Convex Invertible Cones & Dissipative Systems

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Sed – Boker, 22$^{\text{nd}}$ March 2017

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MAIN MESSAGE

Engineering/Physical motivation:

"Dissipativity" (continuous time) $\Longrightarrow$ (hidden) CIC structure.

Math motivation:

Positive (real) functions of non-commuting variables.

Can you please suggest additional Convex Invertible Cones ?

Semi-formal definition

Over a real unital algebra:

- A set C is called a "Convex Cone" if it is closed under positive scaling and summation, e.g.

$$\{A, B, C \ \ldots\} \in \mathrm{C} \implies (A + 2B + \tfrac{2}{3}C) \in \mathrm{C}$$

- A set is called "Invertible", if whenever the inverse of an element exists, this inverse also belongs to the set.
- Intersection of properties = a "Convex Invertible Cone" cic

Example:
$A, B, C \ \ldots \ n \times n$ matrices. Whenever the inverses exist

$$\left(\left(\left(A + 2B^{-1}\right)^{-1} + \tfrac{2}{3}C\right)^{-1} + 3B\right) \in \mathrm{cic}(A, B, C \ \ldots)$$

A step-by-step construction

$\mathbf{X}$ a generating set

$$\mathbf{X}_o := \mathbb{R}_+ \cdot \mathbf{X} \qquad \mathbf{X}_{j+1} = \mathbb{R}_+ \cdot \mathrm{conv}\left(\mathbf{X}_j \bigcup {\mathbf{X}_j}^{-1}\right) \qquad \mathrm{cic}(\mathbf{X}) = \bigcup_{j=0}^{\infty} \mathbf{X}_j$$

under the convention that ${\mathbf{X}_j}^{-1} := \{x^{-1} \ : \ x \in \mathbf{X}_j \quad \exists x^{-1}\}$.

Example: $\mathbf{X}_o = \{A, B, C\} \implies$

$$\left(\left(\left(A + 2B^{-1}\right)^{-1} + \tfrac{2}{3}C\right)^{-1} + 3B\right) \in \mathbf{X}_3 \subset \mathrm{cic}(A, B, C)$$

$\{2A, B^{-1}, C\}$ another generating set.

Rational functions of non-commuting variables Positivity ?

First motivation - a feedback loop

G, H $\quad m \times m$-valued rational functions

A step-by-step construction of $\quad \text{cic}(G,\ H)$

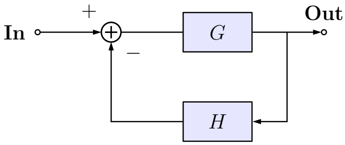
$$\mathbf{X}_o = \{G,\ H\}$$

$$\det(G)\not\equiv 0 \quad \Longrightarrow \quad \left(G^{-1} + H\right) \quad \in \quad \mathbf{X}_1$$

$$\left.\begin{array}{c} \det(G)\not\equiv 0 \\ \det\left(G^{-1}+H\right)\not\equiv 0 \end{array}\right\} \quad \Longrightarrow \quad \left(G^{-1} + H\right)^{-1} \quad \in \quad \mathbf{X}_2$$

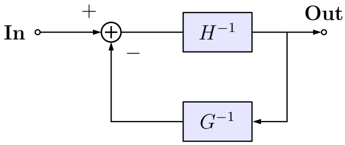
Feedback - CIC (cont.)

$$\mathbf{Out} = \left(\boldsymbol{G}^{-1} + \boldsymbol{H}\right)^{-1} \mathbf{In}$$



Feedback - CIC (cont.)

