Recursive Interpolation and Partial Realization: a polynomial approach

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in honor of
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Problem:

Homogeneus polynomial intepolation (simple nodes).

Given distinct points

$$\lambda_0, ...\lambda_N \in \mathbb{C}$$

and row vectors

$$w_0, ..., w_N \in \mathbb{C}^m$$

find recursively, for each $0 \leq n \leq N$, a polynomial matrix $\Gamma_n(s)$ such that

$$w_i \Gamma_n(\lambda_i) = 0 \qquad \text{for } 0 \leq i \leq n$$

Why is this problem interesting?

- Lagrange and Newton interpolation
- Rational Interpolation
- Partial Realization
- Exact Identification

At the bottom there is always

$$w_i \Gamma_n(\lambda_i) = 0 \qquad \text{for } 0 \le i \le n$$

- **Newton, Lagrange, Hermite, Schur, Szegö, Nevanlinna, Pick, Löwner...**
- **Antoulas, Anderson, Ball, Willems... (rational interpolation)**
- **Dym, Ball, Gohberg, Rodman, Nudelman, Alpay... (rational and Schur interpolation)**
- **Kalman, Rissanen, Fuhrmann, Antoulas, Gragg, Lindquist, Kailath... (partial realization)**
- **Byrnes, Georgiou, Lindquist, Kimura, Nagamune... (degree constrained interpolation)**
- **De Moor, Willems, Markowski (identification)**

All approaches (with some non trivial qualifiers) can be recast as

$$w_i \Gamma_n(\lambda_i) = 0 \qquad \text{for } 0 \leq i \leq n$$

The solution looks very simple!

...in this approach (as we shall see in two slides).

In fact, too simple!

So, why is the problem NOT interesting?

(at least with this approach...)

$$w_i \Gamma_n(\lambda_i) = 0 \qquad \text{for } 0 \le i \le n$$

- Interpolation nodes must be simple
- Solutions are highly nonunique.
- There is no control on the degree of the interpolants.
- Lack of coprimeness (see Applications below).

Simple solution

Want to solve

$$w_i \Gamma_n(\lambda_i) = 0 \qquad \text{for } 0 \le i \le n$$

with w_i m-dimensional row vectors, $\Gamma_i(s)$ an $m \times m$ polynomial matrix, and λ_i distinct.

First step: find $\Gamma_0(s)$ such that

$$w_0 \Gamma_0(\lambda_0) = 0$$

Denote

$$w_0 = [w_{0,1}, ..., w_{0,m}]$$

and choose j such that $w_{0,j} \neq 0$.

Set

$$\Gamma_0(s) := \begin{bmatrix} w_{0,j} & 0 & \dots & 0 & & 0 \\ 0 & w_{0,j} & 0 & \vdots & & \vdots \\ \vdots & \ddots & \ddots & & & \vdots \\ -w_{0,1} & -w_{0,j2} & \dots & (s-\lambda_0) & \dots & -w_{0,m} \\ \vdots & \dots & 0 & & \ddots & 0 \\ 0 & \dots & 0 & 0 & \dots & w_{0,j} \end{bmatrix}$$

Clearly $w_0\Gamma_0(\lambda_0) = 0$. **Notice that** $\det \Gamma_0(s) = K(s-\lambda_0)$.

Suppose now $\Gamma_{n-1}(s)$ satisfies

$$w_i \Gamma_{n-1}(\lambda_i) = 0 \qquad i = 0, \ldots, n-1$$

and we want to add w_n, λ_n. Define

$$\boxed{\epsilon_n = [\epsilon_{n,1}, \ldots, \epsilon_{n,m}] := w_n \Gamma_{n-1}(\lambda_n)}$$

Select any entry j such that $\epsilon_j \neq 0$ and define $\tilde{\Gamma}_n(s)$ as

$$\tilde{\Gamma}_n(s) := \begin{bmatrix} \epsilon_{n,j} & 0 & \dots & & & 0 \\ 0 & \epsilon_{n,j} & 0 & & & \vdots \\ \vdots & \ddots & \ddots & & & \vdots \\ -\epsilon_{n,1} & -\epsilon_{n,2} & \dots & (s-\lambda_n) & \dots & -\epsilon_{n,m} \\ \vdots & \dots & 0 & & \ddots & 0 \\ 0 & \dots & 0 & & & \epsilon_{n,j} \end{bmatrix}$$

