Schanuel’s Conjecture and Algebraic Roots of Exponential Polynomials

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Abstract

In this paper, I will prove that assuming Schanuel’s conjecture, an exponential polynomial with algebraic coefficients can have only finitely many algebraic roots. Furthermore, this proof demonstrates that there are no unexpected algebraic roots of any exponential polynomial. This implies a special case of Shapiro’s conjecture: if \( p(x) \) and \( q(x) \) are two exponential polynomials over the algebraic numbers, each involving only one iteration of the exponential map, and they have common factors only of the form \( \exp(g) \) for some exponential polynomial \( g \), then \( p \) and \( q \) have only finitely many common zeros.

1 Introduction

In the 1960’s, Schanuel made the following conjecture:

Conjecture 1. If \( \{z_1, \ldots, z_n\} \subset \mathbb{C} \), then \( \text{td}_\mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \), where \( \text{td}_\mathbb{Q} \) is the transcendence degree over \( \mathbb{Q} \), is at least the \( \mathbb{Q} \) linear dimension of \( \{z_1, \ldots, z_n\} \)

While there are proofs of special cases of this statement (e.g. Lindemann-Weierstrass Theorem), Schanuel’s conjecture is as yet unproven. In [6], Zilber constructs an algebraically closed exponential field known as pseudoexponentiation which satisfies the analog of Schanuel’s conjecture. We will make use of the following generalization of Schanuel’s conjecture.
Definition 2. An algebraically closed exponential field $K$ satisfies Schanuel’s conjecture if for any $\{z_1, ..., z_n\} \subset K$, the $\text{td}_\mathbb{Q}(z_1, ..., z_n, \exp(z_1), ..., \exp(z_n))$ is at least the $\mathbb{Q}$ linear dimension of $\{z_1, ..., z_n\}$.

In this paper we will give various consequences of Schanuel’s conjecture. Since Zilber’s construction satisfies this as well as the more general conditions we will set on the algebraically closed exponential field, these results are theorems of pseudoexponentiation.

We will use the notation $\mathbb{Q}^{alg}$ to refer to the algebraic closure of the rational numbers. The goal of this paper is to prove the following theorem:

Theorem 3. Suppose $p(x)$ is an exponential polynomial in $\mathbb{Q}^{alg}[x]^E$. Then Schanuel’s conjecture implies that $p(x)$ has finitely many algebraic zeros.

We will define $\mathbb{Q}^{alg}[x]^E$ in the following section.

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2 The Exponential Polynomial Ring

We will begin with the following definitions.

Definition 4. In this paper, a (total) E-ring is a $\mathbb{Q}$-algebra $R$ with no zero divisors, together with a homomorphism $\exp : \langle R, + \rangle \rightarrow \langle R^*, \cdot \rangle$.

A partial E-ring is a $\mathbb{Q}$-algebra $R$ with no zero divisors, together with a $\mathbb{Q}$-linear subspace $A(R)$ of $R$ and a homomorphism $\exp : \langle A(R), + \rangle \rightarrow \langle R^*, \cdot \rangle$. $A(R)$ is then the domain of $\exp$.

An E-field is an E-ring which is a field.

We say $S$ is a partial E-ring extension of $R$ if $R$ and $S$ are partial E-ring, $R \subset S$, and for all $r \in A(R)$, $\exp_S(r) = \exp_R(r)$.

NOTE: In particular, E-rings and E-fields have characteristic 0.

Recall the following construction of $K[X]^E$, the exponential polynomial ring over an E-ring $K$ on the set of indeterminates $X$: (see [5],[1])

If $R$ is a partial E-ring, we can construct $R'$, a partial E-ring extension of $R$, with the following properties:
The domain of the exponential map in $R'$ is precisely $R$.

The kernel of the exponential map in $R'$ is precisely the kernel of the exponential map in $R$.

