^{2i} \alpha =$$

$$(-1)^{\frac{n-1}{2}} (\tan^{n-1} \alpha - \binom{n}{n-2} \tan^{n-3} \alpha + \ldots -$$

$$(-1)^{\frac{n-1}{2}} \binom{n}{3} \tan^2 \alpha + (-1)^{\frac{n-1}{2}} \binom{n}{1}).$$

This means that the equation

$$\lambda^{\frac{n-1}{2}} - \binom{n}{2} \lambda^{\frac{n-3}{2}} + \binom{n}{4} \lambda^{\frac{n-5}{2}} - \ldots + (-1)^{\frac{n-1}{2}} \binom{n}{n-1} = 0 \tag{11}$$

has $\frac{n-1}{2}$ roots: $\lambda_k = \tan^2 \frac{k\pi}{n},\ k = 1, 2, \ldots, \frac{n-1}{2}$. Note that (11) is the characteristic equation for the difference equation

$$y(p) = \binom{n}{2} y(p-1) - \binom{n}{4} y(p-2) + \ldots -$$

$$(-1)^{\frac{n-1}{2}} \binom{n}{n-1} y(p - \frac{n-1}{2}) \tag{12}$$

which, consequently, has a closed solution

$$y(p) = \sum_{k=1}^{\frac{n-1}{2}} (\tan^2 \frac{k\pi}{n})^p = \sigma(n, p).$$

Now, using Newton's formulas for equation (11),

$$\sigma(n, 1) = \binom{n}{2},$$

$$\sigma(n,2) = \binom{n}{2}\sigma(n,1) - 2\binom{n}{4},$$

$$\sigma(n,3) = \binom{n}{2}\sigma(n,2) - \binom{n}{4}\sigma(n,1) + 3\binom{n}{6}, \ etc. \tag{13}$$

We conclude that $\sigma(n,p)$ is a polynomial in n of degree $2p$. Note that, by induction, all these polynomials are integer-valued and thus we have another independent proof of Theorem 1. To find the leading terms of these polynomials, we carry out some transformations of (1). Let $m = \frac{n-1}{2}$ and $l = m - k$. Changing the order of the summands in (1), and noting that

$$\frac{(m-l)\pi}{2m+1} + \frac{(2l+1)\pi}{4m+2} = \frac{\pi}{2},$$

we have

$$\sigma(n,p) = \sum_{l=0}^{m-1} \cot^{2p} \frac{(2l+1)\pi}{4m+2}. \tag{14}$$

Further, we have

$$\sigma(n,p) = \sum_{0 \le l \le \sqrt{m}} \cot^{2p} \frac{(2l+1)\pi}{4m+2} +$$

$$\sum_{\sqrt{m} < l \le m-1} \cot^{2p} \frac{(2l+1)\pi}{4m+2} = \Sigma_1 + \Sigma_2. \tag{15}$$

Let $p > 1$. Let us estimate the second sum Σ_2. The convexity of $\sin x$ on $[0, \frac{\pi}{2}]$ gives the inequality $\sin x \ge \frac{2}{\pi}x$. Therefore, for summands in the second sum, we have

$$\cot^{2p} \frac{(2l+1)\pi}{4m+2} < \sin^{-2p} \frac{(2l+1)\pi}{4m+2} <$$

$$(\frac{2m+1}{2l+1})^{2p} < (\frac{2m+1}{2\sqrt{m}+1})^{2p} < m^p.$$

This means that $\Sigma_2 < m^{p+1} < m^{2p}$ and has no influence on the leading term. Note that

$$\frac{(2l+1)\pi}{4m+2} cot \frac{(2l+1)\pi}{4m+2} \to 1$$

uniformly over $l \le \sqrt{m}$. Thus

$$\Sigma_1 = \sum_{0 \le l \le \sqrt{m}} (\frac{(4m+2)}{(2l+1)\pi})^{2p} + \alpha(m) =$$

$$(\frac{(4m+2)}{\pi})^{2p} \sum_{0 \le l \le \sqrt{m}} \frac{1}{(2l+1)^{2p}} + \alpha(m),$$

where $\alpha(m) \leq \varepsilon\sqrt{m}$. Thus the coefficient of the leading term of the polynomial $\sigma(n,p)$ is

$$\lim_{m\to\infty}\frac{\Sigma_1}{n^{2p}} = (\frac{2}{\pi})^{2p}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^{2p}} =$$

$$(\frac{2}{\pi})^{2p}(\zeta(2p) - \sum_{l=1}^{\infty}\frac{1}{(2l)^{2p}}) =$$

$$(\frac{2}{\pi})^{2p}(\zeta(2p) - \frac{1}{2^{2p}}\zeta(2p)) = \frac{2^p(2^{2p}-1)}{\pi^{2p}}\zeta(2p).$$