Set $\Gamma_n(s) := \Gamma_{n-1}(s)\tilde{\Gamma}_n(s)$. Now it is clear that

$$w_n \Gamma_n(\lambda_n) = w_n \Gamma_{n-1}(\lambda_n) \tilde{\Gamma}_n(\lambda_n) = \epsilon_n \tilde{\Gamma}_n(\lambda_n) = 0$$

In conclusion, in this simple case, the answer is quite easy: at each step,

- **Compute**

$$\epsilon_n = w_n \Gamma_{n-1}(\lambda_n)$$

- **Compute, for j such that $\epsilon_{n,j} \neq 0$,**

$$\tilde{\Gamma}_n(s) := I\epsilon_{n,j} - e_j^T[\epsilon_n + e_j(s - \lambda_n)]$$

(e_j is the j-th row unit vector).

Notice that $\det \Gamma_n(s) = K \prod_{i=0}^{n}(s - \lambda_i)$.

It is not new: see *Boros, Sayed, and Kailath, IEEE TAC 1999*.
There are, of course, many non recursive solution (se e.g. *BGR, Chapter II*).

Shortcomings...

- **If we pick a different column or change the order of $\{\lambda_i, w_i\}, i = 0, \ldots, N$ we obtain different polynomial matrices $\Gamma_n(s)$.**

- **We cannot impose conditions on derivatives of $\Gamma_n(s)$ (or, yes, we can, but it is quite complicated, see BSK).**

- **Other minor problems...**

Non uniqueness

This can be addressed in different ways: providing a selection mechanism for the columns and an order for the interpolation nodes...

Not very satisfactory!

Want something canonical...

Inner functions are much better!

Well, if not inner, J-inner, or J-unitary: they all provide a unique solution (up to a constant term). They need more structure, though...

Still, there is the problem of the conditions on the derivatives. The canonical way to handle this problem is to imbed the data into a Hilbert space and orthogonalize them.

Again, this can be done with inner functions, using Beurling's theorem (or J-inner functions, using RKHS).

Write data as $\mathcal{A}_n = \mathrm{diag}\{\lambda_0, ..., \lambda_n\}$ **and** $W_n := \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{bmatrix}$.

Then the idea is to orthogonalize the set

$$\left\{ \frac{1}{s-\lambda_0} w_0, \frac{1}{s-\lambda_1} w_1, ..., \frac{1}{s-\lambda_n} w_n \right\}$$

in a suitable inner product. This set is spanned by the rows of

$$(sI - \mathcal{A}_n)^{-1} W_n$$

and orthonormalization is achieved by diagonalizing the solution of a Lyapunov (or Stein) equation, and Blaschke factors are computed... All can be done recursively and even if $\mathcal{A}$ **is not diagonalizable.**

In our simple example, though, all we need is

$$\epsilon_n = w_n \Gamma_{n-1}(\lambda_n)$$

which makes sense only if the algebraic multiplicity of λ_n is one.

Is there a general way to extract the "errors" ϵ_n directly form $(\mathcal{A}_n, W_n)$? An to use this procedure even if $\mathcal{A}_n$ is not diagonalizable?

The answer is obviously YES!

Recall that, if $\mathcal{A} = \mathrm{diag}\{\lambda_0, \lambda_1, ..., \lambda_n\}$ and $W = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{bmatrix}$,

then, for any polynomial $\gamma(s)$, the matrix $\gamma(\mathcal{A})$ is also diagonal and thus

$$\begin{bmatrix} \gamma(\lambda_0)w_0 \\ \gamma(\lambda_1)w_1 \\ \vdots \\ \gamma(\lambda_n)w_n \end{bmatrix} = \gamma(\mathcal{A})W$$

Write $\Gamma(s)$ as

$$\Gamma(s) = \begin{bmatrix} \gamma_{1,1}(s) & \gamma_{1,2}(s) & \cdots & \gamma_{1,m}(s) \\ \gamma_{2,1}(s) & \ddots & & \vdots \\ \vdots & & & \\ \gamma_{m,1}(s) & \cdots & & \gamma_{m,m}(s) \end{bmatrix}$$