If $y_i \notin A(R)$ for $i = 1, \ldots, n$, then $\text{td}_R(\exp_{R'}(\bar{y}))$ in $R'$ will be exactly the $\mathbb{Q}$-linear dimension of $\bar{y}$ over $A(R)$.

$R'$ is generated as a ring by $R \cup \exp(R)$.

For $K$ an E-ring and $X$ a set of indeterminates, let $K[X]$ be the partial E-ring where $A(K[X]) = K$. Then the exponential polynomial ring over $K$, $K'[X]^E$, is simply the union of the chain

$$K[X] = R_0 \hookrightarrow R_1 \hookrightarrow R_2 \hookrightarrow R_3 \hookrightarrow R_4 \hookrightarrow \cdots$$

where $R_{n+1} = R_n'$.

This construction yields a natural notion of height.

**Definition 5.** For $p$ an exponential polynomial

$$\text{height}(p) = \min \{i : p \in R_i\}$$

**Example 6.** The exponential polynomial $p(x_1, x_2) = \exp(\exp(\frac{x_1}{2} + \frac{x_2}{2})) + x_1^3$ in $\mathbb{C}[x_1, x_2]^E$ has height 2.

Fix $K$, an algebraically closed exponential field. Let $\mathbb{Q}^{alg}[x]^E$ be the exponential subring of $K[x]^E$ generated by $\mathbb{Q}^{alg}[x]$.

**NOTE:** The definition of $\mathbb{Q}^{alg}[x]^E$ depends entirely on $K$. We are fixing this algebraically closed exponential field at this point to avoid cumbersome notation. When we assume Schanuel’s conjecture, we are assuming $K$ satisfies Schanuel’s conjecture as in the introduction. Since we have not as of yet specified anything about the exponential map on $K$, it is worth noting that $\mathbb{Q}^{alg}$ may not be an exponential field (as in the case $K = \mathbb{C}_{\exp}$) in which case $\mathbb{Q}^{alg}[x]^E$ is not an exponential polynomial ring over $\mathbb{Q}^{alg}$. Thus, when we refer to the height of an element of $\mathbb{Q}^{alg}[x]^E$, we refer to its height in $K[x]^E$.

The following lemma is easy and useful.
Lemma 7. Let $\hat{Q}$ be the union of the chain

$$Q_0 \hookrightarrow Q_1 \hookrightarrow \ldots$$

where $Q_0 = \mathbb{Q}^{alg}$ and $Q_{i+1} = [Q_i \cup \exp(Q_i)]$, the subring of $K$ generated by $Q_i$ and the exponential image of $Q_i$.

Clearly $\hat{Q}$ is an $E$-ring.

Then $\mathbb{Q}^{alg}[x]_E = \hat{Q}[x]^E$, the free $E$-ring over $\hat{Q}$ on $x$.

This filtration of $\hat{Q}$ yields a natural notion of depth.

Definition 8. Let $q \in \hat{Q}$. Then

$$\text{depth}(q) = \min\{i : q \in Q_i\}.$$ 

Lemma 9. Let $s \in Q_i$.

Then there are $q_1, \ldots, q_n \in \mathbb{Q}^{alg}$ and $s_1, \ldots, s_m \in Q_{i-1}$ such that

- For all $1 \leq j \leq m$, $s_j$ is algebraic over the set
  $$\{q_1, \ldots, q_n, \exp(q_1), \ldots, \exp(q_n), \exp(s_1), \exp(s_m)\}$$

- $s$ is algebraic over
  $$\{q_1, \ldots, q_n, \exp(q_1), \ldots, \exp(q_n), \exp(s_1), \ldots, \exp(s_m)\}$$

- $q_1, \ldots, q_n, s_1, \ldots, s_m$ are $\mathbb{Q}$-linearly independent.

