Conformal Geometry and Elliptic Operators

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Sde Boker, 2017
March 20-24

[0] (Joint works with Viktor Burenkov and Alexander Ukhlov)

This talk devoted to connections between conformal (hyperbolic) geometry of bounded simply connected planar domains and the Neumann eigenvalues of the p-Laplace operator $(1 < p \leq 2)$. The main results concerned to a large class of bounded non convex domains under some additional restrictions on its conformal (hyperbolic) geometry. This new class of planar domains includes quasidiscs (images of the unit disc under quasiconformal homeomorphism of the plane). Situation in space domains will be discussed also.

The two main problems will be discussed:

1. Lower estimates for the principal Neumann eigenvalues of the (p)-Laplace operator.

2. Spectral stability for Neumann-Laplace operators.

The classical Neumann-Laplace spectral problem is:

$$\begin{cases} -\Delta u = \mu u \text{ in } \Omega, \\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0. \end{cases}$$

The weak statement of the spectral problem is: *a function $u \in W^{1,2}(\Omega)$ solves the previous problem iff*

$$\int\limits_{\Omega} \nabla u(x) \cdot \nabla v(x) \; dvol = \mu \int\limits_{\Omega} u(x) v(x) \; dvol, \forall v \in W^{1,2}(\Omega).$$

Here $W^{1,2}$ is the space of integrable functions with the bounded energy integral $\int_{\Omega} |\nabla u(x)|^2 dvol$.

By the Max-Min principle the first nontrivial Neumann eigenvalue of the Laplace operator μ_1 can be characterized as

$$\mu_1(\Omega) = \min \left\{ \frac{\int\limits_{\Omega} |\nabla u(x)|^2 \ dvol}{\int\limits_{\Omega} |u(x)|^2 \ dx} : u \in W^{1,2}(\Omega) \setminus \{0\}, \int\limits_{\Omega} u \ dvol = 0 \right\}.$$

It means that the $\mu_1(\Omega)^{-\frac{1}{2}}$ is the best constant $B_{2,2}(\Omega)$ in the following Sobolev-Poincaré inequality

$$\inf_{c\in\mathbb{R}} \|f - c \mid L_2(\Omega)\| \leq B_{2,2}(\Omega)\|\nabla f \mid L_2(\Omega)\|, \quad f \in W^{1,2}(\Omega).$$

Recall that the Sobolev space $W^{1,p}(\Omega)$, $1 \leq p < \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^1_p(\Omega)\| = \left(\int_\Omega |f(x)|^p \, dvol \right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla f(x)|^p \, dvol \right)^{\frac{1}{p}}.$$

The previous assertions are correct for the first nontrivial Neumann eigenvalue of the (p)-Laplace operator in terms of $W^{1,p}$-spaces.

Short historic remarks for the Neuman-Laplace operator

1. Lord Rayleigh, The theory of sound, London, 1894/96 (formulation of the spectral problem)
2. H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen, Math. Ann. 71, 1912. (Asymptotic of eigenvalues).
3. G. Pólya, G. Szegö, Isoperimetic inequalities in mathematical physics, Princeton University Press, 1951. (Upper estimates)
4. L. E. Payne, H. F. Weinberger, An optimal Poincar inequality for convex domains, Arch. Rat. Mech. Anal., 5, 1960. (Lower estimates for convex domains in terms of its diameters)

The classical results by L. E. Payne and H. F. Weinberger (1960) give the lower estimates of the first non-trivial eigenvalue of the Neumann Laplacian in convex domains in terms of its diameters:

$$\mu_1[\Omega] \geq \frac{\pi^2}{d(\Omega)^2}.$$

Infinitesimally a conformal homeomorphism is a similarity. It means that

$$\nabla(f(\varphi(x,y)) = (\nabla f)(\varphi(x,y))|\varphi'_z(x,y)$$

for any conformal homeomorphism $\varphi : \Omega \to \Omega'$ and any smooth function $f : \Omega' \to R$. Hence $|\varphi'_z(x,y)|^2 = J((x,y);\varphi)$. This equality can be used as an alternative definition of conformal mappings. Here $z = x + iy$.