Then, for each i,

$$w_i \Gamma(\lambda_i) = \left[\sum_{j=1}^{m} w_{i,j} \gamma_{j,1}(\lambda_i), \ldots, \sum_{j=1}^{m} w_{i,j} \gamma_{j,m}(\lambda_i) \right]$$

$$\begin{bmatrix} w_0\Gamma(\boldsymbol{\lambda}_0) \\ w_1\Gamma(\boldsymbol{\lambda}_1) \\ \vdots \\ w_n\Gamma(\boldsymbol{\lambda}_n) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^m \gamma_{j,1}(\boldsymbol{\lambda}_{\mathbf{0}})w_{\mathbf{0},j} & \cdots & \sum_{j=1}^m \gamma_{j,m}(\boldsymbol{\lambda}_{\mathbf{0}})w_{\mathbf{0},j} \\ \sum_{j=1}^m \gamma_{j,1}(\boldsymbol{\lambda}_{\mathbf{1}})w_{\mathbf{1},j} & \cdots & \sum_{j=1}^m \gamma_{j,m}(\boldsymbol{\lambda}_{\mathbf{1}})w_{\mathbf{1},j} \\ & & \\ \sum_{j=1}^m \gamma_{j,1}(\boldsymbol{\lambda}_{\boldsymbol{n}})w_{\boldsymbol{n},j} & \cdots & \sum_{j=1}^m \gamma_{j,m}(\boldsymbol{\lambda}_{\boldsymbol{n}})w_{\boldsymbol{n},j} \end{bmatrix}$$

$$= \left[\sum_{j=1}^m \gamma_{j,\mathbf{1}}(\boldsymbol{\mathcal{A}})\boldsymbol{W}_j, \ldots, \sum_{j=1}^m \gamma_{j,\boldsymbol{m}}(\boldsymbol{\mathcal{A}})\boldsymbol{W}_j\right]$$

Thus we need to formally define:

$$\mathcal{A} := \text{diag}\{\lambda_0, ..., \lambda_n\} \qquad W =: [W_1, ...W_m]$$

Then the conditions

$$w_i\Gamma(\lambda_i) = 0 \qquad \text{for } i = 0, ..., n$$

on $\Gamma(s)$ are equivalent to

$$= \left[\sum_{j=1}^{m} \gamma_{j,1}(\mathcal{A})W_j, ..., \sum_{j=1}^{m} \gamma_{j,m}(\mathcal{A})W_j\right] = 0$$

Also, if $\Gamma(s)$ satisfies the conditions

$$w_i \Gamma(\lambda_i) = 0 \qquad \text{for } i = 0, \ldots, n-1$$

and we define

$$w_n \Gamma(\lambda_n) = \epsilon_n$$

we can write

$$= \left[\sum_{j=1}^{m} \gamma_{j,1}(\mathcal{A}) W_j, \ldots, \sum_{j=1}^{m} \gamma_{j,m}(\mathcal{A}) W_j \right] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

So far, nothing new. We already knew all that from the start!

The news is that $\mathcal{A}$ does not need to be diagonal!

Theorem 1 *Let $(\mathcal{A}, W)$ be a controllable pair with $W = [W_1, ..., W_m]$ and let $\gamma(s) = [\gamma_1(s), ..., \gamma_m(s)]^T$ be a polynomial vector. The following are equivalent:*

1.

$$(sI - \mathcal{A})^{-1} W \gamma(s) = h(s)$$

is polynomial.

2.

$$\sum_{j=1}^{m} \sum_{i=0}^{p_j} \gamma_{i,j} \mathcal{A}^i W_j = \sum_{j=1}^{m} \gamma_j(\mathcal{A}) W_j = 0$$

where $\gamma_j(s) = \sum_{i=0}^{p_j} \gamma_{i,j} s^i$

Recursive step

Recall that, given ϵ_n, λ_n, we have defined $\tilde{\Gamma}_n(s)$ as:

$$\tilde{\Gamma}_n(s) := \begin{bmatrix} \epsilon_{n,j} & 0 & \dots & & & 0 \\ 0 & \epsilon_{n,j} & 0 & & & \vdots \\ \vdots & \ddots & \ddots & & & \vdots \\ -\epsilon_{n,1} & -\epsilon_{n,2} & \dots & (s-\lambda_n) & \dots & -\epsilon_{n,m} \\ \vdots & \dots & 0 & & \ddots & 0 \\ 0 & \dots & 0 & & & \epsilon_{n,j} \end{bmatrix}$$

Nesting

Definition 2 *We say that the set of interpolation conditions* $(\mathcal{A}_{n-1} W_{n-1})$ *is* ***nested*** *in* $(\mathcal{A}_n, W_n)$ *if*

$$\mathcal{A}_n = \begin{bmatrix} \mathcal{A}_{n-1} & 0 \\ a_n & \lambda_n \end{bmatrix} \qquad W_n = \begin{bmatrix} W_{n-1} \\ w_n \end{bmatrix}$$

and both pairs are controllable.

