Matrix convex sets, Dilations and CP maps

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Operator Theory, Sde Boker, March 2017
In honor of Abie Feintuch

We use matrix convex sets $\mathcal{S} = \cup_n \mathcal{S}_n \subseteq \cup_n M_n(\mathbb{C})^d$ (as in Wittstock and Effros-Winkler) to study

- the existence of interpolating unital completely positive maps mapping one given tuple of operators to another and
- the existence of certain dilations of a given tuple of operators to a normal commuting tuple.

Some of these results follow from understanding the dependence of $\mathcal{S}$ on $\mathcal{S}_1$.
We were inspired by the recent work of Helton, Klep, McCullough and Schweighofer on free spectrahedra and an older work of Arveson on matrix ranges.

Matrix convex sets

Definition

(1) For $d \in \mathbb{N}$, a free set (in d free dimensions) $\mathcal{S}$ is a disjoint union $\mathcal{S} = \cup_n \mathcal{S}_n$ where $\mathcal{S}_n \subseteq M_n^d$.

(2) An nc set is a free set that is closed under direct sums and under simultaneous unitary conjugation.

(3) An nc set is matrix convex if whenever $X = (X_1, \ldots, X_d) \in \mathcal{S}_n$ and $\phi \in UCP(M_n, M_m)$, then $\phi(X) := (\phi(X_1), \ldots, \phi(X_d)) \in \mathcal{S}_m$ (equivalently, $\mathcal{S}$ is closed under matrix convex combinations [Effros-Winkler]).

Examples

1. The free spectrahedron associated to $A = (A_1, \ldots, A_d) \in B(H)^d$ is $\mathcal{D}_A = \cup \mathcal{D}_A(n)$ where

$$\mathcal{D}_A(n) = \{X \in M_n^d : Re \sum_j A_j \otimes X_j \leq I\}.$$

2. If A_j are self adjoint,

$$\mathcal{D}_A^{sa}(n) = \{X \in (M_n)_{sa}^d : \sum_j A_j \otimes X_j \leq I\}.[HKM]$$

3. Matrix ranges: If $A \in B(H)^d$, $\mathcal{W}(A) = \cup \mathcal{W}_n(A)$ where

$$\mathcal{W}_n(A) = \{\phi(A) : \phi \in UCP(C^*(S_A), M_n)\}.[A]$$

Note: Here S_A is the operator system generated by $\{A_j\}$ and $C^*(S_A)$ is the unital C^*-algebra generated by $\{A_j\}$.

Example

Let $A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Then

$$A_1 \otimes X_1 + A_2 \otimes X_2 = \begin{pmatrix} X_1 & 0 & 0 & 0 \\ 0 & X_2 & 0 & 0 \\ 0 & 0 & -X_1 & 0 \\ 0 & 0 & 0 & -X_2 \end{pmatrix}$$

and

$$\mathcal{D}_A^{sa}(n) = \{X = (X_1, X_2) \in (M_n)_{sa}^2 : \|X_i\| \leq 1\}.$$

Also,

$$\mathcal{W}_n(A) = \{(K_1 - K_3, K_2 - K_4) : K_i \geq 0, \sum K_i = I\}$$

$$\subseteq \{X = (X_1, X_2) \in (M_n)_{sa}^2 : \|X_1 \pm X_2\| \leq 1\}.$$

The relation between the matrix range and the free spectrahedron is given by **polarity**. For an nc set $\mathcal{S} \subseteq \cup_n (M_n)^d_{sa}$ define

$$\mathcal{S}^\bullet_n = \{X \in (M_n)^d_{sa} : \sum A_j \otimes X_j \leq I \quad \text{for all } A \in \mathcal{S}\}.$$

Similarly, it can be defined for the non-self-adjoint case but, for now on, I will stick with the self adjoint case.

Theorem

(1) $(\mathcal{W}(A) \cup \{0\})^\bullet = \mathcal{W}(A)^\bullet = \mathcal{D}^{sa}_A$.