Basic Lemma. *Any conformal mapping $\varphi : \Omega \to \Omega'$ preserves the energy integral:*

$$\int_{\Omega} |\nabla (f \circ \varphi)|^2 dvol = \int_{\Omega} |\nabla (f \circ \varphi)|^2 \frac{|\varphi_z'(x,y)|^2}{|J((x,y);\varphi)|} |J((x,y);\varphi)| dvol$$

$$= \int_{\Omega'} |\nabla f|^2 dvol$$

If $\varphi : \Omega \to \Omega'$ is not conformal but is weakly differentiable, then

$$\int_\Omega |\nabla(f \circ \varphi)(x,y)|^2 dvol$$

$$= \int_\Omega |\nabla(f \circ \varphi)|(x,y)^2 \frac{|\varphi'_z(x,y)|^2}{|J(x,y);\varphi)} |J((x,y);\varphi)| dvol$$

$$\leq sup_{(x,y)\in\Omega} \frac{|D\varphi(x,y)|^2}{|J((x,y);\varphi)|} \int_{\Omega'} |\nabla(f|(u,v)^2 dvol.$$

The quantity

$$Q := sup_{(x,y)\in\Omega} \frac{|D\varphi(x,y)|^2}{|J((x,y);\varphi)|}$$

is called a coefficient of quasiconformality (a quasiconformal dilatation).
If Q is bounded a corresponding homeomorphism is called quasiconformal. It is one of the classical definitions (by B.V.Shabat). Quasiconformal homeomorphisms quasi-preserve the energy integral.

Let us give simple illustration to our method. Consider the ellipse $E \subset \mathbb{R}^2$: $(x, y) \in R^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1,\ a \geq b$. The linear mapping

$$\varphi(x, y) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

maps the unit disc $\mathbb{D}$ onto E with the qusiconformality coefficient $Q(\mathbb{D}) = \frac{b}{a}$. Using this change of variable in Sobolev-Poincaré inequality we obtain $\mu_1(E) \geq \frac{(j'_{1,1})^2}{a^2}$. This estimate is new and it is better then the classical estimate for convex domains $\mu_1(E) \geq \frac{\pi^2}{d(E)^2}$, because $d(E) = 2a$ and $2j'_{1,1} > \pi$, $j'_{1,1} \approx 1.84118$.

The suggested method is based on the following diagram proposed in (V.G. and L. Gurov, 1994, V.G. and A. Ukhlov, 2009).

$$\begin{array}{ccc} W^{1,p}(\Omega) & \xrightarrow{(\varphi^{-1})^*} & W^{1,q}(\mathbb{D}) \\ \downarrow & & \downarrow \\ L^s(\Omega) & \xleftarrow{\varphi^*} & L^r(\mathbb{D}) \end{array}$$

Here the operator φ^* is a bounded composition operator on Lebesgue spaces induced by a homeomorphism φ of Ω and $\mathbb{D}$ and the operator $(\varphi^{-1})^*$ is a bounded composition operator on Sobolev spaces.

We suggest the estimates in terms of the hyperbolic radii for a large class of bounded non-convex domains with some additional restrictions on the hyperbolic geometry that we call a conformal regularity:

A simply connected planar domain Ω with non-empty boundary is called a conformal α-regular domain if there exists a Riemann mapping $\varphi : \Omega \to \mathbb{D}$:

$$\int_{\mathbb{D}} |(\varphi^{-1})'(w)|^{\alpha} \, dvol < \infty \quad \text{for some} \quad \alpha > 2.$$

In the case of conformal α-regular domains $\Omega \subset \mathbb{C}$ we have embedding

$$L^r(\Omega, h) \hookrightarrow L^s(\Omega), \quad s = \frac{\alpha - 2}{\alpha} r :$$