Theorem 3 *Suppose* $\Gamma_{n-1}(s)$ *interpolates the data* $(\mathcal{A}_{n-1}, W_{n-1})$*, that is*

$$(sI - \mathcal{A}_{n-1})^{-1} W_{n-1} \Gamma_{n-1}(s) \qquad \text{is polynomial}$$

Suppose $(\mathcal{A}_{n-1} W_{n-1})$ *is nested in* $(\mathcal{A}_n, W_n)$. *Then, denoting* $W_n = [W_{n,1}, ..., W_{n,m}]$,

$$= \left[\sum_{j=1}^{m} \gamma_{j,1}^{n-1}(\mathcal{A}_n) W_{n,j}, ..., \sum_{j=1}^{m} \gamma_{j,m}^{n-1}(\mathcal{A}_n) W_{n,j} \right] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \epsilon_n \end{bmatrix}$$

for some $\epsilon_n \neq 0$. *Set* $\Gamma_n(s) := \Gamma_{n-1}(s)\tilde{\Gamma}_n(s)$, *where* $\tilde{\Gamma}_n(s)$ *is defined as above. Then*

$$(sI - \mathcal{A}_n)^{-1} W_n \Gamma_n(s) \qquad \text{is polynomial}$$

Uniqueness

We have seen that $\Gamma(s)$ interpolates the data $(\mathcal{A}, W)$ if the coefficients of the polynomial in each of its columns provide a linear combination of the columns

$$W_1, \mathcal{A}W_1, \ldots, \mathcal{A}^{p_1}, W_2, \ldots, \mathcal{A}^{p_m}W_m$$

which vanishes.

Here $p_1, \ldots, p_m$ are integers: is there a "best choice" of them? Is it unique?

The answer is:
Controllabily Indexes

For $i = 1, ..., m$, we set κ_i to be the smallest integer such that

$$\mathcal{A}^{\kappa_i} W_i \in \text{span}$$

$$\{W_1, ..., W_m, \mathcal{A}W_1, ..., \mathcal{A}^{\kappa_i} W_{i-1}, \mathcal{A}^{\kappa_i - 1} W_i, ..., \mathcal{A}^{\kappa_i - 1} W_m\}$$

Not hard to see that

$$M^{\kappa_1-1,\ldots,\kappa_m-1} =$$

$$\{W_1, \ldots, \mathcal{A}^{\kappa_1-1}W_1, W_2, \ldots, \mathcal{A}^{\kappa_2-1}W_2, \ldots, W_m, \ldots, \mathcal{A}^{\kappa_m-1}W_m\}$$

is a basis for $\mathbb{C}^{n+1}$ **(**$\dim \mathcal{A} = n+1$**).**

Thus there exists a unique representation of $\mathcal{A}^{\kappa_1} W_1$ in terms of the basis $M^{\kappa_1-1,\ldots,\kappa_m-1}$. Equivalently, there is a unique $\underline{\gamma}_1$ s.t.

$$[W_1, \ldots, \mathcal{A}^{\kappa_1} W_1, W_2, \ldots, \mathcal{A}^{\kappa_2-1} W_2, \ldots, \mathcal{A}^{\kappa_m-1} W_m]\underline{\gamma}_1 = 0$$

and the coefficient of $\mathcal{A}^{\kappa_1} W_1$ is 1. For short:

$$M^{\kappa_1,\kappa_2-1\ldots,\kappa_m-1}\underline{\gamma}_1 = 0$$

Similarly, there is a unique $\underline{\gamma}_2$ s.t.

$$M^{\kappa_1-1,\kappa_2,\kappa_3-1\ldots,\kappa_m-1}\underline{\gamma}_2 = 0$$

and the coefficient of $\mathcal{A}^{\kappa_1} W_2$ is 1. And so on...