It is left to note that, using $\zeta(2p) = \frac{|B_{2p}|2^{2p-1}\pi^{2p}}{(2p)!}$, we have that the leading coefficient is defined by (2). □

4. Several Numerical Results

In 2002, Chen [1], using generating functions, presented a rather complicated method for finding formulas for $\sigma(n,p)$ for every positive p. Similar results appeared in Chu [2]. However, using Newton's formulas (13) for equation (11), we can effectively find the required formulas in a polynomial form. From (1), $\sigma(1,p) = 0$, so $\sigma(n,p) \equiv 0 \pmod{n(n-1)}$. Let

$$\sigma^*(n,p) = 2\sigma(n,p)/(n(n-1)).$$

By (13), the first polynomials $\{\sigma^*(n,p)\}$ are

$$\sigma^*(n,1) = 1,$$

$$\sigma^*(n,2) = \frac{n^2+n}{3} - 1,$$

$$\sigma^*(n,3) = \frac{2(n^2+n)(n^2-4)}{15} + 1,$$

$$\sigma^*(n,4) = \frac{(n^2+n)(17n^4-95n^2+213)}{315} - 1,$$

$$\sigma^*(n,5) = \frac{2(n^2+n)(n^2-4)(31n^4-100n^2+279)}{2835} + 1.$$

It is well-known (cf. Problem 85 in [9]) that integer-valued polynomials have integer coefficients in the binomial basis $\{\binom{n}{k}\}$. The first integer-valued polynomials $\{\sigma(n,p)\}$ represented in the binomial basis have the form

$$\sigma(n,1) = \binom{n}{2},$$

$$\sigma(n,2) = \binom{n}{2} + 6\binom{n}{3} + 4\binom{n}{4},$$

$$\sigma(n,3) = \binom{n}{2} + 24\binom{n}{3} + 96\binom{n}{4} + 120\binom{n}{5} + 48\binom{n}{6},$$

$$\sigma(n,4) = \binom{n}{2} + 78\binom{n}{3} + 836\binom{n}{4} + 3080\binom{n}{5} + 5040\binom{n}{6} + 3808\binom{n}{7} + 1088\binom{n}{8}.$$

Note that the recursion (12) presupposes a fixed n. In general, by (12), we have

$$\sigma(n,p) = \binom{n}{2}\sigma(n,p-1) - \binom{n}{4}\sigma(n,p-2) + ... -$$

$$(-1)^{\frac{n-1}{2}}\binom{n}{n-1}\sigma(n,p-\frac{n-1}{2}),\ p \geq \frac{n-1}{2}. \tag{16}$$

From (1), $\sigma(n,0) = \frac{n-1}{2},\ n = 3,5,\ldots,$ and then by (13) we have the recursions

$$\sigma(3,p) = 3\sigma(3,p-1),\ p \geq 1,\ \sigma(3,0) = 1;$$

$$\sigma(5,p) = 10\sigma(5,p-1) - 5\sigma(5,p-2),\ p \geq 2,\ \sigma(5,0) = 2,\ \sigma(5,1) = 10;$$

$$\sigma(7,p) = 21\sigma(7,p-1) - 35\sigma(7,p-2) + 7\sigma(7,p-3),\ p \geq 3,$$

$$\sigma(7,0) = 3,\ \sigma(7,1) = 21,\ \sigma(7,2) = 371;$$

$$\sigma(9,p) = 36\sigma(9,p-1) - 126\sigma(9,p-2) + 84\sigma(9,p-3) - 9\sigma(9,p-4),\ p \geq 4,$$

$$\sigma(9,0) = 4,\ \sigma(9,1) = 36,\ \sigma(9,2) = 1044,\ \sigma(9,3) = 33300;\ \textit{etc.}$$