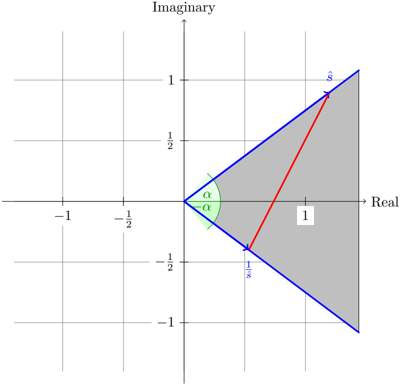
$$\textbf{Out} = \left(G^{-1} + H\right)^{-1} \textbf{In}$$



A CIC generated by a scalar

$\hat{s} \in \mathbb{C}_r$ **the open right half plane**

$\frac{1}{\hat{s}} = \frac{\hat{s}^*}{|\hat{s}|^2} \quad \Longrightarrow \quad \text{cic}(\hat{s}) = \mathbb{R}_+ \cdot \text{conv}\left(\hat{s} \ , \ \frac{\hat{s}^*}{|\hat{s}|^2}\right)$



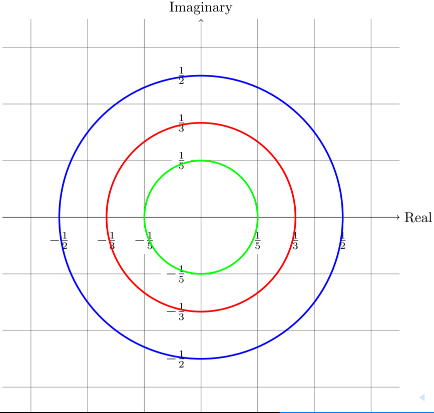
Disks at the origin

Disks at the origin

$\overline{\mathbb{D}}\left(0,\frac{1}{2}\right)$ — **blue**

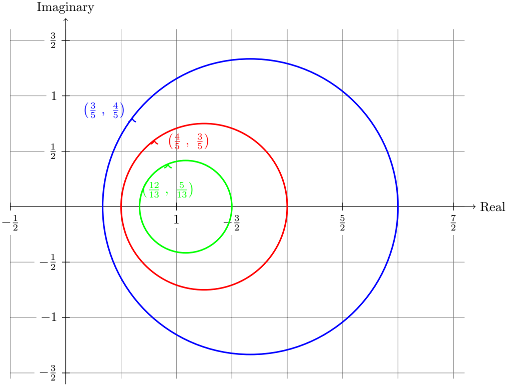
$\overline{\mathbb{D}}\left(0,\frac{1}{3}\right)$ — **red**

$\overline{\mathbb{D}}\left(0,\frac{1}{5}\right)$ — **green**



Invertible Disks

$$\textbf{Cayley Transform} \quad (I - \overline{\mathbb{D}})(I + \overline{\mathbb{D}})^{-1} \Longrightarrow \begin{array}{ll} \overline{\mathbb{D}}\left(\frac{5}{3}, \frac{4}{3}\right) & - \ \textbf{blue} \\ \overline{\mathbb{D}}\left(\frac{5}{4}, \frac{3}{4}\right) & - \ \textbf{red} \\ \overline{\mathbb{D}}\left(\frac{13}{12}, \frac{5}{12}\right) & - \ \textbf{green} \end{array}$$



s a complex variable.

$\mathbb{C}_r$ $(\overline{\mathbb{C}_r})$ open (closed) right half plane.

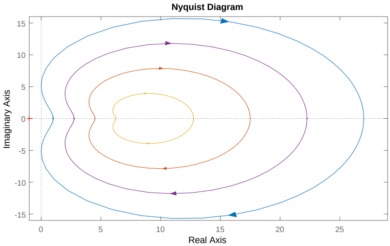
$\mathcal{PR} := \{\ f(s)$ real rational : within $\mathbb{C}_r$ analytic & $\mathrm{Re}\,(f(s)) \geq 0\ \}$

The Nyquist plot is within $\overline{\mathbb{C}_r}$ (infinite gain margin)

A Nyquist plot

$f(s) = \frac{(s+3)^3}{(s+1)^3}$ $\qquad$ $g(s) \equiv 8$

f	$\frac{3}{4}f + \frac{1}{4}g$	$\frac{1}{2}(f+g)$	$\frac{1}{4}f + \frac{3}{4}g$	$1 \to 8$
light blue	purple	orange	yellow	



Scalar $\mathcal{PR}$ a CIC with two commuting generators

s a complex variable.

