# Time-varying multiplicative perturbations of well-posed LTI systems

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In memory of U. Helmke and R. Kalman

We are concerned with a time-varying linear system resulting from (right) multiplicative perturbation of a well-posed LTI system $\Sigma$ (corresponding to $P(t) = I$):

$$\Sigma_r^v \quad \begin{cases} \dot{x}(t) = AP(t)x(t) + Bu(t), & 0 \leq t < \infty, \\ y(t) = \overline{C}x(t) + Du(t). \end{cases}$$

| | |
|---|---|
| $A$ | operator semigroup generator |
| $B$ | the *control operator*, $B \in \mathcal{L}(U, X_{-1})$ |
| $\overline{C}$ | an extension of the *observation operator* $C \in \mathcal{L}(X_1, Y)$ |
| $D$ | the input-output operator, $D \in \mathcal{L}(U, Y)$ |
| $P$ | $P : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous |
| $X$, $U$, $Y$ | state space, input space, output space |
| $X_1$ | $\mathcal{D}(A)$ endowed with graph norm |

# Motivation

**Moving Object Problem:** Offer a well-posed LTV system framework for the system involving the electromagnetic field around a moving object. We have to play with Maxwell's equations:

$$\begin{cases} \frac{\partial \mathbf{B}}{\partial t} = -\mathrm{rot}\mathbf{E}, \\ \frac{\partial \mathbf{D}}{\partial t} = -\mathbf{J} + \mathrm{rot}\mathbf{H}, \end{cases} \qquad \mathrm{div}\mathbf{B} = 0, \quad \mathrm{div}\mathbf{D} = \rho.$$

The (approxiamte) *constitutive relations* is adopted:

$$\mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}.$$

Table: The notation in Maxwell's equations

| symbol | physical meaning | symbol | physical meaning |
|---|---|---|---|
| $\mathbf{E}$ | *electric field intensity* | $\mathbf{H}$ | *magnetic field intensity* |
| $\mathbf{D}$ | *electric displacement* | $\mathbf{B}$ | *magnetic flux density* |
| $\rho$ | *electric charge density* | $\mathbf{J}$ | *electric current density* |
| $\varepsilon$ | *permittivity* | $\mu$ | *permeability* |

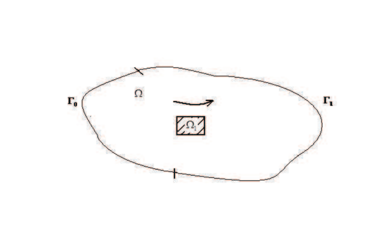


Figure: Schematic of our “moving object problem”. An object $\Omega_1$ moving in the electromagnetic field within domain $\Omega$. The governing equations are hence Maxwell’s equations. A typical example from the real world is a rotor rotating in a motor/generator. The case when the object is at rest corresponds to a well-posded LTI systems and when the object is moving, we must adopt an LTV system.

# How it comes the multiplicative perturbation?

Detailed descriptions on the LTI system corresponding to the non-moving scenario (the system generator, control input, observed output, etc.) can be found in the surveys:

G. Weiss and O. Staffans, Maxwell's equations as a scattering passive linear system, SICON'13

O. Staffans and G. Weiss, A physically motivated class of scattering passive linear systems, SICON'12

For our moving object problem, the state space is $X = \mathcal{E} \times \mathcal{E}$ and $\mathcal{E} = L^2(\Omega; \mathbb{R}^3)$ and the state is $\left[\begin{smallmatrix}\mathbf{B}\\ \mathbf{D}\end{smallmatrix}\right]$ ($\langle \frac{1}{\mu}\mathbf{B}, \mathbf{B}\rangle + \langle \frac{1}{\varepsilon}\mathbf{D}, \mathbf{D}\rangle$ is twice the physical energy stored in the system).

