Standard versus strict bounded real lemma with infinite-dimensional state space

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Discrete-time LTI i/s/o system

$$\Sigma\colon \begin{cases} \mathbf{x}(n+1) = A\mathbf{x}(n) + B\mathbf{u}(n), \quad x(0) = x_0 \\ \mathbf{y}(n) = C\mathbf{x}(n) + D\mathbf{u}(n) \end{cases}$$

induced by system matrix $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$

$\mathcal{X}$ = states , $\mathcal{U}$ = inputs , $\mathcal{Y}$ = outputs

$\mathbf{x}(0) = 0$, $\mathbf{u} = \{\mathbf{u}(n)\}_n \geq 0$ arbitrary,
$\widehat{\mathbf{u}}(\lambda) = \sum_{n \geq 0} \mathbf{u}(n)\lambda^n$ arbitrary $\Rightarrow \widehat{\mathbf{y}}(\lambda) = F_\Sigma(\lambda) \cdot \widehat{\mathbf{u}}(\lambda)$ where
$F_\Sigma(\lambda) := D + \lambda C(I - \lambda A)^{-1} B$ analytic on a nhd of 0

BRL problem: Characterize in terms of A, B, C, D when F_Σ extends to $F_\Sigma \colon \mathbb{D} \underset{holo}{\to} \mathcal{L}(\mathcal{U}, \mathcal{Y})$ with $\|F_\Sigma\|_{H^\infty} \leq 1$?
(equivalently, $\|\mathbf{y}\|_{\ell^2_{\mathcal{Y}}} \leq \|\mathbf{u}\|_{\ell^2_{\mathcal{U}}}$?)

Outline:

1. Standard and Strict Bounded Real Lemma: finite-dimensional case, including Willems' storage function formulation
2. Infinite-dimensional case
3. Explicit identification of maximum and minimum storage functions in terms of controllability/observability operators
4. Time-varying systems as an application of time-invariant systems

1. Solutions of BRL problem: finite-dimensional case

Theorem 1 (Bounded Real Lemma)

Assume (i) $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$ finite-dimensional,
(ii) (C, A) observable $(\cap_{k \geq 0} \operatorname{Ker} CA^k = \{0\})$,
(iii) (A, B) controllable $(\overline{\operatorname{span}} \operatorname{Ran} A^k B = \mathcal{X})$.

Then $F_\Sigma \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ $(F_\Sigma \colon \mathbb{D} \underset{holo}{\to} \mathcal{L}(\mathcal{U}, \mathcal{Y})$ with $\|F_\Sigma\|_{H^\infty} \leq 1)$ $\Leftrightarrow$
$\exists\, H \succ 0$ satisfying KYP ineq

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} H & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \preceq \begin{bmatrix} H & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}$$

Theorem 2 (Strict Bounded Real Lemma)

Assume (i) $\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$ finite-dimensional
(ii) A stable $(\sigma(A) \subset \mathbb{D})$

Then $F_\Sigma \in \mathcal{S}^\circ(\mathcal{U}, \mathcal{Y})$: $F_\Sigma \colon \Omega \supset \overline{\mathbb{D}} \underset{holo}{\to} \mathcal{L}(\mathcal{U}, \mathcal{Y})$ with $\|F_\Sigma\|_{H^\infty} < 1$

$\Leftrightarrow \exists\, H > 0$ so that $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} H & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \prec \begin{bmatrix} H & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}$

Petersen-Anderson-Jonckheere 1991

Construction/Interpretation of solution H of KYP

The Arov approach

Start with a contractive system node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ realizing $S = F_\Sigma$.

There is a canonical way to construct the maximal solution H_a and minimal solution H_r of the KYP: the optimal and $*$-optimal storage functions on $\mathcal{X}$.

The Willems approach

Assume we have a realization $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ for $S = F_\Sigma$ Look for a storage function $S \colon \mathcal{X} \to \mathbb{R}^+$, i.e., S such that
$S(\mathbf{x}(n+1)) - S(\mathbf{x}(n)) \le \|\mathbf{u}(n)\|^2 - \|\mathbf{y}(n)\|^2$ &
$\min_{x \in \mathcal{X}} S(x) = S(0) = 0$

BRL via storage functions

Given (A, B, C, D) -system Σ , define available and required storage functions S_a and S_r by

$S_a(x_0) = \sup_{\mathbf{u} \in \mathcal{U}, n_1 \geq 0} \sum_{n=0}^{n_1} (\|\mathbf{y}(n)\|^2 - \|\mathbf{u}(n)\|^2)$

$S_r(x_0) = \inf_{\mathbf{u} \in \mathcal{U}, n_{-1} < 0} \sum_{n=n_{-1}}^{-1} (\|\mathbf{u}(n)\|^2 - \|\mathbf{y}(n)\|^2)$

subject to the system dynamical constraints along with $\mathbf{x}(n_{-1}) = 0$, $\mathbf{x}(0) = x_0$

Theorem 3 (Storage function version of BRL)

Given $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ with $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ finite-dimensional, (C, A) observable and (A, B) controllable.