In conclusion: given $(\mathcal{A}, W)$ with controllability indexes $\kappa_1, \kappa_2, ..., \kappa_m$, we can set

$$M^{\kappa_1,\kappa_2,...,\kappa_m} = [W_1, ..., \mathcal{A}^{\kappa_1} W_1, W_2, ..., \mathcal{A}^{\kappa_2} W_2, ..., \mathcal{A}^{\kappa_m} W_m]$$

Then there exists a unique set of m linearly independent vectors $\underline{\gamma}_1, ..., \underline{\gamma}_m$ (with zeros added in the appropriate positions) s.t.

$$M^{\kappa_1,\kappa_2,...,\kappa_m} \left[\underline{\gamma}_1, ..., \underline{\gamma}_m\right] = 0$$

and the coefficient of $\mathcal{A}^{\kappa_i} W_i$ of γ_j is $\delta_{i,j}$.

Going back to polynomial matrices...

Theorem 4 *Given* $(\mathcal{A}, W)$ *controllable with indexes* $\kappa_1, \kappa_2, \ldots, \kappa_m$, *there exists a UNIQUE polynomial matrix* $\Gamma(s)$ *such that*

1. $(sI - \mathcal{A})^{-1} W \Gamma(s)$ *is polynomial.*
2. $\Gamma(s)$ *has column degrees* $\kappa_1, \kappa_2, \ldots, \kappa_m$ *and*

$$\deg \gamma_{j,j}(s) = \kappa_j \qquad \gamma_{j,j}(s) \text{ is monic}$$

$$\deg \gamma_{i,j}(s) \leq \min(\kappa_i - 1, \kappa_j) \qquad \text{for } 1 \leq i, j \leq m, i \neq j$$

Remark: The coefficients of the polynomials $\gamma_{i,j}(s)$ are those of the appropriate section of the vectors $\underline{\gamma}_j$ from the previous slide.

In particular:

- **The highest column degree coefficient matrix is upper triangular.**
- $\det \Gamma(s) = \det(sI - \mathcal{A})$
- $\Gamma(s)$ **is column reduced.**

$\Gamma(s)$ **is a Minimal Fundamental Solution: any other solution can be derived by this one by polynomial multiplication on the right.**

Recursion with uniqueness

Theorem 5 *Let* $\Gamma_{n-1}(s) = [\gamma_1^{n-1}(s), ..., \gamma_m^{n-1}(s)]$ *satisfy the conditions of Theorem (4) for* $(\mathcal{A}_{n-1}, W_{n-1})$ *nested in* $(\mathcal{A}_n, W_n)$*. Define* ϵ_n *as in Theorem (3) and let* j *be s.t.* κ_{n-1}^j *is the smallest index for which* $\epsilon_n^j \neq 0$*. Then* $\Gamma_n(s)$ *defined as*

$$\gamma_i^n(s) = \gamma_i^{n-1}(s) - \frac{\epsilon_n^i}{\epsilon_n^j}\gamma_j^{n-1}(s) \qquad i \neq j \qquad (1)$$

and (for suitable coefficients $\rho_{l,j}^n$*),*

$$\gamma_j^n(s) = (s - \lambda_n)\gamma_j^{n-1}(s) - \sum_{l \neq j} \rho_{l,j}^n \gamma_l^n(s) \qquad (2)$$

also satisfies the conditions of Theorem (4) for $(\mathcal{A}_n, W_n)$*.*

All polynomial interpolants are equal (up to a unit), but some are more equal than others...

In conclusion: given the sequence of nested interpolation problems $(\mathcal{A}_n W_n)$, we have

- **a canonical polynomial interpolant for each n and**
- **a recursive algorithm to compute them all.**

Applications

- Scalar Recursive Partial Realization
- Tangential Extension
- Tangential Newton and Hermite Interpolation
- System Identification
- Subspace Methods (to develop)
- Coding Theory ?
- Econometrics

Scalar Recursive Rational Interpolation (and Partial Realization)

Given a sequence $v_0, \ldots, v_N$, the problem is to construct recursively rational functions of the form

$$Q_n(s) = \sum_{i=0}^{n} v_i s^i + s^{n+1} \tilde{Q}_n(s)$$

Let, for $0 \le n \le N$,

$$\mathcal{A}_n = \begin{bmatrix} 0 & \dots & & & 0 \\ 1 & & & & \vdots \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \end{bmatrix} \quad W_n = [U_n, -V_n] = \begin{bmatrix} 1 & -v_0 \\ 0 & -v_1 \\ \vdots & \vdots \\ 0 & -v_n \end{bmatrix}$$

Take right coprime factorization

$$(sI - \mathcal{A}_n)^{-1} W_n = H_n(s) D_n^{-1}(s)$$

with $D_n(s)$ satisfying conditons of Theorem (5).