Thus

$$\sigma(3,p) = 3^p, \tag{17}$$

and a few terms of the other sequences $\{\sigma(n,p)\}$ are

$$n=5)\quad 2, 10, 90, 850, 8050, 76250, 722250, 6841250, 64801250,$$

$$613806250, 5814056250, \ldots;$$

$$n=7)\quad 3, 21, 371, 7077, 135779, 2606261, 50028755, 960335173,$$

$$18434276035, 353858266965, 6792546291251, \ldots;$$

$$n=9)\quad 4, 36, 1044, 33300, 1070244, 34420356, 1107069876,$$

$$35607151476, 1145248326468, 36835122753252, \ldots;$$

$$n=11)\quad 5, 55, 2365, 113311, 5476405, 264893255, 12813875437,$$

$$619859803695, 29985188632421, 1450508002869079, \ldots.$$

5. Applications to Digit Theory

For $x \in \mathbb{N}$ and odd $n \geq 3$, let $S_n(x)$ be the sum

$$S_n(x) = \sum_{0 \leq r < x:\ r \equiv 0 \pmod{n}} (-1)^{s_{n-1}(r)}, \tag{18}$$

where $s_{n-1}(r)$ is the digit sum of r in base $n-1$. Note that $S_3(x)$ equals the difference between the numbers of multiples of 3 with even and odd binary digit sums (or multiples of 3 from sequences A001969 and A000069 in [15]) in the interval $[0, x)$.

Moser (cf. [8], Introduction) conjectured that

$$S_3(x) > 0. \tag{19}$$

Newman [8] proved this conjecture. Moreover, he obtained the inequalities

$$\frac{1}{20} < S_3(x)x^{-\lambda} < 5, \tag{20}$$

where

$$\lambda = \frac{\ln 3}{\ln 4} = 0.792481\ldots\ . \tag{21}$$

In connection with Newman's remarkable results, we will call the qualitative result (19) a "weak Newman phenomenon" (or "Moser–Newman phenomenon"), while an estimating result of the form (20) will be called a "strong Newman phenomenon."

In 1983, Coquet [3] studied a very complicated continuous and nowhere differentiable fractal function $F(x)$ with period 1 for which

$$S_3(3x) = x^{\lambda} F\left(\frac{\ln x}{\ln 4}\right) + \frac{\eta(x)}{3}, \tag{22}$$

where

$$\eta(x) = \begin{cases} 0, \text{if } x \text{ is even}, \\ (-1)^{s_2(3x-1)}, \text{if } x \text{ is odd}. \end{cases} \tag{23}$$

He obtained

$$\limsup_{x \to \infty,\ x \in \mathbb{N}} S_3(3x)x^{-\lambda} = \frac{55}{3}\left(\frac{3}{65}\right)^{\lambda} = 1.601958421\ldots\ , \tag{24}$$

$$\liminf_{x \to \infty,\ x \in \mathbb{N}} S_3(3x)x^{-\lambda} = \frac{2\sqrt{3}}{3} = 1.154700538\ldots\ . \tag{25}$$

In 2007, Shevelev [13] gave an elementary proof of Coquet's formulas (24)–(25) and gave sharp estimates in the form

$$\frac{2\sqrt{3}}{3}x^{\lambda} \leq S_3(3x) \leq \frac{55}{3}\left(\frac{3}{65}\right)^{\lambda} x^{\lambda}, \quad x \in \mathbb{N}. \tag{26}$$

Shevelev also showed that the sequence $\{(-1)^{s_2(n)}(S_3(n) - 3S_3(\lfloor n/4 \rfloor))\}$ is periodic with period 24, taking the values $-2, -1, 0, 1, 2$. This gives a simple recursion for $S_3(n)$. In 2008, Drmota and Stoll [4] proved a generalized weak Newman phenomenon, showing that (19) is valid for the sum (18) for all $n \geq 3$, at least beginning with $x \geq x_0(n)$. Our proof of Theorem 1 allows us to treat a strong form of this generalization, but only in "full" intervals with even base $n-1$ of the form $[0, (n-1)^{2p})$ (see also the preprint of Shevelev [14]).

Theorem 3. *For* $x_{n,p} = (n-1)^{2p},\ p \geq 1,$ *we have*

$$S_n(x_{n,p}) \sim \frac{2}{n} x_{n,p}^{\lambda}\ , \quad \sigma(n,p) \sim x_{n,p}^{\lambda} \quad (p \to \infty), \tag{27}$$

where

$$\lambda = \lambda_n = \frac{\ln\cot(\frac{\pi}{2n})}{\ln(n-1)}. \tag{28}$$

Proof. Employing (10) and (18), we have

$$S_n(x_{n,p}) = \frac{2}{n}\sigma(n,p),\ p \geq 1. \tag{29}$$

Thus, choosing the maximum exponent in (1) as $p \to \infty$, we find

$$S_n(x_{n,p}) \sim \frac{2}{n}\tan^{2p}\frac{(n-1)\pi}{2n} =$$

$$\frac{2}{n}\cot^{2p}\frac{\pi}{2n} = \exp(\ln\frac{2}{n} + 2p\ln\cot\frac{\pi}{2n}) =$$

$$\exp(\ln\frac{2}{n} + 2p\lambda\ln(n-1)) = \exp(\ln\frac{2}{n} + \ln x_{n,p}^{\lambda}) = \frac{2}{n}x_{n,p}^{\lambda}. \tag{30}$$