Then $F_\Sigma \in \mathcal{S}(\mathcal{U}, \mathcal{Y}) \Leftrightarrow \Sigma$ has a storage function S

Moreover, if $F_\Sigma \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, then S_a and S_r are quadratic storage functions ($S_a(x) = \langle H_a x, x \rangle_{\mathcal{X}}$, $S_r(x) = \langle H_r x, x \rangle_{\mathcal{X}}$) so that $\epsilon^2 \|x\|_{\mathcal{X}}^2 \leq S_a(x) \leq S(x) \leq S_r(x)$ for any other storage function S

2. Infinite-dimensional controllability/observability

Definition: Infinite-dimensional observability/controllability

$\text{Rea}(A|B) := \text{alg span}_{k \geq 0}\{\text{Ran}\, A^k B\}$ (alg. reachability subspace)

$\overline{\text{Rea}}(A|B) := \overline{\text{span}}_{k \geq 0}\{\text{Ran}\, A^k B\}$ (approx. reachability subspace)

$\text{Obs}(C|A) := \text{alg span}_{k \geq 0}\{\text{Ran}\, A^{*k} C^*\}$ (alg. observ. subspace)

$\overline{\text{Obs}}(C|A) := \overline{\text{span}}_{k \geq 0}\{\text{Ran}\, A^{*k} C^*\}$ (approx. observ. subspace)

We say that (A, B) is exactly controllable if $\text{Rea}(A|B) = \mathcal{X}$

We say that (A, B) is (approx.) controllable if $\overline{\text{Rea}}(A|B) = \mathcal{X}$

We say that (C, A) is exactly observable if $\text{Obs}(C|A) = \mathcal{X}$

We say that (C, A) is (approx.) observable if $\overline{\text{Obs}}(C|A) = \mathcal{X}$

Infinite-dimensional BRL

Theorem 1′ (Inf.-dim. standard BRL: first version)

Given $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ with (C, A) exactly observable and (A, B) exactly controllable, ($\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$ all equal to Hilbert spaces). Then $S = F_\Sigma \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ $\Leftrightarrow \exists$ bounded & boundedly invertible solution $H \succ 0$ to the KYP inequality $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^* \left[\begin{smallmatrix} H & 0 \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] - \left[\begin{smallmatrix} H & 0 \\ 0 & I \end{smallmatrix}\right] \preceq 0$

Theorem 1″ (Inf.-dim. standard BRL: second version)

Given $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ with (C, A) (approx.) observable and (A, B) (approx.) controllable ($\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$ all equal to Hilbert spaces). Then $F_\Sigma \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ $\Leftrightarrow \exists$ generalized solution H to KYP, i.e., a possibly unbounded selfadjoint $H \succ 0$ with dense domain $\mathcal{D}(H) \subset \mathcal{X}$ such that $\left\| \left[\begin{smallmatrix} H^{1/2} & 0 \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \right\|^2 - \left\| \left[\begin{smallmatrix} H^{1/2} & 0 \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \right\|^2 \geq 0$ $\forall\ x \in \mathcal{D}(H^{1/2})$, $u \in \mathcal{U}$

Arov-Kaashoek-Pik 2006; cont.-time version: Arov-Staffans 2007;

Strict infinite-dimensional BRL

Theorem $2'$

Given $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ such that A is exp. stable $(\sigma(A) \subset \mathbb{D})$

Then $S = F_\Sigma \in \mathcal{S}^\circ(\mathcal{U}, \mathcal{Y}) \Leftrightarrow \exists\, H \succ 0$ such that

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & I_\mathcal{Y} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} H & 0 \\ 0 & I_\mathcal{U} \end{bmatrix} \prec 0$$

Proof

Follow the proof of Petersen-Anderson-Jonckheere . Use that $S \in \mathcal{S}^\circ(\mathcal{U}, \mathcal{Y})$ to inflate $\mathcal{U}$ to $\mathcal{X} \oplus \mathcal{U}$ and $\mathcal{Y}$ to $\mathcal{X} \oplus \mathcal{Y}$ with $B \mapsto \begin{bmatrix} \epsilon I_\mathcal{X} & B \end{bmatrix}$ and $C \mapsto \begin{bmatrix} \epsilon I_\mathcal{X} \\ C \end{bmatrix}$ to get new system which is exactly controllable and exactly observable. Then apply Theorem $1'$

Proof of Yakubovich 1973–1974: Not clear that H both bounded and boundedly invertible

Dichotomous version: Ben-Artzi–Gohbesrg–Kaashooek 1995

3. Preliminaries toward characterizing H_a and H_r

Given $\Sigma \sim M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \colon \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$

Assumptions:

- (C, A) (approx.) observable
- (A, B) (approx.) controllable
- $F = F_\Sigma \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$

Notation:

- $\mathfrak{L}_F = [F_{i-j}]_{i,j \in \mathbb{Z}} \colon \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\mathcal{Y}}(\mathbb{Z})$
- $\mathfrak{T}_F = [F_{i-j}]_{i,j \in \mathbb{Z}_+} \colon \ell^2_{\mathcal{U}}(\mathbb{Z}_+) \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$
- $\widetilde{\mathfrak{T}}_F = [F_{i-j}]_{i,j \in \mathbb{Z}_-} \colon \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_-)$
- $\mathfrak{H}_F = [F_{i-j}]_{i \in \mathbb{Z}_+, j \in \mathbb{Z}_-} \colon \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$

where $F_n = 0,\ n < 0;\ F_n = D,\ n = 0;\ F_n = CA^{n-1}B,\ n > 0$

Then $\mathfrak{L}_F = \begin{bmatrix} \widetilde{\mathfrak{T}}_F & 0 \\ \mathfrak{H}_F & \mathfrak{T}_F \end{bmatrix} \colon \begin{bmatrix} \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \\ \ell^2_{\mathcal{Y}}(\mathbb{Z}_+) \end{bmatrix} \to \begin{bmatrix} \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \\ \ell^2_{\mathcal{Y}}(\mathbb{Z}_+) \end{bmatrix}$

