THOUGHTS ON SOME OPTIMAL CONTROL PROBLEMS

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Things you see from here,
you don’t see from there.

WHY THINK ABOUT OPTIMAL CONTROL NOW?

- **Better late than never.**
- **Finding the right resolution in which to approach system theory is our goal. If the resolution is too low, the theory may not be practically applicable. On the other hand, if the resolution is too high, we may see the trees but miss the forest. Abstract vs. concrete.**
- **Gain better understanding of the subject by clarifying the relations between different aspects of control theory, e.g., robust stabilization, model reduction, optimal regulator and estimation problems. This has the potential of leading to a grand unification of control theory.**

CONT.

- Open the possibility of extending methods to other settings as: classes of infinite dimensional systems, special classes of systems (positive real, bounded real).
- Extend optimal control theory to deal with complexity, that is, to networks of systems, using local optimality results for the nodes as well as the interconnection data.

BASIC IDEAS FOR REACHING A TARGET

- **Use coprime factorizations, functional models.**
- **Shift/translation realizations.**
- **Construct the reachability map $\mathcal{R} : \mathcal{U} \longrightarrow \mathcal{X}$.**
- **Derive representation of $\mathrm{Ker}\,\mathcal{R}$.**
- **Reduce the reachability map to**
  $\boxed{R : \mathcal{U}/\mathrm{Ker}\,\mathcal{R} \longrightarrow \mathcal{X}.}$
- **Assuming reachability, this is a module isomorphism.**
- **Invert R, using doubly coprime factorizations.**
- **For a discrete time system, this leads to time optimal controllers.**

WHAT HAPPENS WHEN ALL SPACES HAVE HILBERT SPACE STRUCTURE?

DUE TO THE AVAILABILITY OF ORTHOGONAL DECOMPOSITION, OPTIMIZATION PROBLEMS ARE GREATLY SIMPLIFIED.

FROM ALGEBRA TO ANALYSIS

Although the technicalities of polynomial model based system theory for discrete time linear systems over an arbitrary field are vastly different from the Hardy space based theory for some classes of continuous-time systems there are strong algebraic similarities. These, with the help of heavy analytic tools, can be used to extend the algebraic approach to a wide variety of optimal control and estimation problems for several classes of, not necessarily rational, analytic functions.

Some of the ideas and results presented owe much to a cooperation with Raimund Ober in the early 1990s and a long term one with Uwe Helmke.

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SOME ANALOGIES

Algebra	**Analysis**
	$L^2(-\infty,\infty;\mathbb{C}^m)$
$\mathbb{F}((z^{-1}))^m$	$L^2(0,\infty;\mathbb{C}^m)\oplus L^2(-\infty,0;\mathbb{C}^m))$
$=\mathbb{F}[z]^m\oplus z^{-1}\mathbb{F}[[z^{-1}]]^m$	$=H^2_-\oplus H^2_+$, **where** $H^2_\pm=\mathcal{F}L^2(0,$
$\mathbb{F}[z]$	$H^\infty_\pm$
$\mathbb{F}[z]^m=X_D\oplus D\mathbb{F}[z]^m$	$H^2_+=H(M_r)\oplus M_rH^2_+$
$\mathbb{F}[z]$**-module structure**	$H^\infty_\pm$ **functional calculus**
Submodules	**Shift/Translation** **invariant subspaces**
$D(z)\mathbb{F}[z]^m$	$S(z)H^2_+$
$D(z)$ **nonsingular**	$S(z)$ **inner**

Algebra	**Analysis**
Polynomial Models	**Shifts/Translations**
$H_G : \mathbb{F}[z]^m \longrightarrow z^{-1}\mathbb{F}[[z^{-1}]]^p$	$H_G : H^2_+ \longrightarrow H^2_-$
Matrix fractions	**DSS Factorizations**
Shift realization	**Translation realization**
$\mathbb{F}[z]$**-homomorphisms**	**Intertwining maps, CLT**
Polynomial Bezout equation	**Carleson Corona Thm**
$\mathbb{F}[z]$**-unimodular embedding**	H^∞**-unimodular embedding**
Inversion	**Inversion**