In view of the constitutive relations, it is sort of natural to imagine that Maxwell's equations is related to the abstract expression $\dot{x} = APx(t)$ where

$$P = \begin{bmatrix} \mu^{-1} & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}.$$

The moving object is assumed to have zero electrical conductivity but not necessarily the same permittivity and permeability as for the surrounding vacuum. Such objects may be insulators used in windings and the materials are, for instance, Teflon, Silicon Dioxide and hard Rubber, measured at low-frequency and room temperature. Accordingly,

$$\varepsilon(t,\mathbf{x}) = \begin{cases} \epsilon_1, & \mathbf{x} \in \Omega_1(t), \\ \epsilon_0, & \text{otherwise}, \end{cases} \qquad \mu(t,\mathbf{x}) = \begin{cases} \mu_1, & \mathbf{x} \in \Omega_1(t), \\ \mu_0, & \text{otherwise}, \end{cases}$$

where the constants $\epsilon_0$ and $\mu_0$ stand for the permittivity and permeability of the surrounding vacuum, the constants $\epsilon_1$ and $\mu_1$ reflect the permittivity and permeability of the rigid body.

Here comes the **tough stuff:** the bounded operator valued function $P : [0,\infty) \to \mathcal{L}(X)$ is only strongly continuous.

# What is "well-posed"?

Roughly, a system is well-posed if on any time interval $[\tau, t]$, for any initial state $x_0$ in the state space and any input function $u$ in a specified space of functions, it has a unique state trajectory $x$ and a unique output function $y$, both defined on $[\tau, t]$, moreover, $y$ must belong to a specified space of functions, and both $x(t)$ and $y$ must depend continuously on $x(\tau)$ and on $u$.

Not an issue and is usually not even mentioned if the state space is finite-dimensional.

# Well-posed LTI sysems & why well-posedness?

A well-posed LTI system is defined, in an abstract way, as a quadruple $\Sigma = \{\Sigma_\tau\}_{\tau\geq 0} = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ where the four families of one-parameter bounded operators $\mathbb{T}$, $\Phi$, $\Psi$, $\mathbb{F}$ satisfy some conditions reflecting time-invariance and causality.

Well-posedness is usually not a goal in itself. However, it opens a potential way for dealing with control and/or estimation problems by trying to mimic the rich finite-dimensional control theory using “operators in place of matrices” at the conceptual level.

# Well-posed LTV systems

Well-posedness has been generalized to linear time-varying (LTV) systems:

* Hinrichsen and Pritchard SICON'94
* Jacob et al. IEOT'95
* Schnaubert SICON'02
* Schnaubert and Weiss MCSS'10 (the one we adopt here)

Similar to the LTI counterpart, a well-posed LTV system is defined as a quadruple $\Sigma = \{\Sigma_\tau\}_{\tau \geq 0} = (\mathbb{U}, \Phi, \Psi, \mathbb{F})$ where the four families of **two-parameter** bounded operators satisfy some functional equations reflecting (mainly) causality.

A *time-varying well-posed system* $\Sigma^v$ on a time interval $J \subset \mathbb{R}$ with input space $U$, state space $X$ and output space $Y$ is a quadruple of operator families

$$\mathbb{U}(t,s) : X \to X, \quad \Phi(t,s) : L^2(J;U) \to X,$$

$$\Psi(t,s) : X \to L^2(J;Y), \quad \mathbb{F}(t,s) : L^2(J;U) \to L^2(J;Y)$$

indexed by $s$, $t \in J$ with $s \leq t$ which satisfy:

(i) $\mathbb{U}$ is an evolution family on $X$ with time interval $J$.

(ii) $\mathbb{U}(t,s)$, $\Phi(t,s)$, $\Psi(t,s)$, $\mathbb{F}(t,s)$ are locally uniformly bounded.

