A Gershgorin approach to networks of linear systems

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Networks of Linear Systems

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based on past joint work with A.J. Pritchard and M. Karow

Outline

Classical inclusion theorems I

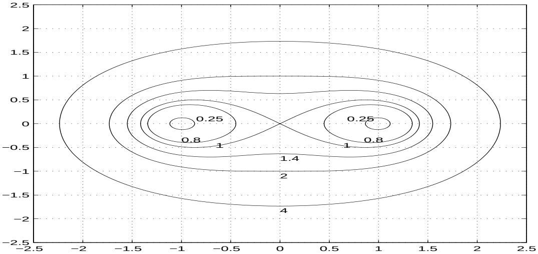
Given $A = [a_{jk}] \in \mathbb{C}^{N\times N}$. Off-diagonal row sums $R_i(A) := \sum_{j\in\underline{n}, j\neq i} |a_{ij}|$

Let $\mathcal{B}(z;\rho) = \{s \in \mathbb{C}; |s - z| \leq \rho\}$ disk with centre $z \in \mathbb{C}$ and radius $\rho > 0$.

Gershgorin's Thm. (1931):

$$\sigma(A) \subset \mathcal{G}_A := \bigcup_{i\in\underline{N}} \mathcal{B}(a_{ii}; R_i(A)) \quad \text{Gershgorin region}$$

Brauer's idea: Use Cassini ovals $\mathcal{B}(z_1, z_2; \rho) = \{s \in \mathbb{C}; |s - z_1|\,|s - z_2| \leq \rho\}$



$z_1 = -1,\ z_2 = 1$

$\rho = 0.25,\ 0.5,\ 1,\ 1.4,\ 2,\ 4$

Brauer's Thm. (1947):

$$\sigma(A) \subset \mathcal{B}_A := \bigcup_{1\leq i<j\leq N} \mathcal{B}(a_{ii}, a_{jj}; R_i(A)R_j(A)) \subset \mathcal{G}_A$$

Brauer region

Classical inclusion theorems II: Brualdi

Brualdi's idea: Given $A = [a_{ij}] \in \mathbb{C}^{N\times N}$, get tighter estimates of $\sigma(A)$ by taking into account which off-diagonal entries of A are nonzero.

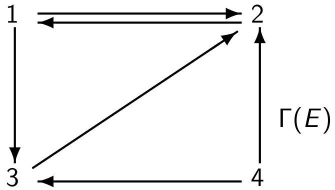
Location of these nonzero entries of A described by indicator matrix:

$$E = E(A) := \{(e_{ij}) \in \{0,1\}^{N\times N}\,;\; e_{ij} = 1 \Leftrightarrow i \neq j \text{ and } a_{ij} \neq 0\}.$$

The directed graph $\Gamma(E)$ associated with E has the vertices $1, 2, \ldots, N$ and a directed arc (j, i) from j to i iff $e_{ij} = 1$. (from column to row index)

Example:

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 9 \\ 3 & 0 & -2 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

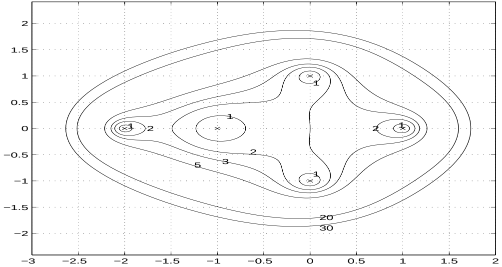


Def.: A path $\gamma = (j_1, \ldots, j_k)$ of distinct vertices in $\Gamma(E)$ is a cycle of length k if there is a directed arc from j_k to j_1 in $\Gamma(E)$, i.e. $e_{j_1 j_k} = 1$.
$\mathcal{Z}(E)$ denotes the set of all nontrivial cycles (i.e. of length $k \geq 2$) of $\Gamma(E)$. E is called weakly irreducible if every vertex is contained in at least one nontrivial cycle.