$f(s) = s$ $\qquad$ $g(s) \equiv 1$

A generating set $\mathbf{X}_0 = \{f, g\}$

$\left(\left(\left((3f + 2g)^{-1} + 6f\right)^{-1} + \frac{1}{5}g\right)\right)^{-1} \in \mathbf{X}_3 \subset \text{cic}(f, g)$

$$\mathcal{PR} = \text{cic}(f, g)$$

Theorem: A maximal CIC of weakly stable real rational functions which includes $g(s) \equiv 1$, is the set $\mathcal{PR}$

Convex Invertible Cones with Non-Commuting generators ?

CIC - Electrical Networks Interpretation

Driving point immittance of R-L-C electrical networks

W. Cauer 1926 **O. Brune 1931**

The set $\mathcal{PR}$ is a Convex Invertible Cone CIC:

closed under (i) positive scaling (ii) addition (iii) inversion.

Electrical networks CIC interpretation of $\mathcal{PR}$

Scaling - transformer ratio

Summation - series connection of impedances

Inversion - impedance / admittance

CICs of Rational Functions

$$\mathcal{PR} := \{ \ f(s) \text{ real rational } : \text{ within } \mathbb{C}_r \text{ analytic } \& \text{ Re}\,(f(s)) \geq 0 \ \}$$

$$\mathcal{PRO} := \{ \ f(s) \in \mathcal{PR} \ : \ f(-s) = -f(s) \ \}$$

Strictly Positive Real

$$\mathcal{SPR} := \{ \ f(s) \text{ real rational } : \ \exists \epsilon > 0 \quad \text{s.t.} \quad f(s-\epsilon) \in \mathcal{PR} \ \}$$

$$\mathcal{SPR}_{\text{DP}} := \{ \ f(s) \in \mathcal{SPR} \ : \ \exists \lim_{s \longrightarrow \infty} f(s) \ \& \text{ is positive } \}$$

CICs of Rational Functions - (cont.)

Observation:

- $\mathcal{PR}$ a maximal (weakly) stable CIC
- $\mathcal{SPR}$ & $\mathcal{PRO}$ non-intersecting subCICs of $\mathcal{PR}$
- $\mathcal{SPR}_{\mathrm{DP}}$ a subCIC of $\mathcal{SPR}$

CICs of Rational Functions - Example

$$f(s) = d + \frac{b}{s+a} \quad a,\ b,\ d \quad \text{parameters}$$

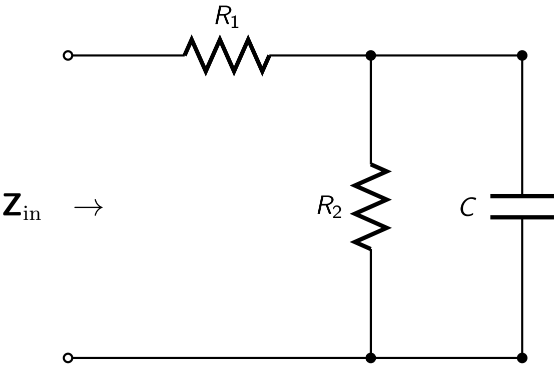
$$f \in \mathcal{PR} \quad \Longleftrightarrow \quad a \geq 0,\ b \geq 0,\ d \geq 0,$$

$$f \in \mathcal{PRO} \quad \Longleftrightarrow \quad a = 0,\ b > 0,\ d = 0,$$

$$f \in \mathcal{SPR} \quad \Longleftrightarrow \quad \text{either} \ \begin{matrix} ab>0 \\ d\geq 0 \end{matrix} \ \text{or} \ \begin{matrix} a=b=0 \\ d>0 \end{matrix}$$

$$f \in \mathcal{SPR}_{\mathrm{DP}} \quad \Longleftrightarrow \quad d > 0 \ \text{ and either } \ ab > 0 \ \text{ or } \ a=b=0.$$

CIC - Electrical Networks Interpretation (cont.)