REACHABILITY MAP AND OPEN LOOP CONTROL

$(A, B) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{n \times m}$ a reachable pair. Our aim is to compute all open loop control sequences that steer the system from the origin to the state $\xi \in \mathbb{F}^n$. Define the <u>reachability map</u> $\mathcal{R}_{(A,B)} : \mathbb{F}[z]^m \longrightarrow \mathbb{F}^n$:

$$\boxed{\begin{array}{rcll} \mathcal{R}_{(A,B)} \sum_{i=0}^{s} u_i z^i & := & \sum_{i=0}^{s} A^i B u_i, & u(z) = \sum_{i=0}^{s} u_i z^i \in \mathbb{F} \\ \mathcal{R}_{(A,B)} u & := & \pi_{zI-A} B u, & u(z) \in \mathbb{F}[z]^m \end{array}}$$

$$\boxed{\begin{array}{rcl} \text{Ker}\, \mathcal{R}_{(A,B)} & = & D\mathbb{F}[z]^m \\ \mathcal{R} & := & \mathcal{R}_{(A,B)}|X_D \simeq \mathbb{F}[z]^m / \text{Ker}\, \mathcal{R}_{(A,B)} \\ u_*(z) & = & \mathcal{R}^{-1}\xi \\ u(z) & = & u_*(z) + D(z)g(z) \end{array}}$$

Set of steering controllers is an affine space.

DCF & INVERSION OF $\mathcal{R}$

$(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ **a reachable pair and let** $(zI - A)^{-1}B = N(z)D(z)^{-1}$ **be coprime factorizations, with** $D(z) \in \mathbb{F}[z]^{m\times m}, N(z) \in \mathbb{F}[z]^{n\times m}$**.**

$$\boxed{\begin{array}{l} \begin{pmatrix} \Theta(z) & \Xi(z) \\ -B & zI - A \end{pmatrix} \begin{pmatrix} D(z) & -\overline{\Xi}(z) \\ N(z) & \overline{\Theta}(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} D(z) & -\overline{\Xi}(z) \\ N(z) & \overline{\Theta}(z) \end{pmatrix} \begin{pmatrix} \Theta(z) & \Xi(z) \\ -B & zI - A \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \end{array}}$$

$$\boxed{\begin{array}{rcl} \overline{\Xi}(z)(zI - A) & = & D(z)\Xi(z) \\ u(z) = \mathcal{R}^{-1}\xi & = & \pi_D\overline{\Xi}\xi. \end{array}}$$

THE SHIFT REALIZATION

$$G(z) = V(z)T(z)^{-1}U(z) + W(z) = D + C(zI - A)^{-1}B$$

$$\boxed{\begin{aligned} A &= S_T \\ B\xi &= \pi_T U\xi, \\ Cf &= (VT^{-1}f)_{-1} \\ D &= G(\infty). \end{aligned}}$$

Realization is reachable $\Leftrightarrow T(z)$ and $U(z)$ left coprime.
Realization is observable $\Leftrightarrow T(z)$ and $V(z)$ right coprime.
Hautus and Kalman criteria.

INVERSION FOR HIGH ORDER SYSTEMS

$(T(z), U(z)) \in \mathbb{F}[z]^{q\times q} \times \mathbb{F}[z]^{q\times m}$ a reachable pair, i.e., left coprime, and let $T(z)^{-1}U(z) = \overline{U}(z)\overline{T}(z)^{-1}$ be coprime factorizations, with $\overline{T}(z) \in \mathbb{F}[z]^{m\times m}, \overline{U}(z) \in \mathbb{F}[z]^{q\times m}$.

$$\boxed{\begin{pmatrix} Y(z) & X(z) \\ -U(z) & T(z) \end{pmatrix} \begin{pmatrix} \overline{T}(z) & -\overline{X}(z) \\ \overline{U}(z) & \overline{Y}(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}}$$

$$\boxed{\begin{array}{rcl} \overline{X}(z)T(z) & = & \overline{T}(z)X(z) \\ u(z) = \mathcal{R}^{-1}\xi & = & \pi_{\overline{T}}\overline{X}\xi. \end{array}}$$

PROGRAM OUTLINE

- Impulse response/transfer functions as H^∞-homomorphisms. The reduced reachability and observability maps. Hankel operators.
- Strictly noncyclic (Close to rational) functions, Douglas-Shapiro-Shields factorizations. Hankels with ”large” kernels, ”small” image. Beurling-Lax representations in terms of inner functions.
- Model operators, intertwining maps and the commutant lifting theorem. Inversion. Intertwining maps and reduced Hankel operators.

OUTLINE (Contd.)