Theorem A. *Let $\Omega \subset \mathbb{R}^2$ be a conformal α-regular domain. Then the spectrum of the Neumann-Laplace operator in Ω is discrete, can be written in the form of a non-decreasing sequence $0 = \mu_0[\Omega] < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots \leq \mu_n(\Omega) \leq \ldots$, and*

$$1/\mu_1(\Omega) \leq \frac{4}{\sqrt[\alpha]{\pi^2}} \left(\frac{2\alpha-2}{\alpha-2}\right)^{\frac{2\alpha-2}{\alpha}} \|\psi' \mid L^\alpha(\mathbb{D})\|^2$$

$$\leq \frac{4}{\sqrt[\alpha]{\pi^2}} \left(\frac{2\alpha-2}{\alpha-2}\right)^{\frac{2\alpha-2}{\alpha}} \left(\int\limits_{\mathbb{D}} \frac{R^2_\Omega(\psi(x))}{(1-|x|^2)^2} dvol\right)^{\frac{2}{\alpha}} \quad (1)$$

where $\psi : \mathbb{D} \to \Omega$ is the Riemann conformal mapping of the unit disc $\mathbb{D} \subset \mathbb{R}^2$ onto Ω and $R_\Omega(\psi(x))$ is a hyperbolic radius of Ω.

A K-quasidisc is the image of the unit disc under a K-quasiconformal mapping of the plane onto itself.

Theorem for Quasidiscs. *Suppose a conformal homeomorphism $\varphi : \mathbb{D} \to \Omega$ maps the unit disc $\mathbb{D}$ onto a K-quasidisc Ω. Then*

$$1/\mu_1[\Omega] \leq B^2_{2\alpha/(\alpha-2),2}[\mathbb{D}] \left(\int\limits_{\mathbb{D}} |\varphi'(x,y)|^\alpha \, dvol \right)^{\frac{2}{\alpha}}$$

$$\leq 4\pi^{-\frac{2}{\alpha}} \left(\frac{2\alpha-2}{\alpha-2} \right)^{\frac{2\alpha-2}{\alpha}} \|\varphi' \mid L^\alpha(\mathbb{D})\|^2$$

for any $2 < \alpha < \frac{2K^2}{K^2-1}$.

Consider the interior of the cardioid (it is not a quasidsic).

Example. *Let Ω_c be the interior of the cardioid. The diffeomorphism*

$$z = \psi(w) = (w+1)^2, z = x + iy,$$

is conformal and maps the unit disc $\mathbb{D}$ onto Ω_c. Then by Theorem A

$$\|\psi' \mid L^{\infty}(\mathbb{D})\| = \max_{w \in \mathbb{D}} 2|w+1| \leq 4.$$

Hence

$$\mu_1(\Omega_c) \geq \frac{(j'_{1,1})^2}{16}.$$

Here $j_{1,1}$ is the first positive zero of the derivative of the Bessel function J'_1.

The quantity

$$Q_p = sup_{(x)\in\Omega} \frac{|D\varphi(x)|^p}{|J(x);\varphi)|}$$

is called a p-dilatation and corresponding homeomorphisms are called p-quasiconformal (or homeomorphisms with bounded p-dilatation). These classes is comparatively new and were introduced in 1994 by G, L.Gurov and A.Romanov in the framework of the composition operator theory for Sobolev spaces W^1_p.

The variational formulation of spectral problems for the p-Laplace operator is based on the Dirichlet (energy) integrals

$$\|u \mid L^1_p(\Omega)\|^p := \int\limits_{\Omega} |\nabla u(x)|^p \, dvol.$$

The p-quasiconformal homeomorphisms induce bounded composition operators for such energy integrals and we use them and their generalizations for estimates of spectrum in rough domains.

Brennan's conjecture

is that for a conformal mapping $\varphi : \Omega \to \mathbb{D}$

$$\int_{\Omega} |\varphi'(x,y)|^{\delta} \, dvol < +\infty, \quad \text{for all} \quad \frac{4}{3} < \delta < 4. \qquad (2)$$

For the inverse conformal mapping $\psi = \varphi^{-1} : \mathbb{D} \to \Omega$ Brennan's conjecture states

$$\iint_{\mathbb{D}} |\psi'(u,v)|^{\gamma} \, dvol < +\infty, \quad \text{for all} \quad -2 < \gamma < \frac{2}{3}. \qquad (3)$$

For bounded domains $-2 < \gamma \leq 2$. The upper bound $\gamma = 2$ is accurate (V.G. and A. Ukhlov, 2009).