Proof. This proof is an easy induction on $i$. The definition of the filtration and finite character of algebraic closure yield the existence of $q_1, \ldots, q_n \in \mathbb{Q}^{alg}$ and $s_1, \ldots, s_m \in Q_{i-1}$ satisfying the first two conditions. To satisfy the third condition, recall that if $q_1, \ldots, q_n, s_1, \ldots, s_m$ are $\mathbb{Q}$-linearly dependent, then $\exp(q_1), \ldots, \exp(q_n), \exp(s_1), \ldots, \exp(s_m)$ are algebraically dependent.

Lemma 10. Assume Schanuel’s conjecture. Then the exponential map on $\mathbb{Q}^{alg}[x]_E$ is injective.
Proof. Since free constructions do not add to the kernel, it suffices to show that the kernel of the exponential map on \( \hat{\mathbb{Q}} \) is \( \{0\} \). To accomplish this, we will induct on depth.

Suppose that \( q \in \mathbb{Q}^{alg} \) and \( \exp(q) = 1 \). Then \( \text{td}_\mathbb{Q}(q, \exp(q)) = 0 \) and by Schanuel’s conjecture, \( q = 0 \). (If \( K \) is the complex exponential field, then this is true by the Lindemann-Weierstrass theorem.) Thus \( Q_0 \cap \ker(\exp_{\hat{\mathbb{Q}}}) = \{0\} \).

Suppose for purposes of induction that \( Q_i \cap \ker(\exp_{\hat{\mathbb{Q}}}) = \{0\} \).

Now suppose that \( s \in Q_{i+1} \cap \ker(\exp_{\hat{\mathbb{Q}}}) \).

Let \( q_1, \ldots, q_n, s_1, \ldots, s_m \) be as above. Then since for all \( j, s_j \) is algebraic over \( \{q_1, \ldots, q_n, \exp(q_1), \ldots, \exp(q_n), \exp(s_1), \ldots, \exp(s_m)\} \), \( s \) is algebraic over \( \{q_1, \ldots, q_n, s_1, \ldots, s_m, \exp(q_1), \ldots, \exp(q_n), \exp(s_1), \exp(s_m)\} \), and \( \exp(s) = 1 \), we have

\[
\text{td}(q_1, \ldots, q_n, s_1, \ldots, s_m, s, \exp(q_1), \ldots, \exp(q_n), \exp(s_1), \ldots, \exp(s_m), \exp(s)) = \text{td}(q_1, \ldots, q_n, \exp(q_1), \ldots, \exp(q_n), \exp(s_1), \ldots, \exp(s_m)).
\]

Thus, since \( q_1, \ldots, q_n \in \mathbb{Q}^{alg} \), we have

\[
\text{td}(q_1, \ldots, q_n, s_1, \ldots, s_m, s, \exp(q_1), \ldots, \exp(q_n), \exp(s_1), \ldots, \exp(s_m), \exp(s)) \leq n + m
\]

Thus, since \( q_1, \ldots, q_n, s_1, \ldots, s_m \) are \( \mathbb{Q} \)-linearly independent, Schanuel’s conjecture implies that \( s \) is \( \mathbb{Q} \)-linearly dependent on \( q_1, \ldots, q_n, s_1, \ldots, s_m \) which in turn implies that \( s \in Q_i \). But we assumed that \( Q_i \cap \ker(\exp_{\hat{\mathbb{Q}}}) = \{0\} \). Thus \( s = 0 \).

\[\square\]

3 Decompositions of \( p \)

We now fix an exponential polynomial \( p(x) \in \mathbb{Q}^{alg}[x]_E \). We will make use of the following filtration, \( F \), of \( \mathbb{Q}^{alg}[x]_E \):

\[\mathbb{Q}^{alg}[x] = R_0 \hookrightarrow R_1 \hookrightarrow \cdots\]

where \( R_{i+1} = [R_i \cup \exp(R_i)] \).

Notice that this filtration contains the filtration of \( \hat{\mathbb{Q}} \); i.e., \( Q_i \in R_i \) for all \( i \in \mathbb{N} \). Thus, the definition of depth generalizes.
Definition 11. We say $p$ has depth $= n$ if $p \in R_n$ and $p \notin R_{n-1}$.