- Shift/translation realizations.
- Kernel and image of Hankel operators and their Beurling-Lax representations.
- Inner functions and their state space representations. Homogeneous Riccati (Lyapunov) equation.
- Normalized coprime factorizations and their unimodular embeddings. ¶
- Characteristic functions.
- Optimal control.

OUTLINE (Contd.)

- **Optimal control and model reduction of some networks of linear systems, from local to global, by interpolation.**
- **Weak controllability, rational (AAK) approximation and suboptimal control.**

INNER FUNCTIONS
STATE SPACE FORMULAS

$M_+ \in H_+^\infty$ **&** $M_+^* = M_+^{-1}$**.**
(A, B) **is reachable and** A **is stable. Homogeneous Riccati equation.**

$$\boxed{XA + A^*X = -XBB^*X}$$

$$\boxed{\begin{aligned} M &= \left(\begin{array}{c|c} A & B \\ \hline -B^*X & I \end{array}\right) \\ M^{-1} &= \left(\begin{array}{c|c} A + BB^*X & B \\ \hline B^*X & I \end{array}\right) \end{aligned}}$$

STABLE SYSTEMS

Any strictly proper, rational function $G(s)$ has a minimal realization of the form

$$\boxed{G(s) = C(sI - A)^{-1}B}$$

(A, B) **reachable,** (C, A) **observable.**

Assuming $G_+(s) \in H_+^\infty$, then A is stable.

Hankel operator $$\boxed{\begin{array}{l} G_+(s) \in H_+^\infty \\ H_{G_+} : H_-^2 \longrightarrow H_+^2 \\ H_{G_+} f = P_+ G_+ f \end{array}}$$

Abstract realization:

$\mathcal{X} = H_-^2 \ominus \mathrm{Ker}\, H_{G_+} = \{\mathrm{Ker}\, H_{G_+}\}^\perp$ **and defining the reachability map** $\mathcal{R}_I : H_-^2 \longrightarrow \{\mathrm{Ker}\, H_{G_+}\}^\perp$ **and observability map** $\mathcal{O}_I : \{\mathrm{Ker}\, H_{G_+}\}^\perp \longrightarrow H_+^2$ **by**

ABSTRACT REALIZATION

$$\mathcal{X} = H^2_- \ominus \text{Ker } H_{G_+} = \{\text{Ker } H_{G_+}\}^\perp$$

$$\mathcal{R}_I : H^2_- \longrightarrow \{\text{Ker } H_{G_+}\}^\perp$$

$$\mathcal{R}_I u = P_{\{\text{Ker } H_{G_+}\}^\perp} u, \qquad u(s) \in H^2_-$$

$$\mathcal{O}_I : \{\text{Ker } H_{G_+}\}^\perp \longrightarrow H^2_+$$

$$\mathcal{O}_I h = H_{G_+} h, \qquad h(s) \in \{\text{Ker } H_{G_+}\}^\perp$$

$$H_{G_+} = \mathcal{O}_I \mathcal{R}_I$$

NORMALIZED COPRIME (DSS) FACTORIZATIONS

$$\boxed{J_S = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}}$$

$$\boxed{\begin{array}{c} J_S - NRCF, \quad G \in H_-^\infty, \quad G = N_S M_S^{-1} \\ \begin{pmatrix} M_S^* & N_S^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M_S \\ N_S \end{pmatrix} = M_S^* M_S = I \end{array}}$$

$$\boxed{\begin{array}{c} J_S - NLCF, \quad G \in H_-^\infty, \quad G = \overline{M}_S^{-1} \overline{N}_S \\ \begin{pmatrix} \overline{M}_S & \overline{N}_S \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{M}_S^* \\ \overline{N}_S^* \end{pmatrix} = \overline{M}_S \overline{M}_S^* = I \end{array}}$$

DSS FACTORIZATION

$G_+(s) = C(sI - A)^{-1}B$, A **stable**

$$\begin{aligned} G_+(s) &= H(s)D(s)^{-1} = \overline{D}(s)^{-1}\overline{H}(s) \\ &= (H(s)E(s)^{-1})(E(s)D(s)^{-1}) \\ &= M_r(s)N_r^*(s) = N_l(s)^* M_l(s) \qquad (DSS) \end{aligned}$$

$$M_l(s) = \left(\begin{array}{c|c} A & B \\ \hline C_0 & I \end{array}\right) \& M_l(s)^{-1} = M_l(s)^* \Rightarrow$$

$$\left(\begin{array}{c|c} A - BC_0 & B \\ \hline -C_0 & I \end{array}\right) \simeq \left(\begin{array}{c|c} A^* & -C_0^* \\ \hline B^* & I \end{array}\right)$$

