Rates of convergence of powers of contractions

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Let T be a power-bounded operator on a complex Banach space X. The Katznelson-Tzafriri theorem says that $\|T^n(I-T)\| \to 0$ if and only if the peripheral spectrum $\sigma(T) \cap \mathbb{T}$ is at most the point 1.

Stolz region. A set $\mathcal{S}$ is called a Stolz region if it is the convex hull of 1 and a disk centered at the origin with radius strictly less than 1. For a Stolz region $\mathcal{S}$ there exists a constant $C > 1$, such that $|1-z| \leq C(1-|z|)$ for every $z \in \mathcal{S}$.

Nagy and Zemánek (1999) and Lyubich (1999) proved that the powers of the operator T have the rate of convergence $\|T^n(I-T)\| = \mathcal{O}(1/n)$ if and only if T satisfies the Ritt resolvent condition

$$\sup_{|\lambda|>1} \|(\lambda-1)R(\lambda,T)\| < \infty.$$

In this case its spectrum $\sigma(T)$ is contained in a Stolz region.

It follows from Nevanlinna's work (2001) that if T is power-bounded and satisfies, for some $\alpha \in [1, 2)$,

$$\sup_{1<|\lambda|<2} |\lambda - 1|^{\alpha} \|R(\lambda, T)\| < \infty,$$

then $\|T^n(I - T)\| = \mathcal{O}(1/n^{(2-\alpha)/\alpha})$. The case $\alpha = 1$ is Ritt's condition.

Léka (2014) has recently constructed, for any $\beta \in (\frac{1}{2}, 1)$, a contraction T in a complex Hilbert space with $\sigma(T) = \{1\}$ and $\|T^n(I - T)\| = \mathcal{O}(1/n^{\beta})$.

Numerical Range. Let T be a power-bounded operator on a Hilbert space H. Its numerical range is $W(T) := \{\langle Tf, f\rangle : \ \|f\| = 1\}$.

By the Hausdorff-Toeplitz theorem it is a convex subset of $\mathbb{C}$, the closure $\overline{W(T)}$ contains the spectrum $\sigma(T)$, and if T is normal then $\overline{W(T)}$ is the convex hull of $\sigma(T)$.

Cachia and Zagrebnov (2001) called a contraction T on a complex Hilbert space *quasi-sectorial* if its numerical range $W(T)$ is included in a Stolz region. They proved that if T is quasi-sectorial, then $\|T^n(I-T)\| = \mathcal{O}(1/n)$. For normal operators this had been proved by Bellow, Jones and Rosenblatt (1989).

Quasi-Stolz set. The geometric construction of a Stolz region is by taking a circle of radius $r < 1$ centered at 0 and drawing two tangent line segments from the point 1 to this circle. Paulauskas (2012) suggests a similar construction, but replacing the tangent line segments by arcs of a *tangent* "parabola-like" curve $x = 1 - b|y|^\alpha$, $1 < \alpha < 2$, $b > 0$, or $\alpha = 2$ and $b > \frac{1}{2}$ (with $|y| \le |y_0| < 1$); we call such a curve a *quasi-parabola*. We denote the obtained convex set by $D(\alpha, b)$, and call it a *quasi-Stolz set*. An operator with numerical range contained in a quasi-Stolz set is called in Paulauskas *generalized quasi-sectorial*. Note that the numerical radius of a generalized quasi-sectorial T is at most 1, so necessarily T is power-bounded with $\sup_n \|T^n\| \le 2$. Note that curves of the form $x = 1 - b|y|^\alpha$ with $\alpha > 2$ and $b > 0$ are outside the unit disk in a neighborhood of $(1, 0)$, so cannot be used.

Lemma. [Paulauskas]. Let $D(\alpha, b)$ be a quasi-Stolz set. Then there exists $K > 0$ such that

$$(n+1)^{1/\alpha} \sup_{\lambda \in D(\alpha,b)} |\lambda^n (1-\lambda)| \leq K \quad \forall n \geq 1.$$

Defintion. A compact set $A \subset \mathbb{C}$ is called a *K-spectral set for T* if there exists a $K_A > 0$ such that for every rational function $u(z)$ with poles outside A we have

$$\|u(T)\| \leq K_A \sup_{z \in A} |u(z)|.$$

Delyon and Delyon (1999). For a power-bounded T, any compact convex set containing $\overline{W(T)}$ is a K-spectral set. A consequence is:

Theorem. Let T be a contraction on a complex Hilbert space with numerical range contained in a quasi-Stolz set $D(\alpha, b)$. Then $\|T^n(I-T)\| = \mathcal{O}(1/n^{1/\alpha})$.

Remark. Paulauskas has obtained the above rate $1/\alpha$ only for diagonal operators, which are necessarily normal.