A circuit realizing the function $\mathrm{Z}_{\text{in}}(s) = R_1 + \frac{\frac{1}{C}}{s+\frac{1}{R_2 C}}$.

$$f(s) = d + \frac{b}{s+a} \qquad a = \frac{1}{R_2 C} \quad b = \frac{1}{C} \quad d = R_1$$

Positive Real - dissipativity

Electro-Mechanical duality

electical	mechanical
current	force
voltage	velocity
transformer	gear transmission
resistor (admittance)	damper
inductor (admittance)	spring
capacitor (admittance)	inerter

Malcolm Smith, Cambridg UK, $\sim$ 2009 : Inerter replaces Mass

Positive Real Odd functions

$$f \in \mathcal{PRO} \quad \Longleftrightarrow \quad f \in \mathcal{PR} \;\; \& \;\; f(-s) = -f(s)$$

$f \in \mathcal{PRO} \quad \Longrightarrow \quad f$ maps each $\mathbb{C}_l$ & $\mathbb{C}_r$ to itself

Driving point immittance of L-C circuits (Lossless, Foster).

R. M. Foster (1924)

$$f \in \mathcal{PRO} \iff f(s) = a_o s + \frac{b_o}{s} + \sum_j \left(a_j s + \frac{b_j}{s} \right)^{-1} \quad a_j, b_j \geq 0.$$

Theorem [Cohen, L. 2007] $\qquad \mathcal{PRO} = \{ \text{cic}(f) \ : \ f(s) = s \}$

$\mathcal{PRO}$ - a subCIC of $\mathcal{PR}$

Positive Real Odd functions (cont.)

Theorem [Cohen, L. 2007] Over a real unital algebra

$$\text{cic}(A) = \{ \, f(A) \ : \ f \in \mathcal{PRO} \, \}$$

Example $A, B \in \mathbb{R}^{n \times n}$

$$B \in \text{cic}(A) \iff B = f(A) \ , \ f \in \mathcal{PRO}$$

Foster $\implies$ $$B = {}_{a_o}A + {}_{b_o}A^{-1}s + \sum_j \left({}_{a_j}A + {}_{b_j}A^{-1}\right)^{-1} \qquad a_j, b_j \geq 0.$$

$$\mathcal{PR} := \{\ f(s) \text{ real rational } : \text{ within } \mathbb{C}_r \text{ analytic \& } \mathrm{Re}\,(f(s)) \geq 0\ \}$$

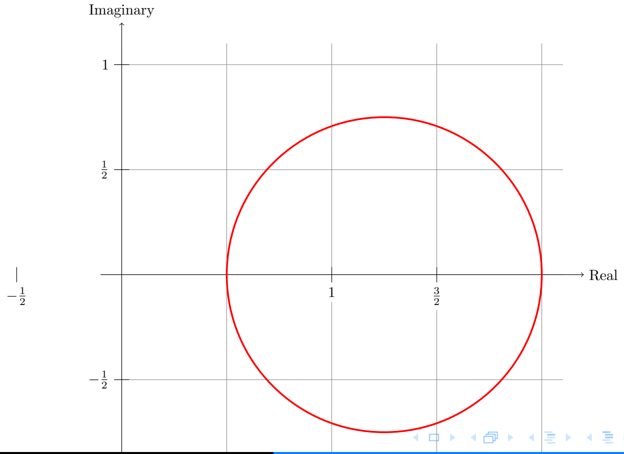
Strictly Positive Real

$$\mathcal{SPR} := \{\ f(s) \text{ real rational } : \exists\epsilon > 0 \quad \text{s.t.} \quad f(s-\epsilon) \in \mathcal{PR}\ \}$$

$$\mathcal{SPR}_{\mathrm{DP}} := \{\ f(s) \in \mathcal{SPR} : \exists \lim_{s \longrightarrow \infty} f(s) \text{ \& is positive }\}$$

Absolutely Stable Rational Functions (cont.)

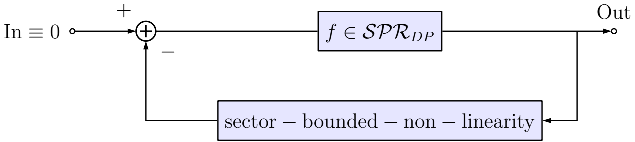
Nyquist plot (in red) of $f(s) = \frac{1}{2}\left(1 + \frac{3}{s+1}\right)$



Absolutely Stable Rational Functions (cont.)