Ker $H_{G_+} = H_-(M_l^*) = \{M_l^* H_-^2\}^\perp \Rightarrow C_0 = -B^* X_+$

$$\boxed{XA + A^*X = -XBB^*X}$$

Im $H_{G_+} = H_+(M_r)$

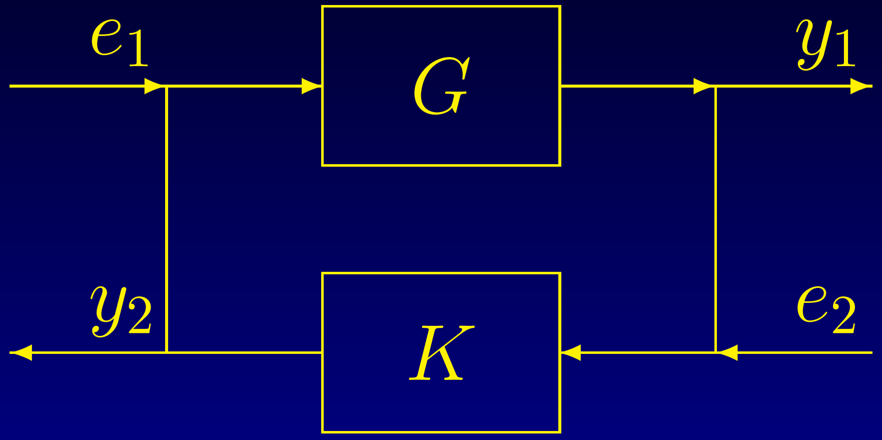
DOUBLY COPRIME FACTORIZATIONS

$$\begin{pmatrix} V_\ell(s) & U_\ell(s) \\ -N_\ell(s) & M_\ell(s) \end{pmatrix} \begin{pmatrix} M_r(s) & -U_r(s) \\ N_r(s) & V_r(s) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\left(\begin{array}{c|cc} A & X_+^{-1}C^* & B \\ \hline CZ_+X_+ & I & 0 \\ -B^*X_+ & 0 & I \end{array}\right), \quad \left(\begin{array}{c|cc} A & -Z_+C^* & Z_+X_+B \\ \hline C & I & 0 \\ -B^*Z_+^{-1} & 0 & I \end{array}\right)$$

$$\boxed{\begin{array}{c} A^*X + XA = -XBB^*X \\ AZ + ZA^* = -ZC^*CZ. \\ X_+, Z_+ > 0 \end{array}}$$

INTERNAL STABILIZATION



The feedback configuration (G, K) is called internally stable if

$$\boxed{\begin{pmatrix} I & G \\ K & I \end{pmatrix}^{-1} = \begin{pmatrix} (I-GK)^{-1} & -(I-GK)^{-1}G \\ -K(I-GK)^{-1} & (I-KG)^{-1} \end{pmatrix} \in H^{\infty}_{+}.}$$

INTERNAL STABILIZATION

$$\boxed{\begin{aligned} G(s) &= M_\ell^{-1} N_\ell = N_r M_r^{-1} \\ K(s) &= V_\ell^{-1} U_\ell = U_r V_r^{-1} \end{aligned}}$$

The following statements are equivalent:

$$\boxed{\begin{aligned} &(V_\ell(s) M_r(s) + U_\ell(s) N_r(s))^{-1} \in H_+^\infty \\ &(M_\ell(s) V_r(s) + N_\ell(s) U_r(s))^{-1} \in H_+^\infty \\ &V_\ell(s) M_r(s) + U_\ell(s) N_r(s) = I, \\ &M_\ell(s) V_r(s) + N_\ell(s) U_r(s) = I. \end{aligned}}$$

DCF and YOULA-KUCERA PARAMETRIZATION

$$\boxed{\begin{pmatrix} V_\ell(s) & U_\ell(s) \\ -N_\ell(s) & M_\ell(s) \end{pmatrix} \begin{pmatrix} M_r(s) & -U_r(s) \\ N_r(s) & V_r(s) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}}$$

$$\boxed{\begin{aligned} K &= UV^{-1} = \overline{V}^{-1}\overline{U} \\ \\ K &= (U_0 - M_r Q)(V_0 + N_r Q)^{-1} \\ &= (\overline{V}_0 - Q\overline{N}_l)^{-1}(\overline{U}_0 + Q\overline{M}_l) \end{aligned}}$$