Observability operators redux

Define $\mathbf{W}_o \colon \mathcal{D}(\mathbf{W}_o) \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$ by
$\mathcal{D}(\mathbf{W}_o) = \{x \in \mathcal{X} \colon \{CA^k x\}_{k \ge 0} \in \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)\}$ and then
$\mathbf{W}_o \colon x \mapsto \{CA^k x\}_{k \ge 0}$ Then

- $\text{Rea}(A|B) \subset \mathcal{D}(\mathbf{W}_o) \;\Rightarrow \mathbf{W}_o$ densely defined (controllability)

Indeed, $x = \sum_{j=-K}^{-1} A^{-j-1} B u(j) \in \text{Rea}(A|B) \;\Rightarrow$
$CA^k x = \sum_j CA^{k-j-1} B \mathbf{u}(j) = \mathfrak{H}_F \mathbf{u}$
with $\mathbf{u}(j) = u(j)$ for $-K \le j \le -1$, $= 0$ for $j < -K$ where
$\|\mathfrak{H}_F\| = \|P_{\ell^2_{\mathcal{Y}}(\mathbb{Z}_+)} \mathfrak{L}_F|_{\ell^2_{\mathcal{U}}(\mathbb{Z}_-)}\| \le \|\mathfrak{L}_F\| = \|\mathfrak{T}_F\| = \|F\|_{H^\infty} \le 1$
$\Rightarrow \mathfrak{H}_F \mathbf{u} \in \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$

Observability operators in more detail

- $\mathbf{W}_o$ is a closed operator (direct check from the definition)
- $\mathbf{W}_o$ is injective (use observability assumption)
- $\mathbf{W}_o^*$ = closed operator (general property of adjoint of densely defined operator)

- $\ell_{\mathrm{fin},\mathcal{Y}}(\mathbb{Z}_+) \subset \mathcal{D}(\mathbf{W}_o^*) \Rightarrow \mathbf{W}_o^*$ densely defined

Indeed, given $\mathbf{y} = \{\mathbf{y}(j)\}_{j \ge 0} \in \ell_{\mathrm{fin},\mathcal{Y}}(\mathbb{Z}_+)$ ($\mathbf{y}(j) = 0$ for $j \ge J$), check $\mathbf{W}_o^* \mathbf{y} = \sum_{j \ge 0} A^{*j} C^* \mathbf{y}(j) \in \mathcal{X}$

- $\operatorname{Ran} \mathbf{W}_o^* \supset \operatorname{Obs}(C|A) \Rightarrow \operatorname{Ran} \mathbf{W}_o^*$ dense
- More generally, $\mathcal{D}(\mathbf{W}_o^*) = \{\mathbf{y} \in \ell^2_{\mathcal{Y}}(\mathbb{Z}_+) \colon \exists x_o \in \mathcal{X}$ with $\lim_{K \to \infty} \langle x, \sum_{k=0}^{K} A^{*k} C^* \mathbf{y}(k) \rangle_{\mathcal{X}} = \langle x, x_0 \rangle_{\mathcal{X}} \}$ and then $\mathbf{W}_o^* \mathbf{y} = x_o$ (computation)

Controllability operators in more detail

Define $\mathbf{W}_c^* \colon \mathcal{D}(\mathbf{W}_c^*) \subset \mathcal{X} \to \ell^2_{\mathcal{U}}(\mathbb{Z}_-)$ by
$\mathcal{D}(\mathbf{W}_c^*) = \{x \in \mathcal{X} \colon \{B^* A^{*-k-1} x\}_{k \le -1} \in \ell^2_{\mathcal{U}}(\mathbb{Z}_-)\}$
and then define $\mathbf{W}_c^* x = \{B^* A^{*-k-1} x\}_{k \le -1}$ and $\mathbf{W}_c = (\mathbf{W}_c^*)^*$

Properties of $\mathbf{W}_c$:

- $\mathbf{W}_c^*$ is closed, injective, densely defined operator and $\mathcal{D}(\mathbf{W}_c^*) \supset \mathrm{Obs}(C|A)$
- $\mathbf{W}_c$ is closed, densely defined with dense range with $\mathcal{D}(\mathbf{W}_c) \supset \ell_{\mathrm{fin},\mathcal{U}}(\mathbb{Z}_-)$; in general,
  $\mathcal{D}(\mathbf{W}_c) = \{\mathbf{u} \in \ell^2_c U(\mathbb{Z}_-) \colon \exists x_c \in \mathcal{X}$ with
  $\lim_{K \to \infty} \langle x, \sum_{-K}^{-1} A^{-k-1} B\mathbf{u}(k) \rangle_{\mathcal{X}} = \langle x, x_c \rangle_{\mathcal{X}}\}$
  and then $\mathbf{W}_c x = x_c$