Theorem B. *Let $\varphi : \mathbb{D} \to \Omega$ be a conformal homeomorphism from the unit disc $\mathbb{D}$ to a conformal α-regular domain Ω and Brennan's Conjecture holds. Then for every $p \in (\max\{4/3, (\alpha+2)/\alpha\}, 2)$ the following inequality is correct*

$$\frac{1}{\mu_p(\Omega)} \leq 2^{\frac{3}{2}} (2\pi)^{\frac{(\alpha-2)q-\alpha p}{\alpha q}} \cdot \|(\varphi)'|L^{\alpha}(\mathbb{D})\|^2 \left(\int\limits_{\mathbb{D}} |\left(\varphi^{-1}\right)'|^{\frac{(p-2)q}{p-q}} \; dvol \right)^{\frac{p-q}{q}}$$

for any $q \in [1, 2p/(4-p))$.

Here $B_{r,q}(\mathbb{D}) \leq 2^{\frac{3}{2}}(2\pi)^{\frac{1}{r}-\frac{1}{q}}$ is the best constant in the Sobolev-Poincaré inequality in the unit disc $\mathbb{D} \subset \mathbb{C}$ and $K_{p,q}(\Omega)$ is the norm of composition operator

$$\left(\varphi^{-1}\right)^{*} : L^{1,p}(\Omega) \to L^{1,q}(\mathbb{D})$$

generated by the inverse conformal mapping $\varphi^{-1} : \mathbb{D} \to \Omega$:

$$K_{p,q}(\Omega) \leq \left(\int\limits_{\mathbb{D}} |\left(\varphi^{-1}\right)'|^{\frac{(p-2)q}{p-q}}\ dvol \right)^{\frac{p-q}{pq}}.$$

Definition

Conformal regular domains Ω_1, Ω_2 are conformal regular equivalent domains if there exists a conformal mapping $\psi : \Omega_1 \to \Omega_2$ such that

$$\iint\limits_{\Omega_1} |(\psi'(z)|^\alpha \; dxdy < \infty \;\&\; \iint\limits_{\Omega_2} |(\psi^{-1})'(w)|^\alpha \; dudv < \infty \qquad (4)$$

for some $\alpha > 2$.

In the conformal regular planar domains $\Omega \subset \mathbb{C}$ the spectrum of the Neumann Laplacian is discrete and can be written in the form of a non-decreasing sequence

$$0 < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots \leq \mu_n(\Omega) \leq \ldots,$$

where each eigenvalue is repeated as many times as its multiplicity.

Stability Theorem Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be conformal regular equivalent domains. Then for any $n \in \mathbb{N}$

$$|\mu_n(\Omega_1) - \mu_n(\Omega_2)| \leq 2c_n \left[C \left(\frac{4\alpha}{\alpha - 2} \right) \right]^2 E_\alpha(\varphi_1, \varphi_2) \| \varphi_1 - \varphi_2 \mid L^{1,2}(\mathbb{D}) \| , \tag{5}$$

where $\Omega_1 = \varphi_1(\mathbb{D})$, $\Omega_2 = \varphi_2(\mathbb{D})$ and

$$c_n = \max\{\mu_n^2(\Omega_1), \mu_n^2(\Omega_2)\} . \tag{6}$$

Here

$$E_{\alpha}(\varphi_1,\varphi_2)=\left(\int\limits_{\mathbb{D}}\max\left\{\frac{|\varphi_1'(z)|^{\alpha}}{|\varphi_2'(z)|^{\alpha-2}},\frac{|\varphi_2'(z)|^{\alpha}}{|\varphi_1'(z)|^{\alpha-2}}\right\}\,dvol\right)^{\frac{1}{\alpha}}<\infty$$

Theorem C. Suppose that there exists a 2-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$, of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ onto $\widetilde{\Omega}$, such that

$$M_2(\Omega) = \operatorname*{ess\,sup}_{x \in \Omega} |J(x, \varphi)|^{\frac{1}{2}} < \infty.$$

Then the spectrum of Neumann-Laplace operator in $\widetilde{\Omega}$ is discrete, can be written in the form of a non-decreasing sequence

$$0 = \mu_0(\widetilde{\Omega}) < \mu_1(\widetilde{\Omega}) \leq \mu_2(\widetilde{\Omega}) \leq \ldots \leq \mu_n(\widetilde{\Omega}) \leq \ldots,$$

and

$$\frac{1}{\mu_1(\widetilde{\Omega})} \leq K_2^2(\Omega) M_2^2(\Omega) \frac{1}{\mu_1(\Omega)}.$$

Denote by H_1 the standard n-dimensional simplex, $n \geq 3$,

$H_1 := \{x \in \mathbb{R}^n : n \geq 3, 0 < x_n < 1, 0 < x_i < x_n,\ i = 1, 2, \dots, n-1\}$.