THE S-CONTROLLER AND S-CHARACTERISTIC

$$\boxed{G_+(s) = N_\ell^*(s)M_\ell(s) = M_r(s)N_r^*(s)}$$

There exists a unique stabilizing controller

$$\boxed{K = V_\ell^{-1}U_\ell = U_rV_r^{-1}}$$

for which R_S, the S-characteristic of G_+, defined by,

$$\boxed{R_S(s) := M_\ell(s)U_\ell^*(s) = U_r^*(s)M_r(s)}$$

is in H_+^∞ and strictly proper.

$$\boxed{\begin{array}{rcl} \mathrm{Ker}\, H_{R_S} & = & M_r^*H_-^2 \\ \mathrm{Im}\, H_{R_S} & = & H_+(M_\ell) \end{array}}$$

HANKEL OPERATORS AND INTERTWINING MAPS

$$\begin{tikzcd}
H_-(M_\ell^*) \arrow[r, "H_{G_+}"] \arrow[d, "M_\ell"'] & H_+(M_r) \arrow[dl, shift left, "W^{-1}"] \\
H_+(M_\ell) \arrow[ur, shift left, "W"] & H_-(M_r^*) \arrow[l, "H_{R_S}"] \arrow[u, "M_r"']
\end{tikzcd}$$

where $W : H_+(M_\ell) \longrightarrow H_+(M_r)$ **and** $W^{-1} : H_+(M_r) \longrightarrow H_+(M_\ell)$ **are given by**

$$\boxed{\begin{aligned} Wg &= P_+N_\ell^* g, \qquad g(s) \in H_+(M_\ell) \\ W^{-1}f &= P_+U_r^* f, \qquad f(s) \in H_+(M_r). \end{aligned}}$$

S-CHARACTERISTIC STATE SPACE REALIZATION

$$\boxed{R_S = \xi_S(G_+) = \left(\begin{array}{c|c} A & Z_+C^* \\ \hline B^*X_+ & 0 \end{array}\right)}$$

$$\begin{aligned} \Xi^{-1}A^* + A\Xi^{-1} &= -Z_+C^*CZ_+ \\ \Theta^{-1}A + A^*\Theta^{-1} &= -X_+BB^*X_+. \end{aligned}$$

$$\Xi_+ = Z_+^{-1}, \qquad \Theta_+ = X_+^{-1}$$

$$\boxed{\xi_S(R_S) = \left(\begin{array}{c|c} A & \Theta_+X_+B \\ \hline CZ_+\Xi_+ & 0 \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right) = G_+(s).}$$

DCF
STATE SPACE REALIZATION

$$
\begin{pmatrix} M_r(s) & -U_r(s) \\ N_r(s) & V_r(s) \end{pmatrix}
$$

$$
= \left(\begin{array}{c|cc} -(A^* + X_+BB^*) & -X_+Z_+C^* & X_+Z_+X_+B \\ \hline CX_+^{-1} & I & 0 \\ -B^*Z_+^{-1}X_+^{-1} & 0 & I \end{array} \right)
$$

$$
= \left(\begin{array}{c|cc} -(A^* + C^*CZ_+) & -C^* & X_+B \\ \hline CZ_+ & I & 0 \\ -B^* & 0 & I \end{array} \right),
$$

OPTIMAL STEERING TO TARGET

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad A\ STABLE \\ 0 &= \lim_{t \to -\infty} x(t) \\ x(0) &= \xi \in \mathbb{C}^n. \end{aligned}$$

$$\mathcal{R}^{-}_{(A,B)} : L^2(-\infty, 0; \mathbb{C}^m) \longrightarrow \mathbb{C}^n$$

$$\mathcal{R}^{-}_{(A,B)} u = \int_{-\infty}^{0} e^{-A\tau} Bu(\tau) d\tau,$$

$$\mathcal{R}^{-,*}_{(A,B)} : \mathbb{C}^n \longrightarrow L^2(-\infty, 0; \mathbb{C}^m)$$

$$\mathcal{R}^{-,*}_{(A,B)} \xi = B^* e^{-A^* \tau} \xi, \qquad \tau \leq 0,$$

$$\text{Ker}\, \mathcal{R}^{-}_{(A,B)} = \{u(t) \in L^2(-\infty, 0; \mathbb{C}^m) | \int_{-\infty}^{0} e^{-A\tau} Bu(\tau) d\tau =$$