(2) *If $\mathcal{S}$ is matrix convex and contains 0 then $\mathcal{S}^{\bullet\bullet} = \mathcal{S}$ [EW]*

(3) *If $0 \in \mathcal{W}(A)$, $(\mathcal{D}^{sa}_A)^\bullet = \mathcal{W}(A)$.*

Theorem

Let $\mathcal{S} \subseteq \cup_n (M_n)^d_{sa}$ be a closed matrix convex set.

(1) *$\mathcal{S} = \mathcal{W}(A)$ for some A iff $\mathcal{S}$ is bounded.*

(2) *$\mathcal{S} = \mathcal{D}^{sa}_B$ for some B iff $0 \in int(\mathcal{S})$.*

We now turn to explore the relationship between a matrix convex set $\mathcal{S}$ and it's "ground level" $\mathcal{S}_1 \subseteq \mathbb{R}^d$.
Let $C \subseteq \mathbb{R}^d$ be closed and convex. Then it is the intersection of some half spaces $H(\alpha, a) := \{x \in \mathbb{R}^d : \sum_1^d \alpha_i x_i \leq a\}$. We now define

$$\mathcal{W}_n^{max}(C) = \{X \in (M_n)_{sa}^d : \sum_1^d \alpha_i X_i \leq aI_n \text{ whenever } C \subseteq H(\alpha, a)\}$$

and

$$\mathcal{W}_n^{min}(C) = \{X \in (M_n)_{sa}^d : X \prec N \text{ commuting normal with } \sigma(N) \subseteq C\}.$$

where $X \prec N$ means that the tuple X is a compression of the (normal) tuple N. In fact, N can be chosen to be a tuple of matrices.

Theorem

If $\mathcal{S}$ is closed and matrix convex then $\mathcal{S}_1 = C$ iff for every n,

$$\mathcal{W}_n^{min}(C) \subseteq \mathcal{S}_n \subseteq \mathcal{W}_n^{max}(C).$$

Note: Set $D_2 = \{(x_1, x_2) : |x_1| + |x_2| \leq 1\}$ then $\mathcal{W}^{min}(D_2) = \mathcal{W}(A)$ for A in a previous example.
In general, if N is a normal commuting tuple then $\mathcal{W}(N) = \mathcal{W}^{min}(\overline{conv(\sigma(N))})$.
We also have

Theorem

For C as above, write $C' = \{x \in \mathbb{R}^d : \sum_j x_j y_j \leq 1 \text{ for all } y \in C\}$. Then

$$\mathcal{W}^{min}(C)^\bullet = \mathcal{W}^{max}(C')$$

and, if $0 \in C$,

$$\mathcal{W}^{max}(C)^\bullet = \mathcal{W}^{min}(C').$$

Let $\overline{\mathbb{B}}_d$ be the closed unit ball in $\mathbb{R}^d$. Note that $\overline{\mathbb{B}}_d = (\overline{\mathbb{B}}_d)'$.
Write **the matrix ball**:

$$\mathfrak{B} = \{X \in M_n(\mathbb{C})^d_{sa} : \sum X_j^2 \leq I\}.$$

then $(\mathfrak{B})_1 = \overline{\mathbb{B}}_d$ and $\mathfrak{B} \subset \mathfrak{B}^\bullet$ but $\mathfrak{B} \neq \mathfrak{B}^\bullet$.
We now define another matrix convex set "supported" on $\overline{\mathbb{B}}_d$:

$$\mathfrak{D} = \{X \in M_n(\mathbb{C})^d_{sa} : \| \sum X_j \otimes (X_j^*)^t \| \leq 1\}$$

and we have

Theorem

$$\mathcal{W}^{min}(\overline{\mathbb{B}}_d) \subset \mathfrak{B} \subset \mathfrak{D} = \mathfrak{D}^\bullet \subset \mathfrak{B}^\bullet \subset \mathcal{W}^{max}(\overline{\mathbb{B}}_d)$$

and all containments are proper if $d > 1$.