Definition 12. We will call a set $T$ of exponential polynomials a decomposition of $p$ if it is a minimal set of exponential polynomials such that:

- $\exists t_1, ..., t_k \in T : p \in \Qalg[x, \exp(t_1), ..., \exp(t_k)]$, the subring of $\Qalg[x]_E$ generated by $x, \exp(t_1), ..., \exp(t_k)$.
- $t_i \in T \Rightarrow \exists t_1, ..., t_l \in T : t_i \in \Qalg[x, \exp(t_1), ..., \exp(t_l)]$.
- There is an $L \in \Z^*$ such that $x^L \in T$.

We will call elements of $T$ $T$-bricks.

Let $p^* \in \Qalg[x, \bar{y}]$ be such that $p^*(x, \exp(x), \exp(t_1), ..., \exp(t_\alpha)) = p$ as in the first part of the definition.

Consider the parallel between exponential polynomials and terms in the language $\mathcal{L} = \{+, -, \cdot, 0, 1, \exp\} \cup \{c_k : k \in \Qalg\}$. This parallel extends to subterms and $T$-bricks. Considering this parallel, notice that every $T$-brick can be written as a polynomial in $x$ and the exponential image of the $T$-bricks of lower height. Furthermore, all decompositions are finite. To satisfy the third bullet consider the following: While there are several terms which correspond to the same polynomial, we can choose one such term and take the least common multiple of the denominators of the rational coefficients of all the elements of $x$ which appear in the term.

Example 13. Consider $p(x) = \exp(\exp(\frac{x}{2}+x^2))+x^3$. Then $T = \{\frac{x}{2}, x^2, \exp(\frac{x}{2}+x^2)\}$ is a decomposition of $p$. Notice that $\frac{x}{2}+x^2$ is not in the decomposition since $\exp(\frac{x}{2}+x^2) = \exp(\frac{x}{2})\exp(x^2)$.

Definition 14. We say that a decomposition $T$ is a refined decomposition if $T$ is $\Q$–linearly independent over $\Qalg$.

Recall the following fact: (See [3])

Lemma 15. Given a decomposition $T$, we can form a refined decomposition $T'$.

We now fix a refined decomposition $T$ of $p$, and let $p^*$ witness this as above. Elements of $T$ will be called $t_i$ for $1 \leq i \leq \alpha$ where $t_i \neq t_j$ for $i \neq j$ and $|T| = \alpha$. 
4 Collapsing Points

Let \( a_i(x) \in \mathbb{Q}^{alg}[x] \) be nonzero polynomials and \( g_i(x) \) exponential polynomials in \( \mathbb{Q}^{alg}[x]_E \) such that \( g_i(x) \neq g_j(x) \) for \( i \neq j \) and

\[
p(x) = \sum_{i=1}^{m} a_i(x) \exp(g_i(x)).
\]

Fix these choices of \( a_i \) and \( g_i \).

**Definition 16.** We say \( p \) collapses at \( \beta \) if either \( a_i(\beta) = 0 \) for all \( i \), or there is some \( i, j, i \neq j \) and \( g_i(\beta) = g_j(\beta) \).

**Theorem 17.** Suppose \( \beta \in \mathbb{Q}^{alg} \) and \( p(\beta) = 0 \). Then Schanuel’s conjecture implies that \( p \) collapses at \( \beta \).

**Proof.** To begin this proof, we will need to set some notation.

Let \( p, g_i, \) and \( a_i \) be as above. Then we have

\[
p(x) = \sum_{i=1}^{m} a_i(x) \exp(g_i(x)) = p^*(x, \exp(t_1(x)), ..., \exp(t_\alpha(x))).
\]

Let the indeterminates \( Y_1, ..., Y_\alpha \) take the place of \( \exp(t_1(x)), ..., \exp(t_\alpha(x)) \). Then we also have

\[
p^*(x, Y_1, ..., Y_\alpha) = \sum_{i=1}^{m} a_i(x) \psi_i(Y_1, ..., Y_\alpha)
\]

and

\[
p^*(\beta, Y_1, ..., Y_\alpha) = \sum_{i=1}^{m} a_i(\beta) \psi_i(Y_1, ..., Y_\alpha)
\]

where \( \psi_i \) is a monomial for all \( i \). Notice that since \( g_i(x) \neq g_j(x) \) for \( i \neq j \), \( \psi_i(Y_1, ..., Y_\alpha) \neq \psi_j(Y_1, ..., Y_\alpha) \) for \( i \neq j \).