REACHABILITY GRAMIAN OPTIMAL CONTROLLER

$$\boxed{\begin{array}{l} W = \mathcal{R}^{-}_{(A,B)}\mathcal{R}^{-,*}_{(A,B)} = \int_{-\infty}^{0} e^{-A\tau}BB^{*}e^{-A^{*}\tau}d\tau, \\ AW + WA^{*} = -BB^{*}. \end{array}}$$

$$\boxed{\begin{array}{c} u_{*}(t) = B^{*}e^{-A^{*}t}W^{-1}\xi = B^{*}W^{-1}x_{*}(t) \\ \\ x_{*}(t) = We^{-A^{*}t}W^{-1}\xi. \end{array}}$$

OPTIMAL CONTROLLER IS A STATE FEEDBACK CONTROLLER

NO STABILITY ASSUMPTION

$$\boxed{G(s) = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right)}$$

minimal realization.

NORMALIZED (LQ) COPRIME FACTORIZATION

$$\boxed{J_L = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}}$$

$$\boxed{\begin{array}{c} J_L - NRCF, \quad G = N_L M_L^{-1} \\ \begin{pmatrix} M_L^* & N_L^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} M_L \\ N_L \end{pmatrix} = I \end{array}}$$

$$\boxed{\begin{array}{c} J_L - NLCF, \quad G = \overline{M}_L^{-1} \overline{N}_L \\ \begin{pmatrix} \overline{M}_L & \overline{N}_L \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \overline{M}_L^* \\ \overline{N}_L^* \end{pmatrix} = I \end{array}}$$

NORMALIZED COPRIME FACTORIZATION

$$G = N_L M_L^{-1} = \overline{M}_L^{-1} \overline{N}_L$$

$$\begin{pmatrix} \overline{V} & \overline{U} \\ -\overline{N}_L & \overline{M}_L \end{pmatrix} \begin{pmatrix} M_L & -U \\ N_L & V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} M_L \\ N_L \end{pmatrix} = \left(\begin{array}{c|c} A - BB^* X & B \\ \hline -B^* X & I \\ C & 0 \end{array}\right).$$

$$A^* X + XA + C^* C - XBB^* X = 0$$

$$\begin{pmatrix} -\overline{N}_L & \overline{M}_L \end{pmatrix} = \left(\begin{array}{c|cc} A - ZC^* C & B & ZC^* \\ \hline -C & 0 & I \end{array}\right)$$

$$AZ + ZA^* + BB^* - ZC^* CZ = 0$$

THE LQG CONTROLLER

$$R_L^* = M_L^* U_L + N_L^* V_L \in H_-^\infty$$

$$\begin{pmatrix} U_L \\ V_L \end{pmatrix} = \begin{pmatrix} M_L \\ N_L \end{pmatrix} R^* + \begin{pmatrix} -\overline{N}_L^* \\ \overline{M}_L^* \end{pmatrix}$$

$$R_L^* = \overline{U}_L \overline{M}_L^* + \overline{V}_L \overline{N}_L^* \in H_-^\infty$$

$$\begin{pmatrix} \overline{V}_L & -\overline{U}_L \end{pmatrix} = R^* \begin{pmatrix} -\overline{N}_L & \overline{M}_L \end{pmatrix} + \begin{pmatrix} M_L^* & N_L^* \end{pmatrix}$$

L-CONTROLLER
STATE SPACE EQs

$$\begin{pmatrix} U_L \\ V_L \end{pmatrix} = \left(\begin{array}{c|c} A - BB^*X & ZC^* \\ \hline -B^*X & 0 \\ C & I \end{array} \right)$$

$$\begin{pmatrix} \overline{V}_L & -\overline{U}_L \end{pmatrix} = \left(\begin{array}{c|cc} A - ZC^*C & B & ZC^* \\ \hline -B^*X & I & 0 \end{array} \right)$$

$$K = U_L V_L^{-1} = \left(\begin{array}{c|c} A - BB^*X - ZC^*C & ZC^* \\ \hline -B^*X & 0 \end{array} \right)$$

THE L-CHARACTERISTIC

$$\boxed{\begin{gathered} R_L^* = \Phi_K S_K^* = S_I^* \Phi_I \\ \operatorname{Ker} H_{R_L^*} = \{S_K H_+^2\}^\perp \\ \operatorname{Im} H_{R_L^*} = \{S_I^* H_-^2\}^\perp \end{gathered}}$$

