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On the Norm of Linear Combinations of Projections and Some Characterizations of Hilbert Spaces

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Dedicated to Albrecht Böttcher on the occasion of his 60th birthday

Abstract. Let $\mathcal{B}$ be a Banach space and let P, Q $(P, Q \neq 0)$ be two complementary projections in $\mathcal{B}$ (i.e., $P + Q = I$). For $\dim \mathcal{B} > 2$ we show that formulas of the kind $\|aP + bQ\| = f(a, b, \|P\|)$ hold if and only if the norm in $\mathcal{B}$ can be induced by an inner product. The two-dimensional case needs special consideration which is done in the last two sections.

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1. Introduction

Let $\mathcal{B}$ denote a Banach space (real or complex) with $\dim \mathcal{B} > 1$. If the norm in the space $\mathcal{B}$ can be induced by some inner product (i.e., $\|f\|^2 = (f, f)$), it is called Hilbert space. Note, right away, that for the problems considered below the completeness of the space plays no role, and therefore, instead of a Banach (resp. a Hilbert) space one may consider below a normed space (resp. a space with the norm generated by an inner product, i.e., a pre-Hilbert space).

An operator P acting in $\mathcal{B}$ is called a projection if $P^2 = P$. If, in addition, $P \notin \{0, I\}$, we call the projection *nontrivial*.

Consider a linear combination $A = aP + bQ$, where $a, b \in \mathbb{C}$ (or $a, b \in \mathbb{R}$) and $Q = I - P$. The following statement was formulated in [7, Theorem 1] and proved in [8, Theorem 1.1] (see also [5, Example 3.12]).

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Theorem 1.1. *Let $\mathcal{H}$ be a Hilbert space and P a nontrivial projection on $\mathcal{H}$. Then*

$$\|aP + bQ\| = \frac{\gamma_+ + \gamma_-}{2}, \text{ where } \gamma_\pm = \sqrt{(|a| \pm |b|)^2 + |a-b|^2(\|P\|^2 - 1)}. \quad (1.1)$$

It follows from (1.1) that

$$\|I - P\| = \|P\| \quad (1.2)$$

(Ljance's formula, [11]) and

$$\|S\| = \|P\| + \sqrt{\|P\|^2 - 1}, \quad \text{where} \quad S := P - Q$$

(Spitkovsky's formula [12], see also [13]). In particular,

$$\|P\| = 1 \quad \Longrightarrow \quad \|Q\| = \|S\| = 1. \quad (1.3)$$

In this paper we obtain some characterizations of Hilbert spaces in terms of projections, and study the Banach spaces for which some analogues of equality (1.1) hold.

We start with the following result.

Theorem 1.2. *If a Banach space $\mathcal{B}$ satisfies condition* (1.1)*, then $\mathcal{B}$ is a Hilbert space.*

Thus, equality (1.1) is a characterization of Hilbert space.

The proof of Theorem 1.2 will be given in Section 3. The following characterization of Hilbert spaces can be extracted from this proof.

Theorem 1.3. *If in a real reflexive Banach space $\mathcal{B}$ the implication*

$$\|P\| = 1 \quad \Longrightarrow \quad \|2P - I\| = 1$$

holds for every projection P, then $\mathcal{B}$ is a Hilbert space.

Equality (1.1) prompted us to formulate

Problem 1.4. *To describe Banach spaces $\mathcal{B}$ which satisfy the following condition:*

Condition 1.5. *There exists a function $f(a, b, x) = f_{\mathcal{B}}(a, b, x)$ such that*

$$\|aP + bQ\| = f_{\mathcal{B}}(a, b, \|P\|) \quad (Q = I - P) \quad (1.4)$$

for all $a, b \in \mathbb{C}$ $(a, b \in \mathbb{R})$ and all nontrivial projections P.

The solution of the Problem 1.4 for the case $\dim \mathcal{B} > 2$ is given by the following.

Theorem 1.6. *Let* $\dim \mathcal{B} > 2$. *There exists a function $f_{\mathcal{B}}(a, b, x)$ such that Condition* 1.5 *holds if and only if $\mathcal{B}$ is a Hilbert space.*

This theorem gives another characterization of Hilbert spaces (for $\dim \mathcal{B} > 2$). Theorem 1.6 follows directly from the following two statements.