Now suppose that for some \( 1 \leq i \leq m, a_i(\beta) \neq 0 \). Then \( p^*(\beta, Y_1, ..., Y_\alpha) \neq 0 \). Since \( p^*(\beta, \exp(t_1(\beta)), ..., \exp(t_\alpha(\beta))) = 0 \) we know that for some \( d < \alpha \),

\[
td(\exp(t_1(\beta)), ..., \exp(t_\alpha(\beta))) = d.
\]
Since $\beta$ is algebraic

$$\text{td}(\exp(t_1(\beta)), ..., \exp(t_\alpha(\beta))) = \text{td}(\beta, \exp(t_1(\beta)), ..., \exp(t_\alpha(\beta))) = d.$$ 

Since the $T$-bricks are algebraic over $x$ and the exponential image of $T$, we know that

$$\text{td}(\beta, \exp(t_1(\beta)), ..., \exp(t_\alpha(\beta))) = \text{td}(\beta, t_1(\beta), ..., \exp(t_\alpha(\beta))).$$

Since $\beta$ is algebraic, this is equal to

$$\text{td}(t_1(\beta), ..., t_\alpha(\beta), \exp(t_1(\beta)), ..., \exp(t_\alpha(\beta)))$$

and we get

$$\text{td}(t_1(\beta), ..., t_\alpha(\beta), \exp(t_1(\beta)), ..., \exp(t_\alpha(\beta))) = d < \alpha.$$

Assuming Schanuel’s conjecture, we know that the $\mathbb{Q}$-linear dimension of $\{t_1(\beta), ..., t_\alpha(\beta)\}$ is at most $d$. Thus we can reorder the $T$-bricks so that for all $\alpha \geq j > d$, we have

$$t_j(\beta) = \sum_{i=1}^{d} \frac{m_{i,j}}{L} t_i(\beta)$$

where $m_{i,j} \in \mathbb{Q}$.

Now let $r(x, Y_1, ..., Y_d)$ be the polynomial with rational exponents such that

$$r(x, Y_1, ..., Y_d) = p^*(x, Y_1, ..., Y_d, \prod_{i=1}^{d} Y_i^{m_{i,d+1}}, ..., \prod_{i=1}^{d} Y_i^{m_{i,\alpha}}).$$

So

$$r(x, Y_1, ..., Y_d) = \sum_{i=1}^{m} a_i(x) \varphi_i(Y_1, ..., Y_d)$$

where for each $i$,

$$\varphi_i(Y_1, ..., Y_d) = \psi_i(Y_1, ..., Y_d, \prod_{i=1}^{d} Y_i^{m_{i,d+1}}, ..., \prod_{i=1}^{d} Y_i^{m_{i,\alpha}})$$

and is thus of the form $\prod Y_i^{q_i}$ for some $q_i \in \mathbb{Q}$.