Classical inclusion theorems III: Brualdi

Def.: Let $z_1, \ldots, z_k \in \mathbb{C}$, $\rho > 0$. Associated Brualdi set (monotonous in ρ)

$$\mathcal{B}(z_1, \ldots, z_k; \rho) = \{s \in \mathbb{C}; |s - z_1| \cdots |s - z_k| \leq \rho\}.$$



Brualdi set

$\mathcal{B}(-2, -1, 1, -i, i; \rho)$

$\rho = 1, 2, 3, 5, 20, 30.$

Brualdi's Theorem (1982): Given $A = [a_{ij}] \in \mathbb{C}^{N \times N}$, define $E = (e_{ij}) \in \{0,1\}^{N \times N}$ by: $e_{ij} = 1$ iff $i \neq j$ and $a_{ij} \neq 0$. If E is weakly irreducible, then

$$\sigma(A) \subset \hat{\mathcal{B}}_A := \bigcup_{(\gamma_1, \ldots, \gamma_k) \in \mathcal{Z}(E)} \mathcal{B}(a_{\gamma_1\gamma_1}, \ldots, a_{\gamma_k\gamma_k}; \prod_{i=1}^{k} R_{\gamma_i}(A))$$

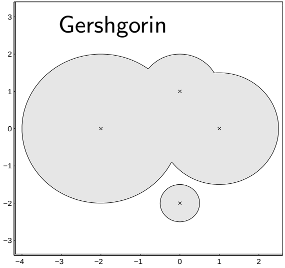
$\hat{\mathcal{B}}_A$ Brualdi region

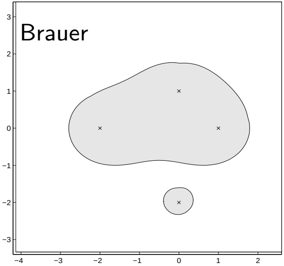
Comparison: Gershgorin, Brauer, Brualdi

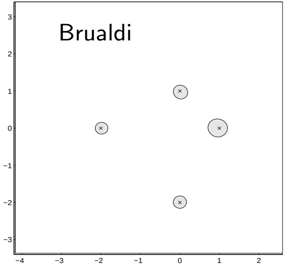
$$A = \begin{bmatrix} 1 & 1.5 & 0 & 0 \\ 0 & i & 0.5 & -0.5i \\ 0 & 0 & -2 & 2 \\ 0.25 & 0.25i & 0 & -2i \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

cycles:
$(1,4,2)$,
$(2,4,3)$







Are the classical inclusion theorems sharp?

Can they be improved without using more information about matrix A than they use?

Gershgorin's and Brauer's Thm: based on knowledge of
a) diagonal entries of A, b) off-diagonal row sums of A.

Inclusion regions $\mathcal{G}_A$ and $\mathcal{B}_A$ increase with $R_i(A)$. Hence they include the spectra of all $\tilde{A}$ with same diagonal as A and with off-diagonal row sums $R_i(\tilde{A}) \leq R_i(A)$: $\sigma(\tilde{A}) \subset \mathcal{G}_A$ and $\sigma(\tilde{A}) \subset \mathcal{B}_A$ for all

$$\tilde{A} \in [A]_{GB} := \{X \in \mathbb{C}^{n\times n}\, ;\ x_{ii} = a_{ii},\ \sum_{j=1, j\neq i}^{N} |x_{ij}| \leq R_i(A)\}$$

Inclusion theorems although stated for single matrices deal with spectra of a matrix set

Question: $\bigcup_{\tilde{A}\in[A]_{GB}} \sigma(\tilde{A}) = \mathcal{G}_A$ **??**, $\quad \bigcup_{\tilde{A}\in[A]_{GB}} \sigma(\tilde{A}) = \mathcal{B}_A$ **??**

Remark: $[A]_{GB}$ scaled ball with centre $D(A) = \mathrm{diag}(a_{11}, \ldots, a_{NN})$ and radius 1 in matrix space $\mathbb{C}^{N\times N}$ provided with the maximum row sum norm $\|\cdot\|_\infty$:

$$[A]_{GB} = \{D(A) + B\Delta\, ;\ \Delta = (\delta_{ij}) \in \mathbf{\Delta}_{offdiag}, \|\Delta\|_\infty \leq 1\},$$

where $B := \mathrm{diag}(R_1(A), \ldots, R_N(A))$ and $\mathbf{\Delta}_{offdiag} := \{\Delta = (\delta_{ij}) \in \mathbb{C}^{N\times N}\, ;\ \delta_{ii} = 0\}$.
"off-diagonal perturbations".