Positive and CP maps

Let $A \in B(H)^d$ and $B \in B(K)^d$ be two d-tuples. Generalizing results of Arveson, we get

Theorem

1. *There is a unital positive map* $\phi : S_A \to S_B$ *that sends* A_i *to* B_i *iff* $\mathcal{W}_1(B) \subseteq \mathcal{W}_1(A)$.
2. *There is a UCP map* $\phi : S_A \to S_B$ *that sends* A_i *to* B_i *if and only if* $\mathcal{W}(B) \subseteq \mathcal{W}(A)$. *In fact, there are isometries* $V_n : K \to H^{(\infty)}$ *such that* $B_i - V_n^* A_i^{(\infty)} V_n \to 0$.
3. *There is a CCP map* $\phi : S_A \to S_B$ *that sends* A_i *to* B_i *if and only if* $\mathcal{W}(B) \subseteq (\mathcal{W}(A))^{\bullet\bullet}$.
4. *There is a completely isometric UCP map* $\phi : S_A \to S_B$ *that sends* A_i *to* B_i *if and only if* $\mathcal{W}(B) = \mathcal{W}(A)$.

Using polarity we generalize some results of Helton, Klep, McCullough and Schweighofer.

Theorem

Suppose $\mathcal{D}_A^{sa}$ is bounded. Then

1. *There is a unital positive map $\phi : S_A \to S_B$ that sends A_i to B_i iff $\mathcal{D}_A^{sa}(1) \subseteq \mathcal{D}_B^{sa}(1)$.*
2. *There is a UCP map $\phi : S_A \to S_B$ that sends A_i to B_i if and only if $\mathcal{D}_A^{sa} \subseteq \mathcal{D}_B^{sa}$.*
3. *There is a CCP map $\phi : S_A \to S_B$ that sends A_i to B_i if and only if $\mathcal{D}_A^{sa} \subseteq \mathcal{D}_B^{sa}$.*
4. *There is a completely isometric UCP map $\phi : S_A \to S_B$ that sends A_i to B_i iff $\mathcal{D}_A^{sa} = \mathcal{D}_B^{sa}$.*

Dilations

The importance of scaled dilations to matrix convex sets inclusion was shown by Helton, Klep, McCullough and Schweighhofer. Here we have the following.

Theorem

Let $\mathcal{S}$ be a closed matrix convex set and $c > 0$. Then TFAE

1. *For all $X \in \mathcal{S}$ there exists a commuting normal d-tuple $T \in \mathcal{S}$ such that cT dilates X. (Note: T_i are matrices).*
2. $\mathcal{S} \subseteq c\mathcal{W}^{min}(\mathcal{S}_1)$.
3. *For any closed matrix convex set $\mathcal{T}$,*

$$\mathcal{S}_1 \subseteq \mathcal{T}_1 \Rightarrow \mathcal{S} \subseteq c\mathcal{T}.$$

In the following, if A is a $d \times d$ matrix and $X = \{X_1, \ldots, X_d\}$ is a d-tuple then AX would be the d-tuple $(\sum_j a_{1,j}X_j, \ldots, \sum_j a_{d,j}X_j)$.

Theorem

Let $\mathcal{S}$ and $\mathcal{T}$ be matrix convex sets. Assume that there is a k-tuple of $d \times d$ rank-one matrices $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ such that $I_d \in conv\{\lambda^{(1)}, \ldots, \lambda^{(k)}\}$ and such that $\lambda^{(m)}\mathcal{S} \subseteq \mathcal{T}$ for all $1 \leq m \leq k$.