□

In particular, in the cases of $n = 3, 5, 7, 9, 11$, we have $\lambda_3 = \frac{\ln 3}{\ln 4} = 0.79248125\ldots$, $\lambda_5 = 0.81092244\ldots,\ \lambda_7 = 0.82452046\ldots,\ \lambda_9 = 0.83455828\ldots, \lambda_{11} = 0.84230667\ldots$, respectively.

To show that

$$1 - \frac{\ln\frac{\pi}{2}}{\ln(n-1)} \leq \lambda_n \leq 1 - \frac{\ln\frac{\pi}{2}}{\ln(n-1)} + \frac{1}{(n-1)\ln(n-1)}, \tag{31}$$

we note the convexity of $\cos x$ on $[0, \frac{\pi}{2}]$, $\cos x \geq 1 - \frac{2}{\pi}x$, and therefore $\cos\frac{\pi}{2n} \geq 1 - \frac{1}{n}$. Noting that $\tan\frac{\pi}{2n} \geq \frac{\pi}{2n} \geq \sin\frac{\pi}{2n}$, we have

$$\frac{2}{\pi}(n-1) \leq \cot\frac{\pi}{2n} \leq \frac{2}{\pi}n$$

and, by (28),

$$1-\frac{\ln\frac{\pi}{2}}{\ln(n-1)}\le\lambda_n\le 1-\frac{\ln\frac{\pi}{2}}{\ln(n-1)}+\frac{\ln(1+\frac{1}{n-1})}{\ln(n-1)},$$

which yields (31), since, for $n\ge 3$, $\ln(1+\frac{1}{n-1})<\frac{1}{n-1}$. Finally, let us show that λ_n is monotonic increasing. For $f(x)=\frac{\ln\cot(\frac{\pi}{2x})}{\ln(x-1)}$, we have

$$\ln(x-1)f'(x)=\frac{\pi}{x^2\sin\frac{\pi}{x}}-\frac{f(x)}{x-1}. \tag{32}$$

As in (31), we also have

$$f(x)\le 1-\frac{\ln\frac{\pi}{2}}{\ln(x-1)}+\frac{1}{(x-1)\ln(x-1)}. \tag{33}$$

On the other hand, since $\sin\frac{\pi}{x}\le\frac{\pi}{x}$, then

$$\frac{\pi(x-1)}{x^2\sin\frac{\pi}{x}}\ge 1-\frac{1}{x},$$

and, by (32), in order to show that $f'(x)>0$, it is sufficient to prove that $f(x)<1-\frac{1}{x}$, or, by (33), to show that

$$1-\frac{\ln\frac{\pi}{2}}{\ln(x-1)}+\frac{1}{(x-1)\ln(x-1)}<1-\frac{1}{x},$$

or

$$\frac{\ln(x-1)}{x}+\frac{1}{x-1}<\ln\frac{\pi}{2}.$$

This inequality holds for $x\ge 7$, and since $\lambda_3<\lambda_5<\lambda_7$, then the monotonicity of λ_n follows. Thus we have the monotonic strengthening of the strong form of a Newman-like phenomenon for base $n-1$ in the intervals considered.

6. An Identity

Since (29) was proved for $x_{n,p}=(n-1)^{2p},\ p\ge 1$, then by (16), for $S_n(x_{n,p})$ in the case $p\ge\frac{n+1}{2}$, we have the relations

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k\binom{n}{2k}\sigma(n,p-k)=$$

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k\binom{n}{2k}S_n((n-1)^{2p-2k})=0.$$

When $p = \frac{n-1}{2}$, the latter relation does not hold. Let us show that in this case, we have the identity

$$\sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k} S_n((n-1)^{n-2k-1}) = (-1)^n,$$

or, putting $n-2k-1 = 2j$, the identity

$$\sum_{j=0}^{\frac{n-1}{2}} (-1)^j \binom{n}{2j+1} S_n((n-1)^{2j}) = 1. \tag{34}$$