TWO MATRIX COMPLETIONS

$$G = N_L M_L^{-1} = \overline{M_L}^{-1} \overline{N_L}$$

$$\begin{pmatrix} -J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} -\overline{N}_L^* \\ \overline{M}_L^* \end{pmatrix} S_I$$

$$\begin{pmatrix} K_1 & K_2 \end{pmatrix} = S_K \begin{pmatrix} M^* & N^* \end{pmatrix}$$

$$\begin{pmatrix} \overline{V}_L & \overline{U}_L \\ -\overline{N}_L & \overline{M}_L \end{pmatrix}, \begin{pmatrix} M_L & -U_L \\ N_L & V_L \end{pmatrix}$$ **UNIMODULAR**

$$\Omega_K = \begin{pmatrix} K_1 & K_2 \\ -\overline{N}_L & \overline{M}_L \end{pmatrix}, \Omega_I = \begin{pmatrix} M_L & -J_1 \\ N_L & J_2 \end{pmatrix}$$ **INNER**

THE KEY DIAGRAM

$$\begin{tikzcd}
\{S_K H^2_+\}^\perp \arrow[r, "H_{R^*}"] \arrow[d, "Z_K"'] \arrow[dr, "H_{\begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix}}" near end] & \{S_I^* H^2_-\}^\perp \arrow[d, "Y_I^*"] \\
\{\Omega_K H^2_+\}^\perp \arrow[r, "H_{\begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix}\begin{pmatrix} -\overline{N} & \overline{M} \end{pmatrix}}"'] \arrow[ur, "H_{(M^* \quad N^*)}" near start] & \{\Omega_I^* H^2_-\}^\perp
\end{tikzcd}$$

$$Z_K f = P_{\{\Omega_K H^2_+\}^\perp} \begin{pmatrix} U_L \\ V_L \end{pmatrix} f$$

$$Y_I \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = P_{\{S_I^* H^2_-\}^\perp} (M^* \quad N^*) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$Y_K \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = P_{\{S_K H^2_+\}^\perp} \begin{pmatrix} -\overline{N} & \overline{M} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

HANKEL TO INTERTWINING MAP

$$\begin{tikzcd}
\{S_K H^2_+\}^\perp \arrow[r, "H_{R^*}"] \arrow[d, "Z_K"'] \arrow[dr, "H_{\begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix}}" near end] & \{S_I^* H^2_-\}^\perp \arrow[r, "S_I"] \arrow[d, "Y_I^*"] & \{S_I H^2_+\}^\perp \arrow[d] \\
\{\Omega_K H^2_+\}^\perp \arrow[r, "H_{\begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix}\begin{pmatrix} -\overline{N} & \overline{M} \end{pmatrix}}"'] \arrow[ur, "H_{(M^* \;\; N^*)}" near start] & \{\Omega_I^* H^2_-\}^\perp \arrow[r, "\Omega_I"'] & \{\Omega_I H^2_+\}^\perp
\end{tikzcd}$$

$$\begin{tikzcd}
\{S_K H^2_+\}^\perp \arrow[r, "P_{\{S_I H^2_+\}^\perp} \Phi_I \times"] \arrow[d, "P_{\{\Omega_K H^2_+\}^\perp} \begin{pmatrix} U_L \\ V_L \end{pmatrix} \times"'] \arrow[dr, "P_{\{\Omega_I H^2_+\}^\perp} \begin{pmatrix} I \\ 0 \end{pmatrix} \times"] & \{S_I H^2_+\}^\perp \arrow[d, "-P_{\{\Omega_I H^2_+\}^\perp} \begin{pmatrix} 0 \\ I \end{pmatrix} \times"] \\
\{\Omega_K H^2_+\}^\perp \arrow[r, "P_{\{\Omega_I H^2_+\}^\perp} \begin{pmatrix} -\overline{N} & \overline{M} \\ 0 & 0 \end{pmatrix} \times"'] \arrow[ur, "P_{\{S_I H^2_+\}^\perp} \begin{pmatrix} K_1 & K_2 \end{pmatrix} \times"'] & \{\Omega_I H^2_+\}^\perp
\end{tikzcd}$$