Now we must compile all the information we have.
\[ r(\beta, \exp(t_1(\beta)), ..., \exp(t_d(\beta))) \]
\[ = p^* \left( \beta, \exp(t_1(\beta)), ..., \exp(t_d(\beta)), \prod_{i=1}^d \exp(m_{i,d+1} t_i(\beta)), ..., \prod_{i=1}^d \exp(m_{i,\alpha} t_i(\beta)) \right) \]
\[ = p^*(\beta, \exp(t_1(\beta)), ..., \exp(t_\alpha(\beta))) \]
\[ = p(\beta) = 0. \]

Since \( \{\exp(t_1(\beta)), ..., \exp(t_d(\beta))\} \) is algebraically independent (over the empty set), we know \( r(\beta, t_d(\beta)) = 0. \)

Since \( r(\beta, Y_1, ..., Y_d) = \sum_{i=1}^m a_i(\beta) \phi_i(Y_1, ..., Y_d) \) and for some \( i, a_i(\beta) \neq 0, \) we know that \( \phi_i(Y_1, ..., Y_d) = \phi_j(Y_1, ..., Y_d) \) for some \( i \neq j. \) But then \( \phi_i(\exp(t_1(\beta)), ..., \exp(t_d(\beta))) = \phi_j(\exp(t_1(\beta)), ..., \exp(t_d(\beta))) \) and we have

\[ \psi_i \left( \exp(t_1(\beta)), ..., \exp(t_d(\beta)), \prod_{k=1}^d \exp(m_{k,d+1} t_k(\beta)), ..., \prod_{k=1}^d \exp(m_{k,\alpha} t_k(\beta)) \right) \]
\[ = \psi_j \left( \exp(t_1(\beta)), ..., \exp(t_d(\beta)), \prod_{k=1}^d \exp(m_{k,d+1} t_k(\beta)), ..., \prod_{k=1}^d \exp(m_{k,\alpha} t_k(\beta)) \right) \]

and

\[ \psi_i(\exp(t_1(\beta)), ..., \exp(t_\alpha(\beta))) = \psi_j(\exp(t_1(\beta)), ..., \exp(t_\alpha(\beta))). \]

Thus \( \exp(g_i(\beta)) = \exp(g_j(\beta)) \) and since \( \exp \) is injective on \( \mathbb{Q}^{alg}[x]_E, \)
\( g_i(\beta) = g_j(\beta). \)

**Corollary 18.** Assume Schanuel's conjecture. Then if \( p \in \mathbb{Q}^{alg}[x]_E, \) then \( p \) has only finitely many algebraic zeros.

**Proof.** This is a simple induction on height.

Base case: \( \text{height}(p) = 0. \) The \( p \) is a polynomial in one variable and has only finitely many zeros. Inductive step: Suppose \( p(\beta) = 0. \) Then \( p \) collapses at \( \beta. \) So \( g_i(\beta) = g_j(\beta) \) for some \( i \neq j \) or \( a_i(\beta) = 0 \) for all \( i. \) Each of these options implies that \( \beta \) is a zero of one of finitely many nontrivial exponential polynomials of lower height. Each of these have only finitely many algebraic zeros by induction. \( \square \)
Corollary 19. Assume Schanuel’s conjecture. Let \( p, q \in \mathbb{Q}^{alg}[x]_E \) so that \( p \) and \( q \) have no common factors aside from units and are both of depth 1. Then \( p \) and \( q \) have only finitely many common zeros.

Proof. Depth 1 exponential polynomials are of the form \( \sum \alpha_i(x) \exp(\beta_i(x)) \) where \( \alpha_i, \beta_i \in \mathbb{Q}^{alg}[x] \).

Let \( T = \{ t_1(x), \ldots, t_\alpha(x) \} \) be a refined decomposition of both \( p \) and \( q \). (Simply require that both can be constructed using \( T \)). Suppose \( p(\beta) = q(\beta) = 0 \). For some integer \( L \), we know that \( x^L \) is one of the \( T \)-bricks. Every \( T \)-brick \( t_i(x) \) is algebraic over \( x^L, \exp(t_1(x)), \ldots, \exp(t_\alpha(x)) \). Therefore we know that

\[
\text{td}(t_1(\beta), \ldots, t_\alpha(\beta), \exp(t_1(\beta)), \ldots, \exp(t_\alpha(\beta))) = \text{td}(\frac{\beta}{L}, \exp(t_1(\beta)), \ldots, \exp(t_\alpha(\beta)))
\]