Proof by duality: Apply the previous results to the adjoint system
$M^* = \left[\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right]$

The operators X_a and X_r

Notation:
$D_{\mathfrak{T}_F^*} = (I - \mathfrak{T}_F \mathfrak{T}_F^*)^{\frac{1}{2}}$ on $\ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$,
$D_{\widetilde{\mathfrak{T}}_F} = (I - \widetilde{\mathfrak{T}}_F^* \widetilde{\mathfrak{T}}_F)^{\frac{1}{2}}$ on $\ell^2_{\mathcal{U}}(\mathbb{Z}_-)$

Lemma

(1) $\exists$ unique closable X_a with $\mathcal{D}(X_a) = \operatorname{Ran} \mathbf{W}_c$ mapping into $\overline{\operatorname{Ran}} D_{\mathfrak{T}_{F^*}} \subset \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$ so that $\mathbf{W}_o|_{\operatorname{Ran} \mathbf{W}_c} = D_{\mathfrak{T}_F^*} X_a$

(2) Dually $\exists$ unique closable X_r with $\mathcal{D}(X_o) = \operatorname{Ran} \mathbf{W}_o^*$ mapping into $\overline{\operatorname{Ran}} D_{\widetilde{\mathfrak{T}}_F} \subset \ell^2_{\mathcal{U}}(\mathbb{Z}_-)$ so that $\mathbf{W}_o^* | \operatorname{Ran} \mathbf{W}_o^* = D_{\widetilde{\mathfrak{T}}_F} X_r$

Construction of X_a

$$I - \mathfrak{L}_F \mathfrak{L}_F = \begin{bmatrix} D^2_{\mathfrak{T}_F^*} & -\widetilde{\mathfrak{T}}_F \mathfrak{H}_F^* \\ -\mathfrak{H}_F \mathfrak{T}_F^* & D^2_{\mathfrak{T}_F^*} - \mathfrak{H}_F \mathfrak{H}_F^* \end{bmatrix} \Rightarrow D^2_{\mathfrak{T}_F^*} \succeq \mathfrak{H}_F \mathfrak{H}_F^*$$

$\Rightarrow$ (Douglas fact. lemma) $\exists$ Y_a with $D_{\mathfrak{T}_F^*} Y_a = \mathfrak{H}_F$, $\|Y_a\| \leq 1$

Set $X_a = Y_a \mathbf{W}_c^{\ddagger}$ where $\mathbf{W}_c^{\ddagger}$ = Moore-Penrose gen. inverse of $\mathbf{W}_c$

Identification of S_a and S_r

Theorem

(1) $S_a(x_0) := \sup_{\mathbf{u} \in \boldsymbol{\mathcal{U}}, n_1 \geq 0} \sum_{n=0}^{n_1} (\|\mathbf{y}(n)\|^2 - \|\mathbf{u}(n)\|^2)$

where $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ = system trajectory, $\mathbf{x}(0) = x_0$ (Willems)

$= \sup_{\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z}_+)} (\|\mathbf{W}_o x_0 + \mathfrak{T}_F \mathbf{u}\|^2_{\ell^2_{\mathcal{Y}}(\mathbb{Z}_+)} - \|\mathbf{u}\|^2_{\ell^2_{\mathcal{U}}(\mathbb{Z}_+)})$, $x_0 \in \operatorname{Ran} \mathbf{W}_c$

$= \|\overline{X}_a x_0\|^2 =: \|H_a^{\frac{1}{2}} x_0\|^2$ for $x_0 \in \operatorname{Ran} \mathbf{W}_c$, i.e., $H_a^{\frac{1}{2}} = \overline{X}_a$

(2) $S_r(x_0) = \inf_{\mathbf{u} \in \boldsymbol{\mathcal{U}}, n_{-1} < 0} \sum_{n=n_{-1}}^{-1} (\|\mathbf{u}(n)\|^2 - \|\mathbf{y}(n)\|^2)$

where $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ = system trajectory, $x(0) = x_0$ (Willems)

$= \inf_{\mathbf{u} \in \mathcal{D}(\mathbf{W}_c), \mathbf{W}_c \mathbf{u} = x_0} \|D_{\widetilde{\mathfrak{T}}_F} \mathbf{u}\|^2_{\ell^2_{\mathcal{Y}}(\mathbb{Z}_-)}$

$= \||\overline{X}_r|^{-1} x_0\|^2 =: \|H_r^{\frac{1}{2}} x_0\|^2$ for $x_0 \in \operatorname{Ran} \mathbf{W}_c$, i.e., $H_r^{\frac{1}{2}} = |X_r|^{-1}$

4. Time-varying i/s/o linear systems

$$\Sigma \colon \begin{cases} \mathbf{x}(n+1) = A_n \mathbf{x}(n) + B_n \mathbf{u}(n), \quad x(0) = x_0 \\ \mathbf{y}(n) = C_n \mathbf{x}(n) + D_n \mathbf{u}(n) \end{cases}$$

System matrix collection $\left\{ M_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \colon \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}_{n \in \mathbb{Z}_+} \right\}$

Recursive solution of system equations:

$\mathbf{x}(n) = C_n A_{n-1} \cdots A_0 x_0 + T_\Sigma \vec{\mathbf{u}}$

where $T_\Sigma \colon \ell^2_{\mathcal{U}}(\mathbb{Z}_+) \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$ given by $T_\Sigma = [F_{i,j}]_{i,j \geq 0}$ where

$$F_{ij} = \begin{cases} 0 \text{ for } i < j, \\ D_n \text{ for } i = j = n, \\ C_i A_{i-1} \cdots A_j B_j \text{ for } i > j \end{cases}$$

Aggregate form of system equations: notation

Set

$\mathbf{A} = \text{diag}_{j\ge 0}[A_j] \in \mathcal{L}(\ell^2_{\mathcal{X}}(\mathbb{Z}_+))$,

$\mathbf{B} = \text{diag}_{j\ge 0}[B_j] \in \mathcal{L}(\ell^2_{\mathcal{U}}(\mathbb{Z}_+), \ell^2_{\mathcal{X}}(\mathbb{Z}_+))$,

$\mathbf{C} = \text{diag}_{j\ge 0}[C_j] \in \mathcal{L}(\ell^2_{\mathcal{X}}(\mathbb{Z}_+), \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$,

$\mathbf{D} = \text{diag}_{j\ge 0}[D_j] \in \mathcal{L}(\ell^2_{\mathcal{U}}(\mathbb{Z}_+), \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$

For $\mathcal{H}$ = any of $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$, set $\vec{\mathcal{H}} = \ell^2_{\mathcal{H}}(\mathbb{Z}_+)$ and define $\sigma_0 \colon \ell^2_{\mathcal{H}}(\mathbb{Z}_+) \to \ell^2_{\vec{\mathcal{H}}}(\mathbb{Z}_+)$ by $\sigma_0 \colon \mathbf{h} = \{\mathbf{h}(n)\}_{n\ge 0} \mapsto \vec{\mathbf{h}} = \{\vec{\mathbf{h}}(n)\}_{n\ge 0}$ where $\vec{\mathbf{h}}(n) = \{\delta_{m,n}\mathbf{h}(n)\}_{m\ge 0}$ (sparse embedding)

Define $\boldsymbol{\mathcal{S}}$ on $\ell^2_{\vec{\mathcal{H}}}(\mathbb{Z}_+)$ by

$$\boldsymbol{\mathcal{S}} \colon \begin{bmatrix} \vec{\mathbf{h}}(0) \\ \vec{\mathbf{h}}(1) \\ \vec{\mathbf{h}}(2) \\ \vdots \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ \vec{\mathbf{h}}(0) \\ \vec{\mathbf{h}}(1) \\ \vdots \end{bmatrix}$$

Time-varying system equations in aggregate form

If $(\mathbf{u}, \mathbf{x}, \mathbf{y}) = \Sigma$ -trajectory, then $(\vec{\mathbf{u}}, \vec{\mathbf{x}}, \vec{\mathbf{y}}) = (\sigma_0 \mathbf{u}, \sigma_0 \mathbf{x}, \sigma_0 \mathbf{y})$ is a trajectory of the time-invariant system

$$\boldsymbol{\Sigma}\colon \left\{ \begin{array}{ccc} \vec{\mathbf{x}}(n+1) & = & \mathbf{A}\vec{x}(n) + \mathbf{B}\vec{\mathbf{u}}(n) \\ \vec{\mathbf{y}}(n) & = & \mathbf{C}\vec{\mathbf{x}}(n) + \mathbf{D}\mathbf{u}(n) \end{array} \right.$$

or in aggregate operator form

$$\Sigma\colon \left\{ \begin{array}{ccc} \boldsymbol{\mathcal{S}}^* \vec{\mathbf{x}} & = & \mathbf{A}\vec{\mathbf{x}} + \mathbf{B}\vec{\mathbf{u}} \\ \vec{\mathbf{y}} & = & \mathbf{C}\vec{\mathbf{x}} + \mathbf{D}\vec{\mathbf{u}} \end{array} \right.$$

Apply shift $\boldsymbol{\mathcal{S}}_{\mathcal{X}}$ to first equation: $\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{S}}^* \vec{\mathbf{x}} = \boldsymbol{\mathcal{S}}\mathbf{A}\vec{\mathbf{x}} + \boldsymbol{\mathcal{S}}\mathbf{B}\vec{\mathbf{u}}$

Note that $\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{S}}^* = I - \iota_0 \iota_0^*$, $\iota_0 \colon x \to \{\delta_{m,0} x\}_{m \ge 0} \Rightarrow$
$(I - \boldsymbol{\mathcal{S}}\mathbf{A})\vec{\mathbf{x}} = \{\delta_{m,0}\mathbf{x}(0)\}_{m \ge 0} + \mathcal{S}\mathbf{B}\vec{\mathbf{u}}$