Brualdi's inclusion theorem sharp?

Brualdi's Thm.: Given $A = [a_{ij}] \in \mathbb{C}^{N\times N}$, define $E = (e_{ij}) \in \{0,1\}^{N\times N}$ by setting $e_{ij} = 1$ iff $i \neq j$ and $a_{ij} \neq 0$. If each $j \in \underline{N}$ is contained in some nontrivial cycle of $\Gamma(E)$, then

$$\sigma(A) \subset \hat{\mathcal{B}}_A := \bigcup_{(j_1,\dots,j_k)\in\mathcal{Z}(E)} \mathcal{B}(a_{j_1j_1},\dots,a_{j_kj_k};\prod_{i=1}^{k} R_{j_i}(A))$$

Brualdi's Thm: based on knowledge of

a) diagonal entries of A, b) off-diagonal row sums of A,

c) location of the nonzero/zero entries of A

$\hat{\mathcal{B}}_A$ increases with $R_i(A)$. Hence $\sigma(\tilde{A}) \subset \hat{\mathcal{B}}_A$ for all $\tilde{A}$ with same diagonal and zero entries as A, and off-diagonal row sums $R_i(\tilde{A}) \leq R_i(A)$:

$$\tilde{A} \in [A]_{Br} := \{D(A) + B\Delta\,;\ \Delta = (\delta_{ij}) \in \mathbf{\Delta}_E, \|\Delta\|_\infty \leq 1\} \quad \Rightarrow \quad \sigma(\tilde{A}) \subset \hat{\mathcal{B}}_A$$

where $\mathbf{\Delta}_E := \{\Delta = (\delta_{ij}) \in \mathbb{C}^{N\times N};\ \delta_{ij} = 0 \text{ if } i = j \text{ or } a_{ij} = 0\}$

"perturbations of structure E"

$B := \operatorname{diag}(R_1(A),\dots,R_N(A))$, $D(A) = \operatorname{diag}(a_{11},\dots,a_{NN})$,

Question: $\displaystyle\bigcup_{\tilde{A}\in[A]_{Br}} \sigma(\tilde{A}) = \hat{\mathcal{B}}_A$ **??**

Subsystems and couplings

For the mathematical description of an autonomous network of linear systems we need to specify the subsystems and the couplings between them.

Given N subsystems $(A_i, B_i, C_i) \in \mathbb{K}^{n_i \times n_i} \times \mathbb{K}^{n_i \times m_i} \times \mathbb{K}^{p_i \times n_i}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$:

$$\Sigma_i = (A_i, B_i, C_i) : \quad \begin{array}{ll} \dot{x}_i(t) & = A_i x_i(t) + B_i u_i(t) \\ y_i(t) & = C_i x_i(t) \end{array}, \quad i \in \underline{N} = \{1, \ldots, N\}.$$

Coupling structure described by an interconnection matrix $E = (e_{ij}) \in \{0, 1\}^{N \times N}$:

Output of Σ_j is coupled to input of Σ_i iff $e_{ij} = 1$

The directed graph $\Gamma(E)$ associated with E has the vertices $\Sigma_1, \ldots \Sigma_N$ and a directed arc from Σ_j to Σ_i iff $e_{ij} = 1$; visualizes the coupling structure of the network.

Def.: A coupling matrix of structure E is a block matrix $\Delta = (\Delta_{ij})_{i,j \in \underline{N}}$ with blocks $\Delta_{ij} \in \mathbb{K}^{m_i \times p_j}$ s.t. $\Delta_{ij} = 0$ if $e_{ij} = 0$.

$$\mathbf{\Delta} := \{\Delta = (\Delta_{ij}); \Delta_{ij} \in \mathbb{K}^{m_i \times p_j}\}, \qquad \mathbf{\Delta}_E := \{\Delta = (\Delta_{ij}) \in \mathbf{\Delta};\ \Delta_{ij} = 0 \text{ if } e_{ij} = 0)\}$$

block matrices | block matrices of structure E

$\Delta = (\Delta_{ij}) \in \mathbf{\Delta}_E$ describes the totality of couplings between the N subsystems:

Couplings: $\quad u_i = \sum_{j=1}^{N} \Delta_{ij} y_j = \sum_{j=1}^{N} e_{ij} \Delta_{ij} y_j, \quad i = 1, \ldots N, \quad \Delta \in \mathbf{\Delta}_E.$

Network equation:

Given N subsystems $(A_i, B_i, C_i) \in \mathbb{K}^{n_i \times n_i} \times \mathbb{K}^{n_i \times m_i} \times \mathbb{K}^{p_i \times n_i}$:

$$\Sigma_i: \quad \dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \quad y_i(t) = C_i x_i(t), \qquad i \in \underline{N},$$

and an interconnection matrix $E = (e_{ij}) \in \{0,1\}^{N \times N}$.

Coupling matrices: $\quad \Delta \in \mathbf{\Delta}_E := \{\Delta = (\Delta_{ij}) \in \mathbf{\Delta};\ \Delta_{ij} = 0 \text{ if } e_{ij} = 0)\}$

Couplings: $\quad$ (Δ_{ij} does not matter if $e_{ij} = 0$)

$$u_i(t) = \sum_{j \in \underline{N}} \Delta_{ij}\, y_j(t) = \sum_{j \in \underline{N}} \Delta_{ij} C_j x_j(t), \quad i \in \underline{N}, \qquad \text{where } \Delta = (\Delta_{ij}) \in \mathbf{\Delta}_E$$

Coupled network equations:

$$\dot{x}_i(t) = A_i x_i(t) + B_i \sum_{j \in \underline{N}} \Delta_{ij} C_j x_j(t), \quad i \in \underline{N}.$$

Network equation in vector form:

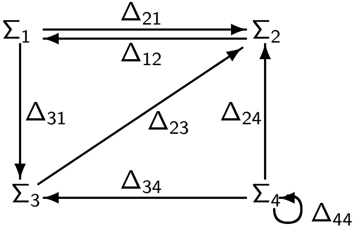
$$\Sigma_\Delta: \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_N \end{bmatrix} = \underbrace{(A + B\Delta C)}_{A(\Delta)} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \begin{aligned} A &= \operatorname{diag}(A_1, \ldots, A_N), \\ B &= \operatorname{diag}(B_1, \ldots, B_N), \\ C &= \operatorname{diag}(C_1, \ldots, C_N), \end{aligned} \quad \begin{aligned} &\Delta = (\Delta_{ij}) \in \mathbf{\Delta}_E, \\ &\Delta_{ij} \in \mathbb{K}^{m_i \times p_j}. \end{aligned}$$

Example: Network of four subsystems

$$\Sigma_i : \quad \dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \quad y_i(t) = C_i x_i(t), \quad i = 1, \ldots, 4.$$

Interconnection structure:

$$E = (e_{ij}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Couplings:

$$u_i(t) = \sum_{j=1}^{4} \Delta_{ij}\, y_j(t), \quad i = 1, \ldots, 4, \qquad \text{where } \Delta_{ij} \in \mathbb{K}^{m_i \times p_j}, \Delta_{ij} = 0 \text{ if } e_{ij} = 0,$$

yield

$$\dot{x}_i(t) = A_i x_i(t) + B_i \sum_{j=1}^{4} \Delta_{ij} C_j x_j(t), \quad i = 1, \ldots, 4, \; \Delta_{ij} = 0 \text{ if } e_{ij} = 0,$$

Coupled Network:

$$\Sigma_\Delta : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \Delta_{12} C_2 & 0 & 0 \\ B_2 \Delta_{21} C_1 & A_2 & B_2 \Delta_{23} C_3 & B_2 \Delta_{24} C_4 \\ B_3 \Delta_{31} C_1 & 0 & A_3 & B_3 \Delta_{34} C_4 \\ 0 & 0 & 0 & A_4 + B_4 \Delta_{44} C_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Spectral value sets and stability radii

Given: N subsystems $\Sigma_i = (A_i, B_i, C_i)$ and interconnection matrix $E \in \{0,1\}^{N\times N}$

$$A = \operatorname{diag}(A_1, \dots, A_N), \quad B = \operatorname{diag}(B_1, \dots, B_N), C = \operatorname{diag}(C_1, \dots, C_N)$$

$$\boldsymbol{\Sigma}_\Delta : \quad \dot{x} = A(\Delta)x = [A + B\Delta)C]\,x, \quad \Delta \in \boldsymbol{\Delta}_E$$