$\mathcal{SPR}_{\mathrm{DP}} \approx$ **absolute stability (Popov)**

$\approx$ **"input & output strictly passive"**



$G, H \quad m \times m -$ valued rational (not necessarily) $\mathcal{PR}$ functions

A step-by-step construction of cic$(G,\ H)$

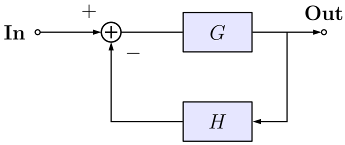
$$\mathbf{X}_o = \{G,\ H\}$$

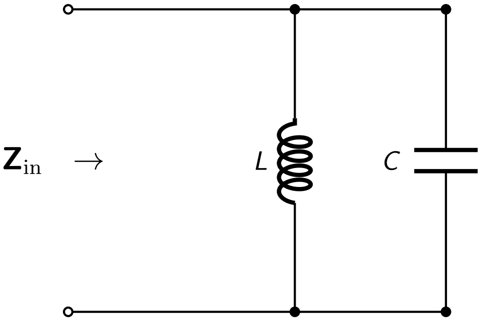
$$\det(G) \not\equiv 0 \quad \Longrightarrow \quad \left(G^{-1} + H\right) \quad \in \quad \mathbf{X}_1$$

$$\left.\begin{array}{c}\det(G) \not\equiv 0 \\ \det\left(G^{-1} + H\right) \not\equiv 0\end{array}\right\} \quad \Longrightarrow \quad \left(G^{-1} + H\right)^{-1} \quad \in \quad \mathbf{X}_2$$

Feedback - CIC (cont.)

$$\mathbf{Out} = \left(\boldsymbol{G}^{-1} + \boldsymbol{H}\right)^{-1} \mathbf{In}$$





A circuit realizing the function $Z_{\mathrm{in}}(s) = \left((sL)^{-1} + sC\right)^{-1}$.

Feedback - CIC (cont.)

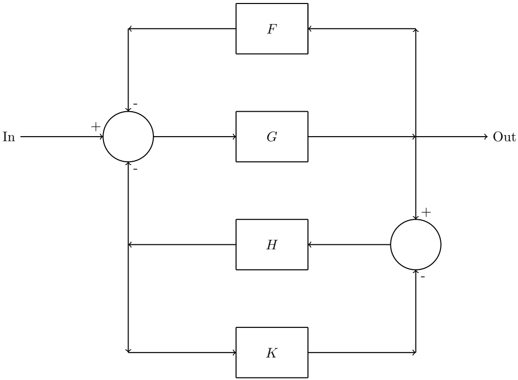
$$\text{Out} = \left(G^{-1} + H\right)^{-1} \text{In}$$

$\mathbf{X}_o = \{G,\ H\}$ $\quad m \times m -$ valued rational (not necessarily) $\mathcal{PR}$ functions

$$\left.\begin{array}{c} \det(G)\not\equiv 0 \\ \det\left(G^{-1}+H\right)\not\equiv 0 \end{array}\right\} \implies \left(G^{-1} + H\right)^{-1} \in \mathbf{X}_2 \subset \text{cic}(G,\ H)$$

How to extend Darlington/Brune/Bott-Duffin etc. synthesis schemes to NON-COMMUTATIVE networks of feedback-loops?

$$\mathbf{Out} = \left(F + G^{-1} + \left(K + H^{-1}\right)^{-1}\right)^{-1} \mathbf{In}$$



Feedback - CIC (cont.)

$\mathbf{X}_o = \{F,\ G,\ H,\ K\}$ $\quad m \times m -$ valued rational (not necessarily) $\mathcal{PR}$ functions

$$\left(F + G^{-1} + \left(K + H^{-1}\right)^{-1}\right)^{-1} \in \mathbf{X}_3 \subset \mathrm{cic}(F,\ G,\ H,\ K)$$

X_3 contains multiple feebback-loops

$\vdots$

cic$(F,\ G,\ H,\ \ldots\ )$ rational functions of non-commuting variables.

How to extend Darlington/Brune/Bott-Duffin etc. synthesis schemes to ellaborate (MIMO) networks of feedback-loops?