Assume $I - \mathcal{S}\mathbf{A}$ invertible $\Rightarrow$
$\vec{\mathbf{x}} = (I - \mathcal{S}\mathbf{A})^{-1}\{\delta_{m,0}\mathbf{x}(0)\}_{m \ge 0} + (I - \mathcal{S}\mathbf{A})^{-1}\mathcal{S}\mathbf{B}\vec{\mathbf{u}}$

and then

$\vec{\mathbf{y}} = \mathbf{C}(I - \mathcal{S}\mathbf{A})^{-1}\{\delta_{m,0}\mathbf{x}(0)\}_{m \ge 0} + T_{\boldsymbol{\Sigma}}\vec{\mathbf{u}}$
where $T_{\boldsymbol{\Sigma}} = \mathbf{D} + \mathbf{C}(I - \mathcal{S}\mathbf{A})^{-1}\mathcal{S}\mathbf{B} \in \mathcal{L}(\ell^2_{\vec{\mathcal{U}}}(\mathbb{Z}_+), \ell^2_{\vec{\mathcal{Y}}}(\mathbb{Z}_+))$

Exponential stability and strict performance

Desired system properties:

- Uniform exponential stability: $\mathbf{u}(j) = 0$ for all $j \geq 0$, $\mathbf{x}(n) = x_0$ arbitrary $\Rightarrow \|\mathbf{x}(j)\| \leq M\rho^j \|x_0\|$ for some $M < \infty$, $\rho < 1$ (independent of choice of n) for all $j \geq n$
- Strict performance: $\mathbf{x}(0) = 0 \Rightarrow \|\mathbf{y}\|_{\ell^2} \leq \rho \|\mathbf{u}\|_{\ell^2}$ for some $\rho < 1$

Theorem

Assume A_n, B_n, C_n, D_n uniformly bounded. Then:

1. Exponential stability holds $\Leftrightarrow \|A_{n+j-1} \cdots A_n\| \leq M\rho^j$ for some $M < \infty$, $\rho < 1$ for all $n, j \geq 0$
2. Strict performance holds $\Leftrightarrow \exists$ uniformly bounded and uniformly boundedly invertible $\{H_n\}_{n \geq 0}$ with $H_n \succ 0$ on $\mathcal{X}$ such that the time-varying KYP holds:
$$\begin{bmatrix} A_n^* & C_n^* \\ B_n^* & D_n^* \end{bmatrix} \begin{bmatrix} H_{n+1} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} H_n & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \prec 0$$

Uniform exponential stability for t-v i/s/o linear systems

Direct verification of (1): Direct computation $\Rightarrow$
$(\mathcal{S}\mathbf{A})^j \{\delta_{k,n} x_0\}_{k \ge 0} = \{\delta_{k,n+j} A_{n+j-1} \cdots A_n x_0\}_{k \ge 0} \Rightarrow$
uniform exponential stability $\Leftrightarrow \|A_{n+j-1} \cdots A_n\| \le M\rho^j$ for some $M < \infty$, $\rho < 1$ (independent of n) for all $n, j \ge 0$

Verification of (2): via time-invariant system theory as follows:

The big time-invariant system

Given a time-variant i/s/o linear system

$$\Sigma: \begin{cases} \mathbf{x}(n+1) &= A_n \mathbf{x}(n) + B_n \mathbf{u}(n), \quad x(0) = x_0 \\ \mathbf{y}(n) &= C_n \mathbf{x}(n) + D_n \mathbf{u}(n) \end{cases}$$

which can be written in aggregate form

$$\Sigma: \begin{cases} \vec{\mathbf{x}} &= \mathcal{S}\mathbf{A}\,\vec{\mathbf{x}} + \mathcal{S}\mathbf{B}\,\vec{\mathbf{u}}, \quad \vec{\mathbf{x}}(0) = \{\delta_{m,0}x_0\}_{m\geq 0} \\ \vec{\mathbf{y}} &= \mathbf{C}\,\vec{\mathbf{x}} + \mathbf{D}\,\vec{\mathbf{u}} \end{cases}$$

where $\vec{\mathbf{x}} = \sigma_0 \mathbf{x}$, $\vec{\mathbf{u}} = \sigma_0 \mathbf{u}$, $\vec{\mathbf{y}} = \sigma_0 \mathbf{y}$

View $\mathbf{M} = \left[\begin{smallmatrix} \mathcal{S}\mathbf{A} & \mathcal{S}\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right] : \begin{bmatrix} \vec{\mathcal{X}} \\ \vec{\mathcal{U}} \end{bmatrix} \to \begin{bmatrix} \vec{\mathcal{X}} \\ \vec{\mathcal{U}} \end{bmatrix}$ as a system matrix

generating the time-invariant i/s/o linear system

$$\boldsymbol{\Sigma}: \begin{cases} \vec{\mathbf{x}}(n+1) &= \mathcal{S}\mathbf{A}\,\vec{\mathbf{x}}(n) + \mathcal{S}\mathbf{B}\,\vec{\mathbf{u}}(n) \\ \vec{\mathbf{y}}(n) &= \mathbf{C}\,\vec{\mathbf{x}}(n) + \mathbf{D}\,\vec{\mathbf{u}}(n) \end{cases}$$

where $\vec{\mathbf{x}}(n) \in \vec{\mathcal{X}} := \ell^2_{\mathcal{X}}(\mathbb{Z}_+)$, $\vec{\mathbf{u}}(n) \in \ell^2_{\mathcal{U}}(\mathbb{Z}_+)$, $\vec{\mathbf{y}}(n) \in \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$
are general rather than sparse (in image of σ_0)