Problem 1: Given a norm $\|\cdot\|$ on $\boldsymbol{\Delta}_E$ and $\rho > 0$, determine the union of the spectra of all the matrices $A + B\Delta C$ where $\Delta \in \boldsymbol{\Delta}_E$ is of norm $\|\Delta\| \leq \rho$:

$$\sigma_{\boldsymbol{\Delta}_E}(A, B, C, E; \rho) = \bigcup_{\Delta \in \boldsymbol{\Delta}_E,\ \|\Delta\| \leq \rho} \sigma(A + B\Delta C)$$

Spectral value set of A of level $\rho > 0$ w.r.t. $A \rightsquigarrow A(\Delta) = A + B\Delta C$, $\Delta \in \boldsymbol{\Delta}_E$.

Problem 2: Suppose that the subsystems $\Sigma_i = (A_i, B_i, C_i)$, $i \in \underline{N}$ are (exponentially) stable. Given a norm $\|\cdot\|$ on $\boldsymbol{\Delta}_E$, determine the largest bound $\delta_{\max} > 0$ such that all the interconnected systems with couplings $\Delta = (\Delta_{ij}) \in \boldsymbol{\Delta}_E$, $\|\Delta\| < \delta_{\max}$ are stable.

$$r_{\boldsymbol{\Delta}_E}(A, B, C, E) = \inf\left\{\|\Delta\|; \Delta \in \boldsymbol{\Delta}_E,\ \sigma(A(\Delta)) \not\subset \mathbb{C}_-\right\}$$

stability radius of A w.r.t. perturbations $A \rightsquigarrow A + B\Delta C$, $\Delta \in \boldsymbol{\Delta}_E$

Solution of Problems 1 and 2 for $\mathbb{K} = \mathbb{C}$

Given: N subsystems $\Sigma_i = (A_i, B_i, C_i)$ and interconnection matrix $E \in \{0,1\}^{N\times N}$

$$G_i(s) = C_i(sI_{n_i} - A_i)^{-1}B_i, \;\; i \in \underline{N}, \quad A = \oplus_{i=1}^N A_i, \quad B = \oplus_{i=1}^N B_i, \quad C = \oplus_{i=1}^N C_i$$

Provide $\mathbf{\Delta}_E := \{\Delta = (\Delta_{ij}) \in \mathbf{\Delta};\; \Delta_{ij} = 0 \text{ if } e_{ij} = 0)\}$ with the norm

$$\|\Delta\| = \max_{i\in\underline{N}} \sum_{j=1}^{N} \|\Delta_{ij}\|_2 = \|\, [\|\Delta_{ij}\|_2]\, \|_\infty$$

Notation: $\|\cdot\|_p$ denotes the operator norm w.r.t. p-norm, $1 \le p \le \infty$.

$\mathcal{Z}(E)$ is the set of cycles of the directed graph $\Gamma(E)$.

If $\gamma = (j_1, \ldots, j_k) \in \mathcal{Z}(E)$, set $m_\gamma(s) := [\|G_{j_1}(s)\|_2 \cdots \|G_{j_k}(s)\|_2]^{1/k}$.

geometric mean.

Theorem: (*i*) $\sigma_{\mathbf{\Delta}_E} = \sigma_{\mathbf{\Delta}_E}(A,B,C,E;\delta) = \bigcup_{\Delta\in\mathbf{\Delta}_E,\, \|\Delta\|\le\delta} \sigma(A + B\Delta C)$

$$= \sigma(A) \cup \bigcup_{\gamma\in\mathcal{Z}(E)} \{s \in \rho(A)\,;\, m_\gamma(s) \ge \delta^{-1}\},$$

(*ii*) $$r_{\mathbf{\Delta}_E}(A,B,C,E) = \left[\max_{\omega\in\mathbb{R}} \max_{\gamma\in\mathcal{Z}(E)} m_\gamma(\imath\omega)\right]^{-1}.$$

Remark: Specialization of (i) to the scalar case ($m_i = n_i = p_i = 1$) yields: Brauer's Thm, Brualdi's Thm. and the fact that their estimates are sharp.