Convex Invertible Cones of matrices - examples

- The set $\overline{\mathrm{P}}_n$ of $n \times n$ positive semi-definite matrices is a CIC.

  The set P_n of $n \times n$ positive definite matrices is

  a non-singular subCIC.

- The set of Hurwitz stable matrices, an invertible cone,

  but not convex

$$A = \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix} \quad B = A^T \quad \Longrightarrow \quad \tfrac{1}{2}(A+B) = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$$

eigenvalues $\tfrac{1}{2}(A+B) = -4, +2$ unstable.

- The set of Hamiltonian matrices $\begin{pmatrix} R & W \\ M & -R^* \end{pmatrix}$ $\begin{smallmatrix} W \in \overline{\mathrm{H}}_n \\ M \in \overline{\mathrm{H}}_m \end{smallmatrix}$ is a CIC:

$$\begin{pmatrix} 0 & -I_m \\ I_n & 0 \end{pmatrix} \begin{pmatrix} R & W \\ M & -R^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} R & W \\ M & -R^* \end{pmatrix} \begin{pmatrix} 0 & -I_m \\ I_n & 0 \end{pmatrix} \quad \text{are in } \overline{\mathrm{H}}_{m+n}$$

$$HA + A^*H = Q$$

$\lambda_j(A) + \lambda_k^*(A) \neq 0 \quad {}_{n \geq j \geq k \geq 1} \quad \Longrightarrow$

For each $Q \in \mathrm{P}_n$ a unique solution $H \in \mathrm{H}_n$

Theorem [M.A. Lyapunov 1892, F.R. Gantmacher 1948]

A is Hurwitz stable $\Longleftrightarrow$ $-H \in \mathrm{P}_n$

CICs & the Lyapunov Equation

$A \in \mathbb{R}^{n \times n}$ $\qquad \lambda_j(A) + \lambda_k^*(A) \neq 0 \quad {\scriptstyle n \geq j \geq k \geq 1}$

(stronger than regular inertia)

$$HA + A^*H = Q \in \mathrm{P}_n$$

- $\alpha > 0 \qquad H\alpha A + \alpha A^* H = \alpha Q$
- $HA^{-1} + A^{-*}H = A^{-*}QA^{-1}$
- $HA + A^*H = Q_a \in \mathrm{P}_n \qquad HB + B^*H = Q_b \in \mathrm{P}_n$

$$H(A+B) + (A+B)^*H = Q_a + Q_b$$

CICs & the Lyapunov Equation (cont.)

$H = H^*$ non-singular $\qquad$ **P** positive definite matrices

$$\mathbf{L}_H = \{ \, A \; : \; HA + A^*H \in \mathbf{P} \, \}$$

$A, \; B, \; C \; \ldots \in \mathbf{L}_H \quad \Longrightarrow$

$$(A + 2B^{-1})^{-1} + \tfrac{1}{2}C + \; \ldots \; \in \mathbf{L}_H$$

Rational functions of k non-commuting variables mapping

$$(\mathbf{L}_I)^k \text{ to } \mathbf{L}_I$$

$H\mathbf{L}_H = \mathbf{L}_I$

CICs & the Lyapunov Equation (cont.)

$$\mathsf{L}_H = \{ \ A \ : \ HA + A^* H \in \mathsf{P} \ \}$$

Differential inclusion : $\frac{dx}{dt} = \begin{cases} Ax \\ Bx \\ Cx \\ \vdots \end{cases}$ **the Devil's switching rule**

$A, \ B, \ C \ \ldots \in \mathsf{L}_H$ **for some** $-H \in \mathsf{P} \quad \Longrightarrow$

the differential inclusion is exponentially stable

$x^* Hx$ **"common quadratic Lyapunov function"** **see e.g.**

T. Laffey, O. Mason, K.S. Narendra, R. Shorten, H. Šmigoc

Matlab : Linear Matrix Inequalities (LMI) -toolbox

$A \in \mathbb{C}^{n\times n}$ Inertia$(A) = (\nu,\ \delta,\ \pi)$ $\quad \nu + \delta + \pi = n$