Lemma. Let T be a power-bounded operator on a complex Banach space X and assume that for some $\alpha \in (1,2]$ and $b > 0$ ($b > \frac{1}{2}$ when $\alpha = 2$) there exist reals $0 \neq t_n \to 0$, such that $(1 - b|t_n|^\alpha + it_n)_{n\geq 1} \subset \sigma(T)$. Then

$$\limsup_{n\to\infty} n^{1/\alpha}\|T^n(I-T)\| \geq (b^{1/\alpha}\,\mathrm{e})^{-1}.$$

Nevanlinna proved that if $\sigma(T) \cap \mathbb{T} = \{1\}$ and 1 is not isolated in $\sigma(T)$, then $\limsup n\|T^n(I - T)\| > 1/\mathrm{e}$. The above Lemma improves this result when additional information is given; however, with $\alpha = 1$, the lemma, though true, is in fact weaker.

Propsition. Let T be a power-bounded operator on a complex Banach space X and assume that for some $\frac{1}{2} < \beta < 1$ we have

$\sup_{n\geq 1} n^\beta\|T^n(I-T)\| = M < \infty$. Then there exists $b > 0$, such that the sepctrum $\sigma(T)$ is contained in a quasi-Stolz region $D(1/\beta, b)$.

Corolarry. Let T be a normal contraction on a complex Hilbert space, and let $1 < \alpha < 2$. Then $\|T^n(I-T)\| = \mathcal{O}(1/n^{1/\alpha})$ if and only if $\sigma(T)$ is contained in a quasi-Stolz region $D(\alpha, b)$ for some $b > 0$.

Proof: Since T is normal, $\overline{W(T)}$ is the convex hull of $\sigma(T)$, so by convexity of quasi-Stolz regions the "if" part follows from the above Theorem. The converse follows from the previous proposition.

Remark. For $\alpha = 1$ we replace a quasi-Stolz region by a Stolz region, and then the corollary holds by Cohen-Cuny-Lin (2014). The "if" part in this case is due to Bellow, Jones and Rosenblatt (1989).

Resolvent conditions.

Proposition. Let T be a power-bounded operator in X. Assume that for some $1 < \alpha < 2$ we have

$$\|R(\lambda, T)\| \leq \frac{C}{|\lambda - 1|^{\alpha}} \qquad \text{for} \quad |\lambda| > 1. \quad (1)$$

Then every $\lambda \in S(\alpha, 1/C) := \{z = x + iy : x > 1 - \frac{1}{C}|y|^{\alpha}\}$ is in $\rho(T)$. For $b < 1/C$, every $\lambda \in S(\alpha, b)$ satisfies $\|R(\lambda, T)\| \leq \frac{C_b}{|\lambda - 1|^{\alpha}}$.

Corollary. Let T be a normal contraction on a complex Hilbert space, and assume that for some $\alpha \in (1, 2)$ the resolvent condition (1) holds. Then $\|T^n(I - T)\| = \mathcal{O}(1/n^{1/\alpha})$.

Proof: For $b > 0$ small enough, the above Proposition yields that $\sigma(T) \subset D(\alpha, b)$. Since T is normal, $\overline{W(T)}$ is the convex hull of $\sigma(T)$, so is also contained in $D(\alpha, b)$. Now apply the above Theorem.

Lemma Let T be a power-bounded operator in a complex Banach space X, and let $\alpha \geq 1$. Then the following are equivalent:

(i) $\|R(\lambda, T)\| \leq C|\lambda - 1|^{-\alpha}$ for $|\lambda| > 1$.

(ii) $\|R(\mathrm{e}^{i\theta}, T)\| \leq c|\theta|^{-\alpha}$ for $0 < |\theta| \leq \pi$.

Theorem – Seifert (2014) Let T be power-bounded on a Hilbert space H with $\sigma(T) \cap \mathbb{T} = \{1\}$ and let $\alpha \geq 1$. Then the following are equivalent:

(i) There exist $\epsilon > 0$ and $C_1 > 0$ such that $\|R(\mathrm{e}^{i\theta}, T)\| \leq C_1|\theta|^{-\alpha}$ for $0 < |\theta| < \epsilon$,

(ii) There exists $C > 0$ such that

$$\|T^n(I - T)\| \leq \frac{C}{n^{1/\alpha}} \quad n \geq 1.$$

Combining Seifert's theorem with the above Lemma we obtain:

Theorem. Let T be power-bounded on a Hilbert space H with $\sigma(T) \cap \mathbb{T} = \{1\}$ and let $\alpha \geq 1$. Then the following conditions are equivalent:

(i) There exists $C > 0$ such that $\|R(\lambda, T)\| \leq C|\lambda - 1|^{-\alpha}$ for $|\lambda| > 1$.

(ii) $\|T^n(I - T)\| = \mathcal{O}(1/n^{1/\alpha})$.

Remarks. 1. The case of $\alpha = 1$ is the known case of Ritt operators.