(iii) The causality conditions: for all $s$, $t \in J$ with $s \leq t$,

$$\Phi(t,s) = \Phi(t,s)\mathbf{P}_{[s,t]}, \quad \Psi(t,s) = \mathbf{P}_{[s,t]}\Psi(t,s),$$

$$\mathbb{F}(t,s) = \mathbf{P}_{[s,t]}\mathbb{F}(t,s) = \mathbb{F}(t,s).\mathbf{P}_{[s,t]}$$

(iv) For all $t$, $\tau$, $s \in J$ with $s \leq \tau \leq t$ and all $u \in L^2(J;U)$, $x_0 \in X$,

$$\begin{aligned} \Phi(t,s)u &= \Phi(t,\tau)u + \mathbb{U}(t,\tau)\Phi(\tau,s)u, \\ = \Psi(t,s)x_0 &= \Psi(t,\tau)\mathbb{U}(\tau,s)x_0 + \Psi(\tau,s)x_0, \end{aligned}$$

$$\mathbb{F}(t,s)u = \mathbb{F}(t,\tau)u + \mathbb{F}(\tau,s)u + \Psi(t,\tau)\Phi(\tau,s)u.$$

# A key concept: evolution family

The concept of Natural evolution family, a generalization of operator semigroup, is introduced to solving the non-autonomous Cauchy problem

$$\dot{x}(t) = A(t)x(t), \qquad t \in J, \qquad t \geq s, \qquad x(s) = x_0,$$

where $A(\cdot)$ is a family of operators on $X$.

Let $J$ be a closed interval and $\mathbb{U} = \{\mathbb{U}(t,s) |\ s,\, t \in J \text{ and } s \le t\}$ a family of operators in $\mathcal{L}(X)$. We call $\mathbb{U}$ an *evolution family* on $X$ with time interval $J$ if the following three conditions hold:

(i) $\mathbb{U}(t,t) = I$ for all $t \in J$.

(ii) $\mathbb{U}(t,s) = \mathbb{U}(t,\tau)\mathbb{U}(\tau,s)$ for all $s,\ \tau,\ t \in J$ with $s \le \tau \le t$.

(iii) For each $z \in X$, the function $(t,s) \to \mathbb{U}(t,s)z$ is continuous.

# Evolution family plays an important role

Two reasons. First, the absence of input and output corresponds to the non-autonomous Cauchy problem. Second, the *Lax-Phillips evolution family* criterion of well-posedness of LTV systems.

**Notation:**

| | |
|---|---|
| $\mathbf{P}_+$ | the projection from $L^2(\mathbb{R};H)$ to $L^2(\mathbb{R}_+;H)$ by truncation |
| $\mathbf{P}_-$ | the projection from $L^2(\mathbb{R};H)$ to $L^2(\mathbb{R}_-;H)$ by truncation |
| $\mathbf{P}_J$ | the truncation operator to the interval $J$ |
| $S_t$ | the left shift operator on $L^2(\mathbb{R};H)$, $(S_t f)(\cdot) = f(\cdot + t)$ |
| $S_t^{\pm}$ | $S_t^{\pm} := \mathbf{P}_{\pm} S_t$ |
| $\mathcal{Y}$ | $L^2(\mathbb{R}_-;Y)$ |
| $\mathcal{U}$ | $L^2(\mathbb{R}_+;U)$ |

# Lax-Phillips evolution family characterization of a well-posed LTV system

Let $\mathbb{U}, \Phi, \Psi, \mathbb{F}$ be operator families on the interval $J$ satisfying the causality conditions. Then they form a time-varying well-posed system $\Sigma^v$ on $J$ if and only if the operators

$$\mathscr{U}(t,s) = \begin{bmatrix} S^-_{t-s} & S^-_t \Psi(t,s) & S^-_t \mathbb{F}(t,s) S_{-s} \\ 0 & \mathbb{U}(t,s) & \Phi(t,s) S_{-s} \\ 0 & 0 & S^+_{t-s} \end{bmatrix}$$

$(s,\, t \in J,\, s \leq t)$ form an evolution family on $\mathcal{H} = \mathcal{Y} \times X \times \mathcal{U}$ with time interval $J$.