Then, for every $X \in \mathcal{S}$ there is a commuting d-tuple T of self-adjoint matrices such that

(1) *$T \in \mathcal{T}$.(In fact, if $X \in \mathcal{S}_n$, $T \in \mathcal{T}_{nk}$.)*

(2) *T dilates X.*

Thus, in particular,

$$\mathcal{W}^{min}(\mathcal{S}_1) \subseteq \mathcal{S} \subseteq \mathcal{W}^{min}(\mathcal{T}_1).$$

Corollary

Let $K_1, K_2 \subseteq \mathbb{R}^d$ be closed convex sets. Assume that there is a k-tuple of $d \times d$ rank-one matrices $\lambda = (\lambda^{(1)}, \dots, \lambda^{(k)})$ such that $I_d \in conv\{\lambda^{(1)}, \dots, \lambda^{(k)}\}$ and such that $\lambda^{(m)} K_1 \subseteq K_2$ for all $1 \leq m \leq k$.
Then

$$\mathcal{W}^{max}(K_1) \subseteq \mathcal{W}^{min}(K_2).$$

In fact, if X is a tuple of operators satisfying all the inequalities that define K_1 then it can be dilated to a normal commuting tuple N with $\sigma(N) \subseteq K_2$.

Example

Every pair (X_1, X_2) of self adjoint contractions can be dilated to a normal commuting pair N with $\sigma(N) \subseteq \{(\alpha_1, \alpha_2) : |\alpha_1 \pm \alpha_2| \leq 2\}$

To see this: $K_1 = [-1, 1]^2$, $\lambda^{(m)} = 2e_m e_m^*$ $(m = 1, 2)$ and K_2 is the diamond above.

Corollary

Let $\mathcal{S}$ be a matrix convex set. Assume that there is a k-tuple of $d \times d$ rank-one matrices $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ such that $I_d \in conv\{\lambda^{(1)}, \ldots, \lambda^{(k)}\}$ and such that $\lambda^{(m)}\mathcal{S} \subseteq C\mathcal{S}$ for all $1 \leq m \leq k$ for some constant $C > 0$. Then

$$\mathcal{S} \subseteq C\mathcal{W}^{min}(\mathcal{S}_1)$$

and, thus, for any closed matrix convex set $\mathcal{T}$,

$$\mathcal{S}_1 \subseteq \mathcal{T}_1 \Rightarrow \mathcal{S} \subseteq \mathcal{T}.$$

Optimal constants

Theorem

Set $\mathcal{S} := \mathcal{W}^{max}(\overline{\mathbb{B}}_d)$. *Then*

1. *For every closed matrix convex set* $\mathcal{T}$ *such that* $\mathcal{S}_1 = \overline{\mathbb{B}}_d \subseteq \mathcal{T}_1$, *we have* $\mathcal{S} \subseteq d\mathcal{T}$. *In particular,* $\mathcal{W}^{max}(\overline{\mathbb{B}}_d) \subseteq d\mathcal{W}^{min}(\overline{\mathbb{B}}_d)$.
2. d *is optimal, i.e. there is a closed matrix convex set* $\mathcal{T}$ *such that* $\mathcal{S}_1 \subseteq \mathcal{T}_1$ *and* $\mathcal{S} \nsubseteq C\mathcal{T}$ *for every* $C < d$.

Note:

$$\mathcal{W}^{max}(\overline{\mathbb{B}}_d) = \{X \in \cup_n M_n(\mathbb{C})^d_{sa} : \sum_j v_j X_j \leq I \text{ for all } v \in \overline{\mathbb{B}}_d\}.$$

Thus, every such X can be dilated to a self adjoint commuting tuple whose joint spectrum lies in $d\overline{\mathbb{B}}_d$.

Example

To see the optimality of d for $d = 2$, set $B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and consider $\mathcal{T} = \mathcal{D}_B^{sa}$.