It is clear than a common factor of \( p^* \) and \( q^* \) would imply a common factor of \( p \) and \( q \). Therefore, since \( p \) and \( q \) have no common factors, any common factor \( r(x, y_1, \ldots, y_\alpha) \) of \( p^* \) and \( q^* \) must satisfy \( r(x, \exp(t_1(x)), \ldots, \exp(t_\alpha(x))) \) is a unit. Thus, if \( r \) is not a unit, one can consider the polynomials of lower total degree \( \frac{p^*}{r} \) and \( \frac{q^*}{r} \) which have the property that for any \( a \)

\[
\frac{p^*}{r}(a, \exp(t_1(a)), \ldots, \exp(t_\alpha(a)) = 0 \iff p(a) = 0
\]

and

\[
\frac{q^*}{r}(a, \exp(t_1(a)), \ldots, \exp(t_\alpha(a)) = 0 \iff q(a) = 0
\]

So we may assume that \( p^* \) and \( q^* \) have no common factors. Thus, since

\[
p^*(\beta, \exp(t_1(\beta)), \ldots, \exp(t_\alpha(\beta))) = q^*(\beta, \exp(t_1(\beta)), \ldots, \exp(t_\alpha(\beta))) = 0
\]

we have that

\[
\text{td}(\beta, \exp(t_1(\beta)), \ldots, \exp(t_\alpha(\beta))) \leq \alpha - 1
\]

and thus

\[
\text{td}(t_1(\beta), \ldots, t_\alpha(\beta), \exp(t_1(\beta)), \ldots, \exp(t_\alpha(\beta))) \leq \alpha - 1.
\]

Thus, Schanuel’s conjecture implies that \( t_1(\beta), \ldots, t_\alpha(\beta) \) are \( \mathbb{Q} \)-linearly dependent. Since \( T \) is a refined decomposition, and comprised of polynomial \( T \)-bricks, we can deduce that \( t_1(\beta), \ldots, t_\alpha(\beta) \) satisfy a non-trivial \( \mathbb{Q} \) linear polynomial, and that \( \beta \) satisfies a nontrivial polynomial over \( \mathbb{Q}^{alg} \). Thus, \( \beta \) is algebraic. By above, there are only finitely many algebraic zeros. \( \square \)
Recall the following conjecture of Shapiro from the 1960’s.

**Conjecture 20.** Suppose

\[ p(x) = \sum_{i=1}^{n} a_i \exp(b_i x) \]

and

\[ q(x) = \sum_{i=1}^{m} c_i \exp(d_i x) \]

where the \( a_i, b_i, c_i, d_i \in \mathbb{C} \). Then, if \( p(x) \) and \( q(x) \) have no common factors in \( \mathbb{C}[x]^E \) aside from units, then \( p(x) \) and \( q(x) \) have only finitely many common zeros in \( \mathbb{C} \).

A special case of Shapiro’s conjecture that has been investigated is the case where \( p(x) \) is a polynomial in \( \exp(b x) \) for some \( b \in \mathbb{C} \). Tijdeman and van der Poorten proved this special case for \( \mathbb{C}_{exp} \) in [4] and D’Aquino, Macintyre, and Terzo recently proved the analog for pseudoexponential fields. The following corollary follows easily from the results in this paper.

**Corollary 21.** (Shapiro’s conjecture over the algebraic numbers.) Assume Schanuel’s conjecture. Suppose

\[ p(x) = \sum_{i=1}^{n} a_i \exp(b_i x) \]

and

\[ q(x) = \sum_{i=1}^{m} c_i \exp(d_i x) \]

where the \( a_i, b_i, c_i, d_i \in \mathbb{Q}^{alg} \). Then, if \( p(x) \) and \( q(x) \) have no common factors aside from units, \( p(x) \) and \( q(x) \) have only finitely many common zeros.