Theorem D. *Let*

$H_g := \{x \in \mathbb{R}^n : n \geq 3, 0 < x_n < 1, 0 < x_i < x_n^{\gamma_i}, i = 1, 2, \dots, n-1\}$

$\gamma_i \geq 1$, $\gamma := 1 + \sum_{i=1}^{n-1} \gamma_i$, $g := (\gamma_1, ..., \gamma_{n-1})$.
Then the spectrum of the Neumann-Laplace operator in the domain H_g is discrete, can be written in the form of a non-decreasing sequence

$$0 = \mu_0(H_g) < \mu_1(H_g) \leq \mu_2(H_g) \leq ... \leq \mu_n(H_g) \leq ... ,$$

and for any $r > 2$ the following inequality holds:

$$\frac{1}{\mu_1(H_g)} \leq$$

$$\inf_a \left(a^2(\gamma_1^2 + ... + \gamma_{n-1}^2 + 1) - 2a \sum_{i=1}^{n-1} \gamma_i \right)$$

$$a \left(\int\limits_{H_1} \left(x_n^{a\gamma - n} \right)^{\frac{r}{r-2}} \, dvol \right)^{\frac{r-2}{r}} B_{r,2}^2(H_1),$$

where $(2n)/(\gamma r) < a \leq (n-2)/(\gamma - 2)$ and $B_{r,2}(H_1)$ is the best constant in the $(r,2)$-Sobolev-Poincaré inequality in the domain H_1.

Recall the analytic description of homeomorphisms which generate bounded composition operators (A. Ukhlov, 1993):

Composition Theorem *A homeomorphism $\varphi : \Omega \to \Omega'$ between two domains $\Omega, \Omega' \subset \mathbb{R}^n$, $n \geq 2$, induces a bounded composition operator*

$$\varphi^* : L^1_p(\Omega') \to L^1_q(\Omega),\ \ 1 \leq q < p < \infty,$$

if and only if $\varphi \in W^1_{1,loc}(\Omega)$, has finite distortion, and

$$K_{p,q}(f;\Omega) = \left(\int\limits_{\Omega} \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|} \right)^{\frac{q}{p-q}} \ dx \right)^{\frac{p-q}{pq}} < \infty.$$

If $p = q$ then the sufficient and necessary analytic condition is:

$$K_p(f;\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} \frac{|D\varphi(x)|^p}{|J(x,\varphi)|} < \infty.$$

(Vodop'yanov and G., 1975, G., Gurov, Romanov 1995)
In the case $p = n$ we have the definition of mappings of bounded distortion and we call mappings that generate bounded composition operators as mappings of bounded (p,q)-distortion.

The homeomorphisms that generate bounded composition operators of Sobolev spaces $L^1_1(\Omega')$ and $L^1_1(\Omega)$ were introduced by V. G. Maz'ya (1969) as a class of sub-areal mappings. This pioneering work established a connection between geometrical properties of homeomorphisms and corresponding Sobolev spaces.

- V. I. Burenkov, V. Gol'dshtein, A. Ukhlov, Conformal spectral stability for the Dirichlet-Laplace operator, Math. Nachr., 288 (2015), 1822–1833.
- V. I. Burenkov, V. Gol'dshtein, A. Ukhlov, Conformal spectral stability estimates for the Neumann Laplacian. Volume 289, Issue 17-18, December 2016 Pages 21332146.
- V. Gol'dshtein, A. Ukhlov, On the First Eigenvalues of Free Vibrating Membranes in Conformal Regular Domains, Rational Mech Anal (2016) 221: 893, DOI :10.1007/s00205-016-0988-9.
- V. Gol'dshtein, A. Ukhlov, Spectral estimates of the p-Laplace Neumann operator in conformal regular domains. Transactions of A. Razmadze Mathematical Institute Volume 170, Issue 1,

THANKS