Set

$$D_n(s) =: \begin{bmatrix} \beta_n^U(s) & \beta_n^U(s) \\ \alpha_n^U(s) & \alpha_n^V(s) \end{bmatrix}$$

and let $\pi_n(s) = [\pi_n^U(s), \pi_n^V(s)]^T$ be a polynomial vector such that

$$D_n(0)\pi_n(0) \neq 0$$

Set

$$\begin{bmatrix} \beta_n^\pi(s) \\ \alpha_n^\pi(s) \end{bmatrix} = \begin{bmatrix} \beta_n^U(s) & \beta_n^U(s) \\ \alpha_n^U(s) & \alpha_n^V(s) \end{bmatrix} \begin{bmatrix} \pi_n^U(s) \\ \pi_n^V(s) \end{bmatrix}$$

Then $\boxed{Q_n^\pi(s) = \frac{\beta_n^\pi(s)}{\alpha_n^\pi(s)}}$ solves the interpolation problem associated to $v_0, ..., v_n$.

Sequence {1, 1, 1, 2, 3, 4, 5...} (Gragg Lindquist 1983)

THE PARTIAL REALIZATION PROBLEM

$\nu(1)=1$ $\nu(2)=3$ $\nu(3)=4$

(1)	1	1	2	3	4	5	...
1	1	(2)	3	4	5	6	...
1	2	3	4	5	6	7	...
2	3	4	(5)	6	7	8	...
3	4	5	6	7	8	9	...

FIG. 1.

Jump from $\nu(1)$ to $\nu(2)$!

Sequence $\{0, 1, 1, 1, 2, 3, 4, 5\ldots\}$ in our setup.

$$\Gamma_0(s) = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \quad \Gamma_1(s) = \begin{bmatrix} s & 0 \\ 1 & s \end{bmatrix} \quad \Gamma_2(s) = \begin{bmatrix} s^2 + s & -s \\ 1 & s - 1 \end{bmatrix}$$

$$\Gamma_3(s) = \begin{bmatrix} s^3 + s^2 + s & -s \\ 1 & s - 1 \end{bmatrix} \quad \Gamma_4(s) = \begin{bmatrix} s^3 + s^2 - s & -s^2 \\ 2s - 1 & s^2 - s \end{bmatrix}$$

$$\Gamma_5(s) = \begin{bmatrix} s^3 - s & -s \\ s^2 + s - 1 & s^3 + s - 1 \end{bmatrix}$$

$$\Gamma_6(s) = \begin{bmatrix} s^4 - s^2 + s & -s \\ s^2 - 2s + 1 & s^3 + s - 1 \end{bmatrix}$$

$$\Gamma_7(s) = \begin{bmatrix} s^4 - s^2 - s & -s^2 \\ s^2 - 2s + 1 & s^4 + s^2 - s \end{bmatrix}$$

In conclusion, a jump in the degree for proper Partial Realizations occurs if

- the entries of the second column are not coprime and
- the degree of the first column is strictly greater than that of the second.

Tangential Extension (well known)

Let $(\mathcal{A}, U, V)$ be given and seek $Q(s)$ such that

$$(sI - \mathcal{A})^{-1}[UQ(s) - V] \qquad \text{is polynomial}$$

Factorize

$$(sI - \mathcal{A})^{-1}[U, -V] = H(s)\Gamma(s)^{-1}$$

and partition $\Gamma(s) = \begin{bmatrix} \Gamma_{11}(s) & \Gamma_{12}(s) \\ \Gamma_{21}(s) & \Gamma_{22}(s) \end{bmatrix}$ Then

$$Q(s) = [\Gamma_{11}(s)\Pi_1(s) + \Gamma_{12}\Pi_2(s)][\Gamma_{21}(s)\Pi_1(s) + \Gamma_{22}\Pi_2(s)]^{-1}$$

interpolates the data $(\mathcal{A}, U, V)$.