It is the generalization of *Lax-Phillips semigroup* associated with a well-posed LTI system. The Lax-Phillips semigroup generator:

$$\mathcal{A} = \begin{bmatrix} \mathfrak{D} & 0 & 0 \\ 0 & A & B\delta_0 \\ 0 & 0 & \mathfrak{D} \end{bmatrix}, \qquad \mathcal{A} \begin{bmatrix} y \\ x \\ u \end{bmatrix} = \begin{bmatrix} y' \\ Ax + Bu(0) \\ u' \end{bmatrix},$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \left[ \begin{smallmatrix} y \\ x \\ u \end{smallmatrix} \right] \in \mathbb{H} \;\middle|\; Ax + Bu(0) \in X,\; y(0) = \overline{C}x + Du(0) \right\},$$

$$\mathbb{H} = \mathcal{H}^1((-\infty, 0); Y) \times X \times \mathcal{H}^1((0, \infty); U).$$

Here, $\mathfrak{D}$ represents the derivative operator and (as usual) $\delta_0$ indicates the Dirac distribution.

# The way we proceed with our story

*Step 1.* For the equation $\dot{x}(t) = AP(t)x(t)$, find an associated evolution family $\mathbb{U}$ and in particular, specify in what sense it solves the equation.

*Step 2.* Reduce well-posedness of $\Sigma_r^v$ to associating an evolution family $\mathscr{U}$ (defined on on $\mathcal{H} = \mathcal{Y} \times X \times \mathcal{U}$) to the evolution equation

$$\dot{z} = \mathcal{A}\mathscr{P}(t)z(t), \qquad \mathscr{P}(t) = \begin{bmatrix} I & 0 & 0 \\ 0 & P(t) & 0 \\ 0 & 0 & I \end{bmatrix},$$

where, $\mathcal{A}$ the Lax-Phillips semigroup generator of $\Sigma$ and $\mathscr{P}$ is only strongly continuous since so is $P : [0, \infty) \to \mathcal{L}(X)$.

Use the result from Step 1 to prove that $\Sigma_r^v$ is well-posed. To this end, we need to retrieve info from the fact that $\mathscr{U}$ solves the equation in $\mathcal{H}_{-1}$.

*Step 3.* Derive the system solution formulas. Explore the system properties, for example, *energy balance inequality*, assuming that the original LTI system is scattering passive. Again, we need to retrieve info from the fact that $\mathscr{U}$ solves the equation in $\mathcal{H}_{-1}$.

# Relevant existing surveys

Schnaubelt and Weiss MCSS'10: multiplicative perturbation (including both the right and the left) of well-posed LTI systems. One of the standing conditions is that $P$ is strongly $C^1$. We need to relax this to "strongly continuous".

Chen and Weiss MCSS'15: additive perturbation

$$\begin{cases} \dot{x}(t) = [A + G(t)]\, x(t) + Bu(t), \quad 0 \le t < \infty, \\ y(t) = \overline{C} x(t) + Du(t) \end{cases}$$

which was intended to model a "moving conductor".

The conductor has the same *permittivity* and *permeability* as for the surrounding vacuum but a possibly high electrical conductivity. Typical materials for such a conductor are some metals, for instance, Aluminium and Cooper, measured at low-frequency and room temperature.

Likewise, $G : [0, \infty) \to \mathcal{L}(X)$ is only strongly continuous. But it is reasonable to imagine that the case is much easier than the multiplicative case. The non-autonomous Cauchy problem $\dot{z} = [\mathcal{A} + \mathscr{G}(t)]\, z(t)$ was solved.

**Standing assumptions in Schnaubelt and Weiss MCSS'10:**

- $A$ is m-dissipative
- $P(t)^* = P(t) \geq 0$ and $P(t)$ is invertible for each $t \geq 0$
- both $P(\cdot)$ and $P(\cdot)^{-1}$ are strongly continuous, namely, both $P(\cdot)z$ and $P(\cdot)^{-1}z \in C(\mathbb{R}_+; X)$ for each $z \in X$.