ν number of eigenvalues in $\mathbb{C}_l$

π number of eigenvalues in $\mathbb{C}_r$

δ number of eigenvalues on $i\mathbb{R}$

$\nu = n \iff$ Hurwitz stable

$\delta = 0 \iff$ Regular inertia

Theorem [A. Ostrowsky & H. Schneider, 1962]

$\lambda_j(A) + \lambda_k^*(A) \neq 0 \quad {}_{n\geq j\geq k\geq 1}$ $\qquad H \in \mathbf{H}_n$

$HA + A^*H \in \mathbf{P}_n \implies$ inertia(A) =inertia(H) (& regular)

$H = H^*$ non-singular P positive definite matrices

$$\mathbf{L}_H = \{ A \ : \ HA + A^*H \in \mathbf{P} \}$$

$\mathbf{L}_H$ a maximal open CIC of matrices

sharing the same inertia (as H)

Maximality

Any B s.t. $HB + B^*H$ has a negative eigenvalue $\Longrightarrow$

$\exists A$ in $\mathbf{L}_H$ s.t. $A + B$ has an eigenvalue on $i\mathbb{R}$

$H = H^*$ non-singular $\qquad$ **P** positive definite matrices

$$\mathbf{L}_H = \{ \, A \; : \; HA + A^*H \in \mathbf{P} \, \}$$

Theorem [T. Ando, 2001, 2004]

A maximal open Convex Invertible Cone of matrices sharing the same inertia (along with additional conditions) is a set $\mathbf{L}_H$.

Conclusion: $\mathbf{L}_H$ a "prototype" of all maximal open non-singular CICs

Positive Real Lemma

State space realization

$F(s)$ $m \times m$-valued rational function

$$\exists \lim_{s \to \infty} F(s) \quad \Longrightarrow \quad F(s) = C(sI - A)^{-1}B + D$$

$\mathcal{PR}$ Lemma = Kalman-Yakubovich-Popov Lemma

$$F \in \mathcal{PR} \qquad \Longleftrightarrow \qquad \exists \quad H \in \mathbf{P}_n \ \text{ s.t.}$$

$$\begin{pmatrix} -H & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} -H & 0 \\ 0 & I_m \end{pmatrix} \in \overline{\mathbf{P}}_{n+m}$$

$$F \in \mathcal{SPR}_{\mathrm{DP}} \qquad \Longleftrightarrow \qquad \exists \quad H \in \mathbf{P}_n \ \text{ s.t.}$$

$$\begin{pmatrix} -H & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} -H & 0 \\ 0 & I_m \end{pmatrix} \in \mathbf{P}_{n+m}$$

Positive real Lemma (cont.)

$F \in \mathcal{PR} \quad \Longleftrightarrow \quad \exists \quad H \in \mathbf{P}_n$ s.t.

$$\begin{pmatrix} -H & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} -H & 0 \\ 0 & I_m \end{pmatrix} \in \overline{\mathbf{P}}_{n+m}$$

- B. Dickinson, Ph. Delsarte, Y. Genin & Y. Kamp, 1985
- Minimality of the realization
- A typical LMI
- $\mathcal{PR}$ a maximal CIC of weakly stable rational functions
- $\mathbf{L}_{\begin{pmatrix} -H & 0 \\ 0 & I_m \end{pmatrix}}$ with $H \in \mathbf{P}_n$ a CIC of realization of $m \times m$-valued $\mathcal{PR}$ functions of McMillan degree of at most n

Partial list of topics left out

- Convex Invertible Sets
- Matrix-valued (generalized) positive rational functions
- Monotone functions and operators $(\tan, \arctan)$
- Nevalinna-Pick interpolation
- Matrix Sign Function (algorithm)
- Lyapunov (partial) Order of matrices
- CICs of matrices sharing common solution to the Sylvester & Riccati equations
- Matrix convexity

Concluding message

Engineering/Physical motivation:

"Dissipativity" (continuous time) $\Longrightarrow$ (hidden) CIC structure.

Math motivation:

Positive (real) functions of non-commuting variables.

(Not necessarily positive) rational functions of non-commuting variables

- **Helton's NC algebra software**
- **First book, D. S. Kaliuzhnyi & V. Vinnikov, 2014**

THANKS FOR YOUR ATTENTION