Time-varying couplings ($\mathbb{K} = \mathbb{C}$)

Given: N exponentially stable subsystems $\mathbf{\Sigma}_i = (A_i, B_i, C_i)$ and interconnection matrix $E \in \{0,1\}^{N\times N}$.

Let $A = \text{diag}(A_1, \dots, A_N), \quad B = \text{diag}(B_1, \dots, B_N), \quad C = \text{diag}(C_1, \dots, C_N)$.

Consider network with time-varying couplings of structure E between $\mathbf{\Sigma}_i$

$$\Sigma_{\Delta(\cdot)} \quad \dot{x} = [A + B\Delta(t)C]x,$$

where $\Delta(\cdot) \in \mathbf{\Delta}_{tv}(E) = \{\Delta(\cdot) : \mathbb{R}_+ \to \mathbf{\Delta}_E\,;\ \Delta(\cdot) \text{ bounded measurable}\}$.
space of time-varying linear couplings of structure E

Provide $\mathbf{\Delta}_E$ with the norm

$$\|\Delta\|_{\mathbf{\Delta}_E} := \max_{i\in \underline{N}} \underbrace{\|[\Delta_{i1}, \dots, \Delta_{iN}]\|_2}_{\text{spectral norm}} = \max_{i\in \underline{N}} \left[\lambda_{\max}\Big(\sum_{j\in \underline{N}} \Delta_{ij}\Delta_{ij}^*\Big)\right]^{1/2}, \quad \Delta \in \mathbf{\Delta}_E,$$

and $\mathbf{\Delta}_{tv}(E)$ with the norm

$$\|\Delta(\cdot)\|_{\mathbf{\Delta}_{tv}} = \operatorname*{ess\,sup}_{t\geq 0} \|\Delta(t)\|_{\mathbf{\Delta}_E}.$$

Problems

Problem 3: For which $r > 0$ does $\|\Delta(\cdot)\|_{\mathbf{\Delta}_{tv}} < r$ imply the exponential stability of $\Sigma_{\Delta(\cdot)}$? In this case the set of systems $\{\Sigma_\Delta; \Delta \in \mathbf{\Delta}_E, \|\Delta\|_{\mathbf{\Delta}_E} < r\}$ is said to be tv-stable. The maximal r with this property is

$$r_{\mathbf{\Delta}_{tv}} = r_{\mathbf{\Delta}_{tv}}(A, B, C, E) := \inf\{\|\Delta(\cdot)\|_{\mathbf{\Delta}_{tv}}; \Delta(\cdot) \in \mathbf{\Delta}_{tv}(E) \text{ and } \Sigma_{\Delta(\cdot)} \text{ not exp. stable}\}.$$

Stability radius of A w.r.t. time-varying couplings/perturbations of structure E

Problem 4: For which data do we have $r_{\mathbf{\Delta}_{tv}}(A, B, C, E) = r_{\mathbf{\Delta}_E}(A, B, C, E)$?

Problem 5: For which $r > 0$ does there exist a joint quadratic Liapunov function for all the systems

$$\Sigma_\Delta : \quad \dot{x} = [A + B\Delta C]x, \quad \Delta \in \mathbf{\Delta}_E, \ \|\Delta\|_{\mathbf{\Delta}_E} < r.$$

In this case the set of systems $\{\Sigma_\Delta; \Delta \in \mathbf{\Delta}_E, \|\Delta\|_{\mathbf{\Delta}_E} < r\}$ is called quadratically stable.