**Conclusions:** roughly speaking, there exists an evolution family which yields the solution in $X$ if $P(\cdot)$ is strongly $C^2$ and in the case of strongly $C^1$, there is still an evolution family but it only gives the solution in the extrapolated space $X_{-1}$

In view of the existing result on right multiplicative perturbation, it is very natural to use the approximation technique. Define the averaged functions $P_n(\cdot)$ by

$$P_n(t)z = n\int_t^{t+\frac{1}{n}} P(\tau)z \,\mathrm{d}\tau.$$

Let $\mathbb{U}_n$ the evolution family associated with $AP_n(\cdot)$. However, it is not easy to "pass to the limit".

The above technique, combined with the theory of *evolution semigroup*, works for the additive perturbation case. This is what we have done in our additive perturbation paper Chen and Weiss MCSS'15.

An evolution semigroup $E(\cdot) = \{E(t)\}_{t\geq 0}$ is defined on the "larger" space $C_0(\mathbb{R}_+; X)$ and associated to the evolution family $\mathbb{U}$ generated by $A(\cdot)$. This theory, introduced by Howland in 1974, offers a semigroup approach to evolution equation $\dot{x} = A(t)x(t)$.

What is helpful to us is that the strong convergence of $\{E_n(\cdot)|\ n = 1, 2 \cdots\}$ implies that of $\{\mathbb{U}_n(\cdot)|\ n = 1, 2 \cdots\}$. For us, $A_n(t) = AP_n(t)$, $\mathbb{U}_n$ is the evolution family associated with $AP_n$. The second Trotter-Kato approximation theorem is applied, what we have actually used is its weaker version.

# What we have obtained

**Assumptions:**

- $A$ is m-dissipative
- $P(t)^* = P(t) \geq 0$ and $P(t)$ is invertible for each $t \geq 0$
- $\exists\, M \geq 1$ such that $\|P(t)z\|^2 \leq \langle P(t)z, z\rangle \leq M^2\|P(t)z\|^2$ for each $t \geq 0$ and every $z \in X$,
- the function $t \to \langle P(t)z, z\rangle$ is non-decreasing on $\mathbb{R}_+$ for each $z \in X$,
- both $P(\cdot)$ and $P(\cdot)^{-1}$ are strongly continuous, namely, both $P(\cdot)z$ and $P(\cdot)^{-1}z \in C(\mathbb{R}_+; X)$ for each $z \in X$.

**Conclusions:** (a) There exists an evolution family $\mathbb{U}$ which solves $\dot{x} = AP(t)$ in $X_{-1}$. (b) The LTV system $\Sigma_r^v$ is well-posed if the original unperturbed LTI system is well-posed.

# How the additional conditions come?

The additional conditions, the price we pay for relaxation from strongly $C^1$ to strongly $C$, are due to the following theorem by Nickle:

The following two statements are equivalent:

(i) The family $\{A(t) |\ t \in [0, T]\}$ is stable with stability constant $\omega = 0$.

(ii) There exist a family of norms $\left\{\|\cdot\|^t \mid t \in [0, T]\right\}$ and a constant $M \geq 1$ such that

  (a) $\|\cdot\| \leq \|\cdot\|^t \leq M\|\cdot\|$ for each $t \in [0, T]$,
  (b) $\|\cdot\|^t \leq \|\cdot\|^s$ if $s \leq t$,
  (c) $\|\mathbb{T}_t(\tau)\|^t \leq 1$ for each $t \in [0, T]$ and all $\tau \geq 0$.

# In the sequel...

The article is still in preparation. To complete it, we need to:

* First, fit the motivating moving object into a well-posed LTV system. We are expected to interpret the monotonicity condition.
* Second, derive *energy balance inequality* of the moving object system (the non-moving LTI system turns out to be *scattering conservative*).

Dropping the monotonicity condition would be a significant improvement.

Thanks for your attention and comments are welcome!