**NOTE** This statement of Shapiro’s conjecture may seem weaker than the conjecture which would only have required that \( p(x) \) and \( q(x) \) have no common factors in the ring \( \mathbb{Q}[\exp(b x) : b \in \hat{\mathbb{Q}}] \) as opposed to assuming they have no common factors in the larger ring \( \mathbb{Q}^{alg}[x]^E \). However, the following argument proves that these are equivalent conditions.

Let \( K \) be an \( E \)-ring of infinite dimension \( \Gamma \) as a \( \mathbb{Q} \)-vector space. By the construction of the exponential polynomial ring we know that if \( b_1, \ldots, b_n \in \mathbb{Q}^{alg} \)
$K$ are $\mathbb{Q}$-linearly independent then $\exp(b_1x), \ldots, \exp(b_nx)$ are algebraically independent. Thus, if $B$ is a basis for $K$ as a $\mathbb{Q}$-vector space, and $\varphi : B \to \Gamma$ is a bijection, it is clear that as rings

$$K[\exp(bx) : b \in K] \cong K[t_{\gamma}^{\frac{1}{d}}, t_{\gamma}^{-\frac{1}{d}} : \gamma \in \Gamma, d \in \mathbb{N}, d \neq 0]$$

via the isomorphism given by $\psi(\exp(qbx)) = t_{\varphi(b)}^{q}$ where $q \in \mathbb{Q}$.

Similarly, for some infinite $\Delta$, where $\Delta \cap \Gamma = \emptyset$,

$$K[x]^{E} \cong K[t_{\gamma}^{\frac{1}{d}}, t_{\gamma}^{-\frac{1}{d}}, t_{\delta}^{\frac{1}{d}}, t_{\delta}^{-\frac{1}{d}} : \gamma \in \Gamma, \delta \in \Delta, d \in \mathbb{N}, d \neq 0]$$

Now suppose that $p(x), q(x) \in K[\exp(bx) : b \in K]$ and they have a common factor $h(x) \in K[x]^{E}$ which is not a unit so that

$$p(x) = h(x)s(x) \text{ and } q(x) = h(x)r(x).$$

There is $d \in \mathbb{N}, d \neq 0$ and finite sets $G \subset \Gamma, D \subset \Delta$ such that

$$\psi(p(x)), \psi(q(x)) \in K[t_{\gamma}^{\frac{1}{d}}, t_{\gamma}^{-\frac{1}{d}} : \gamma \in G]$$

and

$$\psi(h(x)), \psi(r(x)), \psi(s(x)) \in K[t_{\gamma}^{\frac{1}{d}}, t_{\gamma}^{-\frac{1}{d}}, t_{\delta}^{\frac{1}{d}}, t_{\delta}^{-\frac{1}{d}} : \gamma \in G, \delta \in D]$$

Now consider any finite set $\mathcal{J} \subset \Gamma$ such that $\mathcal{J} \cap G = \emptyset$ and $|\mathcal{J}| = |D|$. Then, as we have only renamed transcendental elements, we have that, as rings,

$$K[t_{\gamma}^{\frac{1}{d}}, t_{\gamma}^{-\frac{1}{d}}, t_{\delta}^{\frac{1}{d}}, t_{\delta}^{-\frac{1}{d}} : \gamma \in G, \delta \in D] \cong$$

$$K[t_{\gamma}^{\frac{1}{d}}, t_{\gamma}^{-\frac{1}{d}}, t_{\lambda}^{\frac{1}{d}}, t_{\lambda}^{-\frac{1}{d}} : \gamma \in G, \lambda \in \mathcal{J}]$$

via an isomorphism which preserves $K[t_{\gamma}^{\frac{1}{d}}, t_{\gamma}^{-\frac{1}{d}} : \gamma \in G]$. This will send $\psi(h)$ to a common factor of $\psi(p)$ and $\psi(q)$ which is not a unit and the image of $\psi^{-1}$ of this common factor will give a common factor of $p$ and $q$ in $K[\exp(bx) : b \in K]$ which is not a unit.

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References