Time-varying couplings: Results I

Data:

$$\Sigma_i = (A_i, B_i, C_i),\ \ \sigma(A_i) \subset \mathbb{C}_-,\ \ G_i(s) = C_i(sI - A_i)^{-1}B_i,\ i \in \underline{N},\ \ E \in \{0,1\}^{N\times N}.$$

Let $A = \mathrm{diag}(A_1, \ldots, A_N)$, $\quad B = \mathrm{diag}(B_1, \ldots, B_N)$, $\quad C = \mathrm{diag}(C_1, \ldots, C_N)$

System with time-varying couplings of structure E

$$\Sigma_{\Delta(\cdot)}: \quad \dot{x} = [A + B\Delta(t)C]x, \quad \Delta(\cdot) \in \mathbf{\Delta}_{tv} = \mathbf{\Delta}_{tv}(E).$$

Theorem 2: Let $D_g = \mathrm{diag}(g_1, \ldots, g_N)$, where $g_i = \max_{\omega\in\mathbb{R}} \|G_i(\imath\omega)\|_2 = \|G_i\|_{H^\infty}$. Then

$$r_{\mathbf{\Delta}_{tv}}(A, B, C, E) = \varrho(D_g^2 E)^{-1/2}.$$

If the (A_i, C_i) are observable and E has no zero column, then the set of systems $\{\Sigma_\Delta\,;\ \Delta \in \mathbf{\Delta}_E,\ \|\Delta\|_{\mathbf{\Delta}_E} \le r\}$ is tv-stable iff there exists a joint quadratic Liapunov function for all the (time-invariant) systems

$$\Sigma_\Delta: \qquad \dot{x} = [A + B\Delta C]x, \quad \Delta \in \mathbf{\Delta}_E, \|\Delta\|_{\mathbf{\Delta}_E} \le r.$$

The system set $\{\Sigma_\Delta\,;\ \Delta \in \mathbf{\Delta}_E,\ \|\Delta\|_{\mathbf{\Delta}_E} < r\}$ tv-stable $\Leftrightarrow$ quadratically stable

Time-varying couplings: Results II

Theorem 3: Suppose that E is irreducible and $g_i > 0$, $i \in \underline{N}$. Then equivalent:

(i) $r_{\mathbf{\Delta}_{tv}}(A, B, C, E) = r_{\mathbf{\Delta}_E}(A, B, C, E)$.

(ii) There exists a joint maximum $\omega_0 \in \mathbb{R}$ of the N functions $\omega \mapsto \|G_i(\imath\omega)\|_2,\ i \in \underline{N}$.

Def.: A system (A, B, C) is called positive if $B \geq 0, C \geq 0$ and A is a Metzler matrix, i.e. $A + rI_n \geq 0$ for some $r > 0$.

Cor.: If the subsystems (A_i, B_i, C_i), $i \in \underline{N}$ are positive and stable, then

$$r_{\mathbf{\Delta}_{tv}}(A, B, C, E) = r_{\mathbf{\Delta}_E}(A, B, C, E) = \varrho(D_g^2 E)^{-1/2}$$

where $D_g = \operatorname{diag}\left(\|C_1 A_1^{-1} B_1\|_2, \ldots, \|C_N A_N^{-1} B_N\|_2\right)$.

Proof: The N functions $\omega \mapsto \|G_i(\imath\omega)\|_2 = \|C_i(\imath\omega I - A_i)^{-1} B\|_2$, $i \in \underline{N}$ have a joint maximum at $\omega_0 = 0$ and

$$g_i = \|G_i(0)\|_2 = \|C_i A_i^{-1} B_i\|_2$$

Summary

Given N stable time-invariant linear subsystems (A_i, B_i, C_i) with a fixed interconnection structure defined by $E \in \{0,1\}^{N\times N}$.

Results:

- Computable formula for the spectral value sets and the stability radius of $A = \mathrm{diag}(A_1, \ldots, A_N)$ w.r.t. time-invariant couplings/perturbations of structure E (Solution of Problems 1 and 2). Applied to 1-dimensional subsystems this formula implies the classical inclusion theorems of Gershgorin, Brauer and Brualdi. Moreover it shows that the theorems of Brauer and Brualdi are sharp.
- Computable formula for the stability radius of $A = \mathrm{diag}(A_1, \ldots, A_N)$ w.r.t. time-varying couplings of structure E. (Solution of Problem 3)
- Necessary and sufficient conditions for the quadratic stability of the set of systems Σ_Δ with couplings of norm $\leq r$ and given structure E.
- Necessary and sufficient conditions for $r_{\mathbf{\Delta}_{tv}(E)}(A, B, C, E) = r_{\mathbf{\Delta}_E}(A, B, C, E)$. (Solution of Problem 4). These conditions are satisfied for positive systems.

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