Indeed, when $j = 0$ we have $S_n(1) = 1$, while by (29), for $p = 0$, we obtain

$$S_n(1) = \frac{2}{n}\sigma(n,\ 0) = \frac{2}{n}\frac{n-1}{2} = \frac{n-1}{n},$$

i.e., the error is $-\frac{1}{n}$, and the error in the corresponding sum is $n(-\frac{1}{n}) = -1$. Therefore, in the latter formula, instead of 0, we have 1. Note that (34) can be written in the form

$$\sum_{j=1}^{\frac{n-1}{2}} (-1)^{j-1} \binom{n}{2j+1} \sigma(n,j) = \binom{n}{2}.$$

7. Explicit Combinatorial Representation

The representation (29) allows us to find an explicit combinatorial representation for $\sigma(n,p)$. We need three lemmas.

Lemma 4. *([11], p. 215) The number of compositions $C(m,n,s)$ of m with n positive parts not exceeding s is given by*

$$C(m,n,s) = \sum_{j=0}^{\min(n,\lfloor\frac{m-n}{s}\rfloor)} (-1)^j \binom{n}{j}\binom{m-sj-1}{n-1}. \tag{35}$$

Since $C(m,n,1) = \delta_{m,n}$ (Kronecker delta), then we have the identity

$$\sum_{j=0}^{\min(n,m-n)} (-1)^j \binom{n}{j}\binom{m-j-1}{n-1} = \delta_{m,n}. \tag{36}$$

Lemma 5. *The number of compositions $C_0(m,n,s)$ of m with n nonnegative parts not exceeding s is given by*

$$C_0(m,n,s)=\begin{cases} C(m+n,n,s+1), \text{if } m\geq n\geq 1, s\geq 2,\\ \sum_{\nu=1}^{m} C(m,\nu,s)\binom{n}{n-\nu}, \text{if } 1\leq m<n, s\geq 2,\\ 1, \text{if } m=0, n\geq 1, s\geq 0,\\ 0, \text{if } m>n\geq 1, s=1,\\ \binom{n}{m}, \text{if } 1\leq m\leq n, s=1.\end{cases} \tag{37}$$

Proof. First, let $s\geq 2,\ m\geq n\geq 1$. Decrease by 1 every part of a composition of $m+n$ with n positive parts not exceeding $s+1$. Then we obtain a composition of m with n nonnegative parts not exceeding s such that zero parts are allowed. Second, let $s\geq 2,\ 1\leq m<n$. Consider $C(m,\nu,s)$ compositions of m with $\nu\leq m$ parts. To obtain n parts, consider $n-\nu$ zero parts, which we choose in $\binom{n}{n-\nu}$ ways. Summing over $1\leq\nu\leq m$ gives the required result. The other cases follow. □

Now let $(n-1)^h\leq N<(n-1)^{h+1},\ n\geq 3$. Consider the representation of N in base $n-1$:

$$N=g_h(n-1)^h+...+g_1(n-1)+g_0,$$

where $g_i=g_i(N),\ i=0,...,h$, are the digits of N, $\ 0\leq g_i\leq n-2$. Let

$$s^e(N)=\sum_{i \text{ is even}} g_i,\ s^o(N)=\sum_{i \text{ is odd}} g_i.$$

Lemma 6. *N is a multiple of n if and only if $s^o(N)\equiv s^e(N) \pmod{n}$.*

Proof. The lemma follows from the relation $(n-1)^i\equiv(-1)^i \pmod{n}, i\geq 0$. □

Now we obtain an explicit combinatorial formula for $\sigma(n,p)$.

Theorem 7. *For $n\geq 3,\ p\geq 1$, we have*

$$\sigma(n,p)=\frac{n}{2}\sum_{j=0}^{(n-2)p}((C_0(j,p,n-2))^2+$$

$$2\sum_{k=1}^{\lfloor\frac{(n-2)p-j}{n}\rfloor}(-1)^k C_0(j,p,n-2)C_0(j+nk,p,n-2)), \tag{38}$$

where $C_0(m,n,s)$ is defined by (37).