OPTIMIZATION PROBLEMS I

$$\mu_1 = \min_{Q\in H_+^\infty} \|\Phi_I + S_I Q\|_\infty = \min_{Q\in H_+^\infty} \|R^* + Q\|_\infty = \min_{Q\in H_+^\infty} \|\Phi_K + Q S_K\|_\infty$$

$$\begin{tikzcd}
\{S_K H_+^2\}^\perp \arrow[r] \arrow[d] \arrow[dr, "\frac{\mu_1}{(1+\mu_1^2)^{\frac{1}{2}}}"] & \{S_I H_+^2\}^\perp \arrow[d] \\
\{\Omega_K H_+^2\}^\perp \arrow[r] \arrow[ur, "\frac{\mu_1}{(1+\mu_1^2)^{\frac{1}{2}}}"] & \{\Omega_I H_+^2\}^\perp
\end{tikzcd}$$

$$= \frac{1}{(1-\sigma_1^2)^{\frac{1}{2}}}$$

$$H_+^\infty \| \begin{pmatrix} U_L \\ V_L \end{pmatrix} + \begin{pmatrix} M \\ N \end{pmatrix} Q \|_\infty$$

$$\in H_+^\infty \| \begin{pmatrix} U_L \\ V_L \end{pmatrix} + \begin{pmatrix} M & J_1 \\ N & J_2 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \|_\infty$$

$$H_+^\infty \| \begin{pmatrix} R^* + Q \\ I \end{pmatrix}$$

$$\frac{1}{(1+\mu_1^2)^{\frac{1}{2}}} = \min_{Q_i\in H_+^\infty} \| \begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \|_\infty$$

$$= \min_{Q_i\in H_+^\infty} \| \begin{pmatrix} M \\ N \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} S_I \|_\infty$$

$$\frac{\mu_1}{1+\mu_1^2} = \sigma_1(1-\sigma_1^2)^{\frac{1}{2}} = \min_{Q_{ij}\in H_+^\infty} \| \begin{pmatrix} -\overline{N} & \overline{M} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\overline{N} & \overline{M} \\ K_1 & K_2 \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \|_\infty$$

$$= \min_{Q_{ij}\in H_+^\infty} \| \begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix} \begin{pmatrix} -\overline{N} & \overline{M} \end{pmatrix} + \begin{pmatrix} Q_{11} & Q_{12} \end{pmatrix} \|_\infty$$

OPTIMIZATION PROBLEMS II

$$\frac{1}{\mu_n} = \min_{Q \in H^\infty_+} \|U_S + S_K Q\|_\infty = \min_{Q \in H^\infty_+} \|R_S^* + Q\|_\infty = \min_{Q \in H^\infty_+} \|S_K^* U_S + Q\|_\infty$$

$$\begin{tikzcd}
\{S_K H^2_+\}^\perp & \{S_I H^2_+\}^\perp \arrow[l] \arrow[dl, "\frac{(1+\mu_n^2)^{\frac{1}{2}}}{\mu_n}"'] \\
\{\Omega_K H^2_+\}^\perp \arrow[u] & \{\Omega_I H^2_+\}^\perp \arrow[l] \arrow[u] \arrow[ul, "\frac{(1+\mu_n^2)^{\frac{1}{2}}}{\mu_n}"]
\end{tikzcd}$$

$$-\frac{1}{2} = (1-\sigma_n^2)^{\frac{1}{2}}$$

$$\in H^\infty_+ \| \begin{pmatrix} -\overline{N} & \overline{M} \end{pmatrix} + S_K \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \|_\infty$$

$$\in H^\infty_+ \| \begin{pmatrix} J_1^* & J_2^* \end{pmatrix} + \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \|_\infty$$

$$(1+\mu_n^2)^{\frac{1}{2}} =$$

$$\min_{Q_i \in H^\infty_+} \| \begin{pmatrix} \Phi_I & I \end{pmatrix} + S_I \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} R \|_\infty =$$

$$\min_{Q_i \in H^\infty_+} \| \begin{pmatrix} R_S^* & S_I^* \end{pmatrix} + \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \|_\infty$$

$$\frac{1+\mu_n^2}{\mu_n} = \frac{1}{\sigma_n(1-\sigma_n^2)}$$

$$= \min_{Q_{ij} \in H^\infty_+} \| \begin{pmatrix} U_L \\ V_L \end{pmatrix} \begin{pmatrix} I & U_S \end{pmatrix} + \begin{pmatrix} M & J_1 \\ N & J_2 \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \|_\infty$$