2. Let $Vf(t) := \int_0^t f(s)ds$ be the Volterra operator on $L_2[0,1]$. It is known that $T = I - V$ is power-bounded with $\sigma(T) = \{1\}$ and $\|T^n(I - T)\| = \mathcal{O}(1/\sqrt{n})$. Paulauskas (private communication) showed that $\sup_{|\lambda|<1} |\lambda - 1|^2 \|R(\lambda, T)\| = \infty$; this shows that even when 1 is isolated in the spectrum, the resolvent estimate need not hold if we approach 1 from within the open unit disk.

Pointwise convergence.

Theorem. Let $D(\alpha, b)$ be a quasi-Stolz region, with $1 < \alpha \le 2$ and $b > 0$ ($b \ge 1/2$ for $\alpha = 2$). If $\bar{D}(\alpha, b)$ is a K-spectral set for a power-bounded operator T on $L_2(\mu)$, in particular (by Del.-Del.) if the numerical range of T is included in $\bar{D}(\alpha, b)$, then for every $f \in L_2(\mu)$ we have

(i) $\| \sup_n \frac{|T^n f|}{n^{1-1/\alpha}} \|_2 < \infty$.

(ii) $\frac{T^n f}{n^{1-1/\alpha}} \to 0$ a.e.

Sketch of proof: The condition $\sup_n n^{1/\alpha} \|T^n(I - T)\| < \infty$ yields that $\frac{T^n f}{n^{1-1/\alpha}} \to 0$ a.e. for f in the dense subspace and we may conclude from (i) by the Banach principle.

For every $f \in L_2(\mu)$ we have $(*)\Big\| \sup_n \frac{|T^n f|}{n^{1-1/\alpha}} \Big\|_2 \le \Big\| \sup_n \frac{1}{n^{2-1/\alpha}} \Big| \sum_{k=0}^n T^k f \Big| \Big\|_2 +$
$\Big\| \sup_n \frac{1}{n^{2-1/\alpha}} \Big| \sum_{k=1}^n k(T^k - T^{k-1}) f \Big| \Big\|_2$.

By Cauchy-Schwarz inequality the second term is dominated by $C_\alpha\Big(\sum_{k=1}^\infty k^{\frac{2-\alpha}{\alpha}}|(T^k - T^{k-1})f|^2\Big)^{\frac{1}{2}}$.

We use the fact that $\bar{D}(\alpha, b)$ is K-spectral to obtain

$$\sum_{k=1}^n k^{\frac{2-\alpha}{\alpha}} \|(T^k - T^{k-1})f\|_2^2 \leq$$

$$K_{D(\alpha,b)} \sup_{z\in D(\alpha,b)} \sum_{k=1}^\infty k^{\frac{2-\alpha}{\alpha}} |z^k - z^{k-1}|^2 \leq$$

$$\kappa_\alpha K_{D(\alpha,b)} \sup_{1\neq z\in D(\alpha,b)} \frac{|1-z|^2}{(1-|z|^2)^{2/\alpha}} < \infty.$$

Since $1 < \alpha \leq 2$, the series $\sum_{n=1}^\infty \frac{\|T^n f\|_2}{n^{2-\frac{1}{\alpha}}}$ converges, and the finiteness of the first term in (*) follows from a modification of Kronecker's lemma.

When $\alpha = 1$, the above proof shows the finiteness of the second term of (*) (this is Stein's argument 1961). However, finiteness of the first term in (*) does not always hold, even for contractions, unless we assume T to be a positive contraction and refer to Akcoglu's theorem. We then have the following extension of Bellow, Jones and Rosenblatt (1989).

Theorem. If a Stolz region $\bar{D}(1, b)$ is a K-spectral set for a positive contraction T on $L_2(\mu)$, in particular (by Del.-Del.) if the numerical range of T is included in $\bar{D}(1, b)$, then for every $f \in L_2(\mu)$ we have

(i) $\| \sup_n |T^n f| \|_2 < \infty$.

(ii) $T^n f \to f^*$ a.e.

Theorem. Let $\delta \in (0,1)$ and put

$$D_\delta = \{z : |z-\delta| < 1-\delta\}.$$

If $\bar{D}_\delta$ is a K-spectral set for a power-bounded operator T on $L_2(\mu)$, in particular (by Delyon-Delyon) if the numerical range of T is included in $\bar{D}_\delta$, then for every $f \in L_2(\mu)$ we have

(i) $\| \sup_n \frac{|T^n f|}{n^{1/2}} \|_2 < \infty$.

(ii) $\frac{T^n f}{n^{1/2}} \to 0$ a.e.

The proof is similar to the above proof.

Based on Chacon's construction Mesiar (1989) obtaibed a positive contraction T on L_1 for which there is a function $0 \leq f_0 \in L_1$ with $\limsup_{n\to\infty} \frac{T^n f_0}{n \log n} = \infty$ a.e. Using this property it is possible to build an example (which was suggested by Y. Derriennic) of

A positive contraction S on L_2 and $g \in L_2$ with $\limsup \frac{S^n g}{\sqrt{n \log n}} \equiv \infty$*.*

It shows that the previous theorem is justified since, even for positive contractions in L_2, property (ii) need not hold in general.