This kind of constructions (not recursive and with simple nodes) can be found already in Antoulas, Ball, Kang and Willems (1990). Fuhrmann (2010) has nodes of higher multiplicity and recursion.

What is new are the uniqueness conditions of $\Gamma_n(s)$ in the recursive algorithm and the detailed analysis of the behaviour of the controllability indexes in this context to obtain proper interpolants (see GM).

Tangential Newton and Hermite Interpolation

Scalar problem (Newton interpolation): given distinct points $\lambda_0, ..., \lambda_N$ and values $v_0, ...v_N$, find recursively the polynomials $p_n(s)$ of degree n which interpolates the data.

Idea: for each n, form the polynomial $q_n(s) = \prod_{i=0}^{n}(s - \lambda_i)$ and compute a coefficient ρ_{n+1} such that

$$p_n(\lambda_{n+1}) + \rho_{n+1} q_n(\lambda_{n+1}) = v_{n+1}$$

Fast computation using divided differences: connected to errors ϵ_n of all the previous interpolants.

Important feature: unique interpolant.

Multivariable (Newton and Hermite): let $(\mathcal{A}_n, U_n, V_n)$ be given and seek matrix $Q_n(s)$ polynomial s.t.

$$(sI - \mathcal{A}_n)^{-1}[U_n Q_n(s) - V_n] \qquad \text{is polynomial}$$

Factorize again

$$(sI - \mathcal{A}_n)^{-1}[U_n, -V_n] = H_n(s)\Gamma_n(s)^{-1}$$

so that

$$\Gamma^n(s) = \begin{bmatrix} \Gamma^n_{11}(s) & \Gamma^n_{12}(s) \\ 0 & I \end{bmatrix}$$

and $\Gamma^n_{11}(s)$ satisfies Theorem (5) for $(\mathcal{A}_n, U_n)$ (connected by a Bezout equation to the general solution)

Then, for any polynomial matrix $[\Pi_1^n(s), \Pi_2^n(s)]$ (satisfying some kernel conditions),

$$Q_n(s) = \Gamma_{11}^n(s)\Pi_1^n(s) + \Gamma_{12}^n\Pi_2^n(s)$$

interpolates the data $(\mathcal{A}_n, U_n, V_n)$.

Fuhrmann (2010) showed that, for tangential Newton interpolation, given $\Gamma_{11}^n(s)$, there exists at each step a unique $\Gamma_{12}^n(s)$ such that

$$\Gamma_{11}^n(s)^{-1}\Gamma_{12}^n(s) \qquad \text{is strictly proper}$$

and obtained a recursion for $\Gamma_{12}^{n+1}(s)$. The recursion on $\Gamma_{11}^{n+1}(s)$ also for Hermite interpolation will complete the circle (work in progress).

Important feature: unique interpolant.

System Identification

(De Moor, Markowski, Willems, 2005, based on Kuijper 1997).

The idea is to identify the kernel of a wide Toeplitz matrix constructed from the input and the outputs.

Nevertheless, kernels are quite big in general and difficult to characterize...

Uniqueness of our kernel representation solves exact identification.

Recursive method for handling noise: orthogonalize the rows of data until they end up in the kernel... (work in progress: have to show consistency, asymptotic efficiency, etc.) and average the coefficients.

Slight problem with the orbits associated to different controllability indexes.

Coding Theory ?

One of the applications of Gragg-Lindquist partial realisation is Coding Theory, and in particular the Berlekamp-Massey algorithm (see Kuijper 1998, Fagnani and Zampieri, 1998).

Two reasons for the question mark:

- We do not know very much about the field.
- Our construction provides (e.g. in the scalar case) two interpolants. The total complexity (the sum of the degrees of the columns), though, is equal to the number of data. So, intuitively, if one column codes well, the other will not be so useful...

Econometrics

One of the main issues with Subspace Methods in Econometrics is the assumption that the data are generated by a causal system of which we observe inputs and outputs. In reality, in Time Series Analysis we are seldom able to make such a distinction (this was one of the motivations for Jan Willems behaviours)

Forni, Hallin, Lippi, Reichlin (2000-2001-2005) provided an alternative very succesfull approach which does not need that assumption. Nevertheless, it is, computationally, quite demanding. We feel our approach can address the problem in a simpler way.