Exp. stability and strict performance for the big t-i system

Exponential stability for $\mathbf{\Sigma}$ $\Leftrightarrow \sigma(\mathcal{S}\mathbf{A}) \subset \mathbb{D}$
$\Leftrightarrow \|A_{n+j-1} \cdots A_n\| \le M\rho^j$ uniformly in $n \ge 0$ for some $M < \infty$ and $\rho < 1$

Reason: Use geometric series to compute
$(I - \lambda \mathcal{S}\mathbf{A})^{-1} = \sum_{j=0}^{\infty} (\mathcal{S}\mathbf{A})^j \lambda^j$ converges on neighborhood of $\overline{\mathbb{D}}$
$\Leftrightarrow \|(\mathcal{S}\mathbf{A})^j\| \le M\rho^j$ where $\|(\mathcal{S}\mathbf{A})^j\| = \sup_{n \ge 0} \|A_{n+j-1} \cdots A_n\|$

Thus exp. stability for $\mathbf{\Sigma}$ $\Leftrightarrow$ exp, stability for Σ

Strict performance for $\mathbf{\Sigma}$ $\Leftrightarrow \exists\ \mathbf{H} \succ 0$ on $\vec{\mathcal{X}} = \ell^2_{\mathcal{X}}(\mathbb{Z}_+)$ so that

$$\begin{bmatrix} (\mathcal{S}\mathbf{A})^* & \mathbf{C}^* \\ (\mathcal{S}\mathbf{B})^* & \mathbf{D}^* \end{bmatrix} \begin{bmatrix} \mathbf{H} & 0 \\ 0 & I_{\vec{\mathcal{Y}}} \end{bmatrix} \begin{bmatrix} \mathcal{S}\mathbf{A} & \mathcal{S}\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} - \begin{bmatrix} \mathbf{H} & 0 \\ 0 & I_{\vec{\mathcal{U}}} \end{bmatrix} \prec 0 \quad \text{(aggregate KYP)}$$

Claim: (i) aggregate KYP $\Leftrightarrow$ time-varying KYP

$$\begin{bmatrix} A_n^* & C_n^* \\ B_n^* & D_n^* \end{bmatrix} \begin{bmatrix} H_{n+1} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} H_n & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \prec 0$$

and (ii) strict performance for $\mathbf{\Sigma}$ $\Leftrightarrow$ strict performance for Σ

Input-output map $T_{\mathbf{\Sigma}}$ for big t-i system

Notation: $\boldsymbol{\mathcal{S}}$ = shift on $\ell^2_{\vec{\mathcal{H}}}(\mathbb{Z}_+)$ ($\mathcal{H}$ = any of $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$)
$\overline{\mathbf{A}} = \bigoplus_0^\infty \mathbf{A}$ on $\ell^2_{\vec{\mathcal{X}}}(\mathbb{Z}_+)$, $\overline{\mathbf{B}} = \bigoplus_0^\infty \mathbf{B}$ from $\ell^2_{\vec{\mathcal{U}}}(\mathbb{Z}_+)$ to $\ell^2_{\vec{\mathcal{Y}}}(\mathbb{Z}_+)$
$\overline{\mathbf{C}} = \bigoplus_0^\infty \mathbf{C}$ from $\ell^2_{\vec{\mathcal{X}}}(\mathbb{Z}_+)$ to $\ell^2_{\vec{\mathcal{Y}}}(\mathbb{Z}_+)$,
$\overline{\mathbf{D}} = \bigoplus_0^\infty \mathbf{D}$ from $\ell^2_{\vec{\mathcal{U}}}(\mathbb{Z}_+)$ to $\ell^2_{\vec{\mathcal{Y}}}(\mathbb{Z}_+)$
Then $T_{\mathbf{\Sigma}} \colon \ell^2_{\vec{\mathcal{U}}}(\mathbb{Z}_+) \to \ell^2_{\vec{\mathcal{Y}}}(\mathbb{Z}_+)$ ($T_{\mathbf{\Sigma}} \colon \vec{\mathbf{u}} \to \vec{\mathbf{y}}$ when $(\vec{\mathbf{u}}, \vec{\mathbf{x}}, \vec{\mathbf{y}})$ = big t-i system trajectory with $\vec{\mathbf{x}}(0) = 0$) given by
$T_{\mathbf{\Sigma}} = \overline{\mathbf{D}} + \overline{\mathbf{C}}(I - \boldsymbol{\mathcal{S}}\overline{\mathbf{A}})^{-1}\boldsymbol{\mathcal{S}}\overline{\mathbf{B}}$

Immediate goal: Understand $T_{\mathbf{\Sigma}}$ (i-o map fior big t-i system) versus T_Σ (i-o map for original t-v system)

The intimate connection between Σ and $\boldsymbol{\Sigma}$

Idea from Foias-Frazho-Gohberg-Kaashoek OT100

$\mathcal{H}$ = coefficient Hilbert space, set $\vec{\mathcal{H}} = \ell^2_{\mathcal{H}}(\mathbb{Z}_+)$ (where $\mathcal{H}$ = any of $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$)

Define $\sigma_k \colon \ell^2_{\mathcal{H}}(\mathbb{Z}_+) \to \ell^2_{\vec{\mathcal{H}}}(\mathbb{Z}_+)$ by