Proof. Consider all nonnegative integers N not exceeding $(n-1)^{2p}-1$ that have $2p$ digits $g_i(N)$ in base $n-1$ (leading zeroes are allowed). Let the sum of the digits of N in the even p positions be j, while for the odd p positions, let the sum be $j+kn$

where k is a positive integer. Then, by Lemma 6, such N are multiples of n. Since in base $n-1$ the digits do not exceed $n-2$, then the number of ways to choose such N, for $k=0$, is $(C_0(j,p,n-2))^2$. In case $k \geq 1$, we should also consider the symmetric case when in the odd p positions the sum of the digits of N is j, while over the even p positions, the sum is $j+kn$ with a positive integer k. For $k \geq 1$ this gives $2C_0(j,p,n-2)C_0(j+kn,p,n-2)$ required N. Furthermore, since n is odd, then if k is odd, $s_{n-1}(N)$ is also odd. If k is even, then $s_{n-1}(N)$ is even. Thus the difference, $S_n((n-1)^{2p})$, between n-multiple Ns with even and odd digit sums is

$$S_n((n-1)^{2p}) = \sum_j ((C_0(j,p,n-2))^2 +$$

$$2\sum_k (-1)^k C_0(j,p,n-2)C_0(j+nk,p,n-2)).$$

Now to obtain (38), note that $0 \leq j \leq (n-2)p$, and for $k \geq 1$, $j+nk \leq (n-2)p$, so that $1 \leq k \leq \frac{(n-2)p-j}{n}$, and that by (29), $\sigma(n,p) = \frac{n}{2}S_n((n-1)^{2p})$. □

Example 8. Let $n=5, p=2$. By Theorem 7, we have

$$\sigma(5,2) = 2.5\sum_{j=0}^{6}((C_0(j,2,3))^2 +$$

$$2\sum_{k=1}^{\lfloor\frac{6-j}{3}\rfloor}(-1)^k C_0(j,2,3)C_0(j+5k,2,3)). \tag{39}$$

We have

$$C_0(0,2,3)=1, C_0(1,2,3)=2, C_0(2,2,3)=3,$$

$$C_0(3,2,3)=4, C_0(4,2,3)=3, C_0(5,2,3)=2, C_0(6,2,3)=1.$$

Thus

$$\sum_{j=0}^{6}((C_0(j,2,3))^2 = 44.$$

In the cases $j=0, k=1$ and $j=1, k=1$ we have

$$C_0(0,2,3)C_0(5,2,3)=2,\ C_0(1,2,3)C_0(6,2,3)=2.$$

Thus

$$2\sum_{j=0}^{6}\sum_{k=1}^{\lfloor\frac{6-j}{3}\rfloor}(-1)^k C_0(j,2,3)C_0(j+5k,2,3)) = -8$$

and, by (39),

$$\sigma(5,2) = 2.5(44-8) = 90.$$

On the other hand, by (1),

$$\sigma(5,2)=\sum_{k=1}^{2}\tan^4\frac{\pi k}{5}=0.278640\ldots+89.721359\ldots=89.999999\ldots$$

Example 9. In case $n=3$, then by Theorem 7 and formulas (17) and (37), we have

$$3^p=\frac{3}{2}\sum_{j=0}^{p}((C_0(j,p,1))^2+$$

$$2\sum_{k=1}^{\lfloor\frac{p-j}{3}\rfloor}(-1)^kC_0(j,p,1)C_0(j+3k,p,1))=$$

$$\frac{3}{2}\sum_{j=0}^{p}(\binom{p}{j}^2+2\sum_{k=1}^{\lfloor\frac{p-j}{3}\rfloor}(-1)^k\binom{p}{j}\binom{p}{3k+j}).$$

Using the well-known formula $\sum_{j=0}^{p}(\binom{p}{j}^2=\binom{2p}{p}$, we obtain the identity

$$\sum_{j=0}^{p}\sum_{k=1}^{\lfloor\frac{p-j}{3}\rfloor}(-1)^k\binom{p}{j}\binom{p}{3k+j}=3^{p-1}-\frac{1}{2}\binom{2p}{p},$$

or, changing the order of summation,

$$\sum_{k=1}^{\lfloor\frac{p}{3}\rfloor}(-1)^k\sum_{j=0}^{p-3k}\binom{p}{j}\binom{p}{3k+j}=3^{p-1}-\frac{1}{2}\binom{2p}{p}.$$

Since (cf. [10], p. 8)

$$\sum_{j=0}^{p-3k}\binom{p}{j}\binom{p}{3k+j}=\binom{2p}{p+3k}, \tag{40}$$

we obtain the identity

$$\sum_{k=1}^{\lfloor\frac{p}{3}\rfloor}(-1)^{k-1}\binom{2p}{p+3k}=\frac{1}{2}\binom{2p}{p}-3^{p-1},\ p\geq 1. \tag{41}$$

Note that (41) was proved by another method by Shevelev [12] and again by Merca [7] (cf. Cor. 8.3)

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