Theorem 1.7. *Let $\mathcal{B}$ be a Banach space satisfying Condition* 1.5*. Then the equality $\|I - P\| = \|P\|$ holds for all nontrivial projections P.*

Theorem 1.8. *Let $\mathcal{B}$ be a Banach space and* $\dim \mathcal{B} > 2$. *If* $\|I - P\| = \|P\|$ *holds for all nontrivial projections P, then $\mathcal{B}$ is a Hilbert space.*

A sketch of the proof of these two theorems is given in [8, Subsection 1.3]. Below more detailed proofs of Theorems 1.7 and 1.8 are given. As for Theorem 1.8, it is more convenient for us to prove the following stronger proposition.

Theorem 1.9. *Let* $\dim \mathcal{B} > 2$. *If for every one-dimensional projection P in $\mathcal{B}$ the condition*

$$\|P\| = 1 \quad \Longrightarrow \quad \|I - P\| = 1 \tag{1.5}$$

holds, then $\mathcal{B}$ is a Hilbert space.

Theorem 1.9 for the real spaces is well known [1, p. 144]. We give in Section 2 another proof of this theorem which works for complex spaces as well.

Remark 1.10. Theorem 1.8 fails if $\dim \mathcal{B} = 2$. There exists a wide class of non-Hilbert norms in two-dimensional (real and complex) spaces, where equality (1.2) holds for any one-dimensional projection P. See Sections 4 and 5.

Remark 1.11. In contrast with the case $\dim \mathcal{B} > 2$, the statement converse to Theorem 1.7 fails in two-dimensional spaces. See Proposition 4.7.

We conclude the introduction with an open

Question 1.12. *Do there exist two-dimensional non-Hilbert Banach spaces for which Condition* 1.5 *holds?*

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2. Proof of Theorems 1.7 and 1.9

2.1. We start with the proof of Theorem 1.7.

Proof. Let Condition 1.5 be fulfilled in some Banach space $\mathcal{B}$. We set in equality (1.4) $a = 0, b = 1$ and obtain $\|Q\| = f_{\mathcal{B}}(0, 1, \|P\|)$. Denote the function $f_{\mathcal{B}}(0, 1, \|P\|)$ by $h(\|P\|)$ and we obtain

$$\|I - P\| = h(\|P\|). \tag{2.1}$$

Let $y_0 \in \mathcal{B}$, $f_0 \in \mathcal{B}^*$ and $f_0(y_0) = \|y_0\| = \|f_0\| = 1$. We choose a non-zero vector z_0 such that $f_0(z_0) = 0$, and consider the one-dimensional projections $P_t = f_0(\cdot)(y_0 + tz_0)$. Then $\|P_t\| = \|y_0 + tz_0\|$ and $\|P_t\| \to +\infty$ when $t \to +\infty$. It follows that $\|P_t\|$ $(t \geq 0)$ takes all values form $[1, +\infty)$, and hence $h(x)$ is defined on $[1, +\infty)$.

Let $x \in [1, +\infty)$ and $\|P\| = x$. Then

$$h(x) = h(\|P\|) = \|I - P\| \geq \|P\| - 1 = x - 1,$$

therefore

$$\lim_{x \to +\infty} h(x) = +\infty.$$

Let us show that $h(x)$ is a monotone function. It is clear that

$$h(h(x)) = h(\|I - P\|) = \|I - (I - P)\| = \|P\| = x,$$

and hence,

$$h(a) = h(b) \quad \Longrightarrow \quad h(h(a)) = h(h(b)) \quad \Longrightarrow \quad a = b.$$

Thus, we proved that $h(x)$ is an increasing function on $[1, \infty)$ and $h(h(x)) = x$. It follows that $h(x) = x$ for all $x \in [1, \infty)$. Indeed, assume that $h(a) \geq a$ for some a. Then $a = h(h(a)) \geq h(a)$, i.e., $h(a) = a$. The same follows if we assume that $h(a) \leq a$ for some a. This proves that $h(x) \equiv x$. Using equality (2.1), we obtain that $\|I - P\| = \|P\|$. □

2.2. In this subsection we prove Theorem 1.9. We start with the following lemma.

Lemma 2.1. *If a Banach space* $\mathcal{B}$ ($\dim \mathcal{B} > 3$) *satisfies the condition of Theorem* 1.9 (*i.e., condition* (1.5)) *then any three-dimensional subspace* $E \subset \mathcal{B}$ *also satisfies the condition of Theorem* 1.9.