$\mathbf{h} = \{\mathbf{h}(n)\}_{n\in\mathbb{Z}_+} \mapsto \sigma_k \mathbf{h} = \{\vec{\mathbf{h}}^{(k)}(n)\}_{n\in\mathbb{Z}_+}$ where

$\vec{\mathbf{h}}^{(k)}(n) = \{\delta_{m,n+k}\mathbf{h}(n)\}_{m\in\mathbb{Z}_+}$ (sparse embedding)

Check: Each σ_k is an isometry and

$\sigma := \text{row}_{k\geq 0}[\sigma_k] \colon \bigoplus_{k=0}^{\infty} \ell^2_{\mathcal{H}}(\mathbb{Z}_+) \to \ell^2_{\vec{\mathcal{H}}}(\mathbb{Z}_+)$ is unitary

Furthermore $\begin{bmatrix} \mathcal{S}\overline{\mathbf{A}} & \mathcal{S}\overline{\mathbf{B}} \\ \overline{\mathbf{C}} & \overline{\mathbf{D}} \end{bmatrix} \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix} = \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix} \begin{bmatrix} \mathcal{S}\mathbf{A} & \mathcal{S}\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ for each $k = 0, 1, 2, \dots$ $\Rightarrow$

$\begin{bmatrix} \mathcal{S}\overline{\mathbf{A}} & \mathcal{S}\overline{\mathbf{B}} \\ \overline{\mathbf{C}} & \overline{\mathbf{D}} \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \left(\bigoplus_0^{\infty} \begin{bmatrix} \mathcal{S}\mathbf{A} & \mathcal{S}\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \right)$, so in particular

$$\begin{bmatrix} \mathcal{S}\overline{\mathbf{A}} & \mathcal{S}\overline{\mathbf{B}} \\ \overline{\mathbf{C}} & \overline{\mathbf{D}} \end{bmatrix} \underset{u}{\cong} \bigoplus_0^{\infty} \begin{bmatrix} \mathcal{S}\mathbf{A} & \mathcal{S}\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

Conclusions

In particular: (i) $\boldsymbol{\mathcal{S}}\overline{\mathbf{A}} \underset{u}{\cong} \bigoplus_0^\infty \mathcal{S}\mathbf{A} \;\Rightarrow \boldsymbol{\mathcal{S}}\overline{\mathbf{A}}$ exp. stable $\Leftrightarrow \mathcal{S}\mathbf{A}$ exp. stable (as already seen directly), and

(ii) $(\overline{\mathbf{D}} + \overline{\mathbf{C}}(I - \boldsymbol{\mathcal{S}}\overline{\mathbf{A}})^{-1}\boldsymbol{\mathcal{S}}\overline{\mathbf{B}})\sigma_k = \sigma_k(\mathbf{D} + \mathbf{C}(I - \mathcal{S}\mathbf{A})^{-1}\mathcal{S}\mathbf{B})$ for each $k = 0, 1, 2, \dots \;\Rightarrow T_{\boldsymbol{\Sigma}}\sigma = \sigma\left(\bigoplus_0^\infty T_\Sigma\right)$

Conclude: $\|T_{\boldsymbol{\Sigma}}\| < 1 \Leftrightarrow \|T_\Sigma\| < 1$

By t-i theory, given that $\boldsymbol{\mathcal{S}}\overline{\mathbf{A}}$ is exp. stable, we know $\|T_{\boldsymbol{\Sigma}}\| < 1$ $\Leftrightarrow \exists\, \mathbf{H} \succ 0$ on $\ell^2_{\vec{\mathcal{X}}}$ so that

$$\begin{bmatrix} (\boldsymbol{\mathcal{S}}\overline{\mathbf{A}})^* & \overline{\mathbf{C}}^* \\ \overline{\mathbf{B}}^* & \overline{\mathbf{D}}^* \end{bmatrix} \begin{bmatrix} \mathbf{H} & 0 \\ 0 & I_{\vec{\mathcal{Y}}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{S}}\overline{\mathbf{A}} & \boldsymbol{\mathcal{S}}\overline{\mathbf{B}} \\ \overline{\mathbf{C}} & \overline{\mathbf{D}} \end{bmatrix} - \begin{bmatrix} \mathbf{H} & 0 \\ 0 & I_{\vec{\mathcal{U}}} \end{bmatrix} \prec 0 \quad \text{(big t-i KYP)}$$

Define $H_k = \sigma_k^* \mathbf{H} \sigma_k \succ 0$ on $\mathcal{X}$ Then big t-i KYP becomes

$$\begin{bmatrix} A_k^* & C_k^* \\ B_k^* & D_k^* \end{bmatrix} \begin{bmatrix} H_k & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} - \begin{bmatrix} H_k & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \prec 0 \quad \text{(t-v KYP)}$$

and t-v KYP $=$ criterion for $\|T_\Sigma\| < 1$

Additional topics

Dichotomy & indefinite solutions of t-v KYP with fixed inertia

References:

- J.A. Ball, G.J. Groenewald, and S. ter Horst, *Standard versus strict bounded real lemma with infinite-dimensional state space*, submitted
- J.A. Ball, G.J. Groenewald, and S. ter Horst, *Standard versus strict bounded real lemma with infinite-dimensional state space II: The dichotomy case*, in preparation

Thanks for your attention!