As $x_1 B_1 + x_2 B_2 = \begin{pmatrix} x_2 & x_1 \\ x_1 & -x_2 \end{pmatrix}$, $\mathcal{T}_1 = \mathcal{D}_B^{sa}(1) = \overline{\mathbb{B}}_2 = \mathcal{S}_1$.
It also shows that $B \in \mathcal{S} = \{X : v_1 X_1 + v_2 X_2 \leq I \text{ for all } v \in \overline{\mathbb{B}}_2\}$.
To see that B is not in $C\mathcal{T}$ (for $C < 2$), note that

$$B_1 \otimes B_1 + B_2 \otimes B_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

and it has 2 as an eigenvalue. If $C < 2$, $\frac{1}{C}(B_1 \otimes B_1 + B_2 \otimes B_2)$ has an eigenvalue larger than 1 and, thus, $\frac{1}{C}B$ is not in $\mathcal{T} = \mathcal{D}_B^{sa}$.

The Box

Theorem

Let $K = \prod_{i=1}^{d}[-s_i, s_i]$, $s_i > 0$. *Then* $\mathcal{W}^{max}(K) \subseteq \sqrt{d}\mathcal{W}^{min}(K)$ *and* $\sqrt{d}$ *is optimal.*

It follows from

Theorem

If $X_1, \ldots, X_d$ *are self adjoint matrices with* $||X_i|| \leq s_i$. *Then they can be dilated to a commuting d-tuple* $N_1, \ldots, N_d$ *with* $||N_i|| \leq s_i\sqrt{d}$.

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The construction is explicit.

Theorem

For $K = conv\{(0,0),(1,0),(0,1)\} \subseteq \mathbb{R}^2$, $\mathcal{W}^{max}(K) = \mathcal{W}^{min}(K)$.

Proof.

$\mathcal{W}^{max}(K) = \{(X_1, X_2) : X_i \geq 0,\ X_1 + X_2 \leq I\}$. Choose T_i such that $T_i T_i^* = X_i$. Then $\tilde{T} = (T_1, T_2)$ is a row contraction. Form

$$\Delta = (I - \tilde{T}^* \tilde{T})^{1/2} = (\Delta_{i,j}).$$

Set

$$V_i = \begin{pmatrix} T_i & 0 & 0 \\ \Delta_{1,i} & 0 & 0 \\ \Delta_{2,i} & 0 & 0 \end{pmatrix}$$

and $N_i = V_i V_i^*$. $N = (N_1, N_2)$ are positive, dilate X, commute and $\sigma(N) \subseteq K$. In fact $N_1 N_2 = N_2 N_1 = 0$. □

Same holds for $d \geq 2$.

Theorem

For $D_d = \{x \in \mathbb{R}^d : ||x||_1 \leq 1\}$,

$$\mathcal{W}^{max}(D_d) \subseteq \sqrt{d}\mathcal{W}^{min}(D_d)$$

and $\sqrt{d}$ *is optimal.*

This can be proved either by using the result for the box and polarity or by using the "positive diamond" above and the fact that if $X \in \mathcal{W}^{max}(D_d)$ then $|X_1| + \cdots + |X_d| \leq \sqrt{d}I$.
"Interpolating" (using the cases $p = \infty, 1, 2$) we get

Theorem

The optimal constant for the unit ball of ℓ_d^p *is* $d^{1/2+\min(1/p, 1-1/p)}$.

Theorem

If K is a convex subset of $\mathbb{R}^d$ satisfying

$$\mathcal{W}^{max}(K) \subseteq C\mathcal{W}^{min}(K)$$

and if A and B are two selfadjoint tuples such that $\mathcal{W}(A)_1 = K$ and such that the unital map ϕ from S_A to S_B sending each A_j to B_j is positive then the unital map from S_A to S_B sending each A_j to $\frac{1}{C}B_j$ is completely positive.

Proof.

The existence of such ϕ implies that $\mathcal{W}(B)_1 \subseteq \mathcal{W}(A)_1 = K$. Thus

$$\mathcal{W}(B) \subseteq \mathcal{W}^{max}(K) \subseteq C\mathcal{W}^{min}(K) \subseteq C\mathcal{W}(A) = \mathcal{W}(CA)$$

and this implies the existence of a unital CP map ψ sending CA_j to B_j.

□

Thank You !