Proof. Let P_0 be an arbitrary one-dimensional projection in E with $\|P_0\| = 1$. Let us write $P_0 = f_0(\cdot)z_0$. We may assume that $\|z_0\| = 1$ and $\|f_0\| = 1$. Since $z_0 \in \operatorname{Im} P_0$ we have $P_0 z_0 = z_0$, and since $P_0 z_0 = f_0(z_0)z_0$ we have $f_0(z_0) = 1$. Using the Hahn–Banach theorem we obtain a functional $f_1 \in \mathcal{B}^*$ such that

$$f_1(u) = f_0(u) \ (u \in E) \quad \text{and} \quad \|f_1\| = \|f_0\|.$$

In particular, $f_1(z_0) = f_0(z_0) = 1$.

If $P := f_1(\cdot)z_0$, then P is a one-dimensional projection in $\mathcal{B}$ and $\|P\| = \|f_1\|\|z_0\| = 1$. Obviously, for $u \in E$ the following equalities hold:

$$Pu = f_1(u)z_0 = f_0(u)z_0 = P_0 u.$$

Since condition (1.5) is satisfied in the space $\mathcal{B}$, we have $\|I - P\| = 1$. Furthermore for $u \in E$ we have $(I - P)u = (I - P_0)u$, and hence $\|I - P_0\| \leq \|I - P\| = 1$. But the norm of a non-zero projection cannot be less than 1. Hence, $\|I - P_0\| = 1$. So, the subspace E also satisfies the condition of Theorem 1.9, and the lemma is proved. □

In the proof of Theorem 1.9 we use the following statement.

Proposition 2.2. *A Banach space* $\mathcal{B}$ *of dimension* ≥ 3 *is a Hilbert space if and only if every two-dimensional subspace* M *of* $\mathcal{B}$ *is the range of a projection* P *of norm* 1.

For the real case this statement was proved by S. Kakutani [10] and for the complex case by F. Bohnenblust [4].

Proof of Theorem 1.9. First we prove the theorem for the case $\dim \mathcal{B} = 3$. Let M be any two-dimensional subspace of $\mathcal{B}$. There exists a functional $f \in \mathcal{B}^*$ such that $\|f\| = 1$ and $\ker f = M$. Furthermore there exists a vector $z \in \mathcal{B}$ such that $f(z) = 1$ and $\|z\| = 1$. Let $P = f(\cdot)z$. Then P is a one-dimensional projection with $\|P\| = 1$.

By (1.5) the norm of $Q = I - P$ also equals 1. Evidently, $\ker P = \ker f = M$, and hence $\operatorname{Im} Q = \ker P = M$. So, for an arbitrary two-dimensional subspace M of $\mathcal{B}$ there exists a projection P with norm 1 and with range M. By Proposition 2.2 $\mathcal{B}$ is a Hilbert space.

Consider now the case $\dim \mathcal{B} > 3$. Let E be an arbitrary three-dimensional subspace of $\mathcal{B}$. By Lemma 2.1 for each one-dimensional projection P_0 in E relation (1.5) also holds. According to the first part of the proof, E is a Hilbert space. Since every three-dimensional subspace of $\mathcal{B}$ is a Hilbert space, it follows that $\mathcal{B}$ is also a Hilbert space. □

3. Proof of Theorem 1.2

We start with the case when $\mathcal{B}$ is a complex space and we will use the following characterization of Hilbert spaces (see [3, p. 314]).

Proposition 3.1. *Let $\mathcal{B}$ be a complex Banach space. Then $\mathcal{B}$ is a Hilbert space if and only if every its one-dimensional subspace is the range of a Vidav-Hermitian projection.*

Recall that an operator $A \in L(\mathcal{B})$ is called *Vidav-Hermitian*, if

$$\|I - itA\| = 1 + o(t),\ t \in \mathbb{R},\ t \to 0.$$

Proof of Theorem 1.2 *for complex spaces.* Let F be an arbitrary one-dimensional subspace of $\mathcal{B}$ and let P be a projection on this subspace with $\|P\| = 1$. Set in (1.1) $a = 1 + it$ and $b = 1$. Then $|a| = \sqrt{1+t^2}$, $|a - b| = t$, and (1.1) implies $\|(1 + it)P + Q\| = (\gamma_+ + \gamma_-)/2$, where $\gamma_\pm = ||a| \pm |b|| = \sqrt{1+t^2} \pm 1$. Hence

$$\|(1 + it)P + Q\| = \sqrt{1+t^2}, \quad \text{i.e.,} \quad \|I + itP\| - 1 \sim \frac{t^2}{2} = o(t).$$

Thus P is a Vidav-Hermitian projection and we can use Proposition 3.1. This proves the theorem. □

We pass to the case when $\mathcal{B}$ is a real space and we will use the following characterization of Hilbert spaces (see [1, p. 34, Statement (4.8)]).

Proposition 3.2. *Let $\mathcal{B}$ be a real Banach space. Then $\mathcal{B}$ is a Hilbert space if and only if for each maximal subspace M of $\mathcal{B}$ there exists an element $z \notin M$ such that $\|z - u\| = \|z + u\|$ for all $u \in M$.*

Recall that a subspace M of $\mathcal{B}$ is called *maximal* if $\dim(\mathcal{B}/M) = 1$.

Proof of Theorem 1.2 *for real spaces.* Let $\mathcal{B}$ be a Banach space satisfying condition (1.1). It suffices to prove this theorem for any fixed two-dimensional subspace $E \subset \mathcal{B}$. Let M be an arbitrary one-dimensional (and thus maximal) subspace of E. We take $f \in E^*$ such that $\|f\| = 1$ and $\ker f = M$. By the Hahn–Banach theorem, there exists a vector z such that $\|z\| = 1$ and $f(z) = 1$. Consider the following two projections: $P = f \otimes z$ and $Q = I - P$. Note that $\|Px\| = \|f(x)z\| = |f(x)|$, therefore $\|P\| = \|f\| = 1$. Denote by S the operator $S = P - Q$. It follows from (1.3) that $\|S\| = 1$. Since, in addition, $S^2 = (P-Q)^2 = I$, it follows that S is an isometry, i.e., $\|Sx\| = \|x\|$ for all $x \in E$. Let u be an arbitrary vector from M and $x = u + z$. Then

$$Px = f(u+z)z = z, \quad Qx = x - Px = u, \quad Sx = Px - Qx = z - u.$$

Therefore

$$\|z - u\| = \|Sx\| = \|x\| = \|u + z\|.$$

It remains to use Proposition 3.2, and the theorem is proved. □

Proof of Theorem 1.3. In the proof of Theorem 1.2 for a real two-dimensional space $\mathcal{B}$ we have used only the following part of equality (1.3):

$$\|P\| = 1 \Longrightarrow \|S\| = 1 \quad (S = 2P - I).$$

The same proof can be used for any real reflexive space $\mathcal{B}$. This proves Theorem 1.3. □

We do not know if Theorem 1.3 holds for complex spaces.

4. Real two-dimensional spaces

In this section we assume that $\mathcal{B}$ is a two-dimensional space. If the space $\mathcal{B}$ satisfies Condition 1.5, then (by Theorem 1.7) equality (1.2) holds for any one-dimensional projection P acting in this space. If in a two-dimensional space $\mathcal{B}$ equality (1.2) holds, we will call this space (for short) *suitable*. In this section we will use the following.

Theorem 4.1. *Let $\mathcal{B}$ be a two-dimensional space. Then the equation $\|P\| = \|I - P\|$ holds for every one-dimensional projection P if and only if for arbitrary vectors $z \in \mathcal{B}$ and $f \in \mathcal{B}^*$ the following equality holds:*

$$\|f\|^* \|z\| = \|\tilde{z}\|^* \|\tilde{f}\|, \tag{4.1}$$

where $\tilde{f} := (-f_2, f_1)$, $\tilde{z} := (-z_2, z_1)$ and $\|\cdot\|^*$ denotes the norm in the space $\mathcal{B}^*$.

Proof. Let P be a one-dimensional projection, then it can be represented in the form $Px = f(x)z$, where $f(z) = 1$ and $\|P\| = \|f\|^*\|z\|$. Denote $Qx = \tilde{z}(x)\tilde{f}$. It is easy to see that

$$\begin{aligned} Px + Qx &= (f_1x_1 + f_2x_2)(z_1, z_2) + (-z_2x_1 + z_1x_2)(-f_2, f_1) \\ &= (f_1z_1 + f_2z_2)(x_1, x_2) = f(z)x = x, \end{aligned}$$

i.e., $Q = I - P$. Thus,

$$P = f \otimes z \Longleftrightarrow I - P = \tilde{z} \otimes \tilde{f} \quad \text{and} \quad \|I - P\| = \|\tilde{z}\|^* \|\tilde{f}\|.$$

1. Let condition (4.1) be fulfilled, then $\|I - P\| = \|\tilde{z}\|^*\|\tilde{f}\| = \|f\|^*\|z\| = \|P\|$.

2. Let $\|I-P\| = \|P\|$ and let two vectors z, f be given. First, we assume that $f(z) = d \neq 0$, denote $z_0 = z/d$. Then $f(z_0) = 1$ and $P = f \otimes z_0$ is a projection. Equality (4.1) follows from the following one:

$$\|f\|^*\|z_0\| = \|P\| = \|I-P\| = \|\tilde{z_0}\|^*\|\tilde{f}\|.$$

Now we assume that $f(z) = 0$. Denote $z_s := z + sw$, where $f(w) = 1$. If $s \neq 0$, then $f(z_s) \neq 0$ and for the pairs f, z_s we just proved equality (4.1). Passing to the limit when $s \to 0$ we obtain equality (4.1) for the pair f, z. □

Recall the following known statement (see, for example, [2, p. 10]), which will be used below in some examples.

Proposition 4.2. *Let the unit ball in $\mathcal{B}$ have only a finite number of extreme points: w_k $(k = 1, \ldots, 2n)$ and $w_{k+n} = -w_k$. Then the norm of each operator $A \in L(\mathcal{B})$ equals*

$$\|A\| = \max_{k=1,\ldots,n} \|Aw_k\|.$$

Example 4.3. Let us take for the unit ball D in $\mathcal{B}$ a regular hexagon with vertices $w_{1,2,3} = \{(1,0), (\pm 1/2, \sqrt{3}/2)\}$ and $w_{k+3} = -w_k$.

Let $f \in \mathcal{B}^*$. Using Proposition 4.2 it is not difficult to show that

$$\|f\|^* = \max\{|f_1 y_1 + f_2 y_2| : (y_1, y_2) \in \{w_{1,2,3}\}\} = \max\left\{|f_1|, \frac{|f_1| + |f_2|\sqrt{3}}{2}\right\}.$$

It follows from this that the unit ball D^* in the space $\mathcal{B}^*$ is the regular hexagon with vertices

$$\tilde{w}_{1,2,3} = \{(0, 2/\sqrt{3}), (\pm 1, 1/\sqrt{3})\} \quad \text{and} \quad \tilde{w}_{k+3} = -\tilde{w}_k.$$

This implies, that the norm of the vector $z = (z_1, z_2) \in \mathcal{B}$ equals

$$\|z\| = \max\{|z_1 t_1 + z_2 t_2| : (t_1, t_2) \in \tilde{w}_{1,2,3}\} = \max\left\{\frac{2|z_2|}{\sqrt{3}}, \frac{|z_2| + |z_1|\sqrt{3}}{\sqrt{3}}\right\}.$$

Thus

$$\|f\|^*\|z\| = \frac{1}{2}\max\left\{2|f_1|, |f_1| + |f_2|\sqrt{3}\right\} \times \frac{1}{\sqrt{3}}\max\left\{2|z_2|, |z_2| + |z_1|\sqrt{3}\right\}. \quad (4.2)$$

Using the same arguments as above we obtain that

$$\|\tilde{z}\|^*\|\tilde{f}\| = \frac{1}{2}\max\left\{2|z_2|, |z_2| + |z_1|\sqrt{3}\right\} \times \frac{1}{\sqrt{3}}\max\left\{2|f_1|, |f_1| + |f_2|\sqrt{3}\right\}. \quad (4.3)$$

It follows from (4.2) and (4.3) that $\|f\|^*\|z\| = \|\tilde{z}\|^*\|\tilde{f}\|$ and from Theorem 4.1 that $\|P\| = \|I-P\|$, i.e., the space $\mathcal{B}$ is suitable.

Remark 4.4. There exists a general procedure, proposed by M.M. Day ([6, Section 6], see also [1, p. 77]), which turns some non-suitable spaces with the norm

$\|(x,y)\|$ into suitable spaces with the norm

$$\|(x,y)\|_1 := \begin{cases} \|(x,y)\| & \text{if} \quad xy \geq 0 \\ \|(-y,x)\|^* & \text{if} \quad xy \leq 0. \end{cases}$$

Example 4.5. Let $E = \ell_1^2$, and $P(x,y) = (x+y,0)$. Then $\|P\| = 1$ and $\|I-P\| > 1$. Thus, the space E (with the standard norm) is not suitable. But the space $\mathcal{B}$ with the norm

$$\|(x,y)\|_1 := \begin{cases} |x|+|y| & \text{if} \quad xy \geq 0 \\ \max(|x|,|y|) & \text{if} \quad xy \leq 0, \end{cases} \tag{4.4}$$

obtained by the Day procedure, is a suitable space.

Remark 4.6. One can prove directly (using the same arguments as in Example 4.3), that the space $\mathcal{B}$ with the norm (4.4) is suitable.

Proposition 4.7. *In contrast with the case* $\dim \mathcal{B} > 2$, *the statement converse to Theorem* 1.7 *fails for two-dimensional spaces.*

Proof. Let $\mathcal{B}$ denote the real two-dimensional Banach space with the norm defined by equality (4.4), and let $A \in L(B)$. The closed unit ball in $\mathcal{B}$ has only six extremal points: $\pm(1,0),\ \pm(0,1),\ \pm(-1,1)$. By Proposition 4.2 the norm of each operator $A \in L(\mathcal{B})$ equals

$$\|A\| = \max_{1\leq k\leq 3}\{\|Aw_k\| :\ w_1 = (1,0);\ w_2 = (0,1);\ w_3 = (-1,1)\}.$$

Consider two collections of operators:

$$P_t := \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad S_t = I - 2P_t = -\begin{bmatrix} 1 & 2t \\ 0 & -1 \end{bmatrix}.$$

Direct calculations show that

$$\|P_t w_1\| = 1;\ \|P_t w_2\| = |t| \quad \text{and} \quad \|P_t w_3\| = |t-1|.$$

It is easy to check that $\max(1,|a|,|a-1|) = (1+|a|+|a-1|)/2$ for any real number a. Therefore

$$\|P_t\| = \max_{k=1,2,3} \|P_t w_k\| = (|t|+|t-1|+1)/2.$$

Analogous calculations show that

$$\|S_t w_2\| = \begin{cases} 2|t|+1 & \text{if} \quad t<0 \\ 1 & \text{if} \quad 0<2t<1 \\ 2t & \text{if} \quad 2t>1 \end{cases};\ \|S_t w_3\| = \begin{cases} 2-2t & \text{if} \quad 2t\leq 1 \\ 1 & \text{if} \quad 1<2t<2 \\ 2t-1 & \text{if} \quad t>1 \end{cases}$$

and $\|S_t w_1\| = 1$. Therefore

$$\|S_t\| = \max_{k=1,2,3} \|S_t w_k\| = |2t-1|+1.$$

If, in particular, $t \in [0,1]$, then $\|P_t\| \equiv 1$, i.e., it does not depend on t, but $\|S_t\|$ is not constant on this segment. This proves that Condition 1.5 fails in $\mathcal{B}$. □

5. Complex two-dimensional spaces

In order to consider the complex case we need a small introduction into the complexification of real Banach spaces and operators on these spaces. Denote by $\mathcal{B}$ a real two-dimensional normed space with the norm $\|\cdot\|$. By definition its compexification $\mathcal{B}^{\mathbb{C}}$ is the direct sum of $\mathcal{B}$ and $i\mathcal{B}$. Let $\|\cdot\|^{\mathbb{C}}$ denote a norm in $\mathcal{B}^{\mathbb{C}}$.

Definition 5.1. The norm $\|\cdot\|^{\mathbb{C}}$ is a complexification of the norm $\|\cdot\|$ if

$$\|(x,y)\| = \|(x+i0, y+i0)\|^{\mathbb{C}}.$$

The complexification of a norm is not unique [9, p. 288, Problem 25].

Definition 5.2. Let $A \in L(\mathcal{B})$ and $z = u + iv$ $\big(u, v \in \mathcal{B},\ z \in \mathcal{B}^{\mathbb{C}}\big)$. The complexification $A^{\mathbb{C}}$ of operator A is defined by the equality $A^{\mathbb{C}}z := Au + iAv$.

Definition 5.3. We say that a complexification of the norm agrees with the complexification of operators if $\|A^{\mathbb{C}}\|^{\mathbb{C}} = \|A\|$ for all $A \in L(\mathcal{B})$.

Among all complexifications of a given norm there exists minimal and maximal [9, p. 289, Problems 26-28]. On the other hand, there exist complexifications which transfer any suitable real space into a suitable complex space. We will see that this property is possessed, in particular, by the complexification with minimal norm. This minimal norm is defined by equality ([9, p. 289, Problem 27])

$$\|z\|^{\mathbb{C}} = \sup_{f \in \mathcal{B}^*} \frac{|f^{\mathbb{C}}(z)|}{\|f\|}, \quad z = u + iv,\ f^{\mathbb{C}}(z) = f(u) + if(v). \tag{5.1}$$

Theorem 5.4. *If a space $\mathcal{B}$ is suitable, then the space $\mathcal{B}^{\mathbb{C}}$ with the norm* (5.1) *is also suitable. Moreover, the complexification* (5.1) *agrees with the complexification of operators.*

In the proof of this theorem we use the following lemma, proved in [14, Lemma 1 and Theorem 1]. For the convenience of the reader we give here its proof.

Lemma 5.5. *The complexification of the norm $\|\cdot\|$, defined in* (5.1)*, agrees with the complexification of operators.*

Proof. Equality (5.1) can be rewritten in the following form:

$$\begin{aligned}\|z\|^{\mathbb{C}} &= \sup_{f \in \mathcal{B}^*} \frac{|f(u) + if(v)|}{\|f\|} = \sup_{f \in \mathcal{B}^*} \frac{\big(f(u)^2 + f(v))^2\big)^{1/2}}{\|f\|} \\ &= \sup_{f,\theta} \frac{|f(u)\cos\theta + f(v)\sin\theta|}{\|f\|} = \max_{\theta}(\|u\cos\theta + v\sin\theta\|).\end{aligned}$$

It follows from here that

$$\|A^{\mathbb{C}}z\|^{\mathbb{C}} = \max_{\theta}(\|Au\cos\theta + Av\sin\theta\|) \le \|A\| \max_{\theta}(\|u\cos\theta + v\sin\theta\|) = \|A\|\|z\|^{\mathbb{C}}.$$

Hence, $\|A^{\mathbb{C}}\|^{\mathbb{C}} \le \|A\|$.

The inverse inequality is evident, and we obtain $\|A^{\mathbb{C}}\|^{\mathbb{C}} = \|A\|$. □

Proof of Theorem 5.4. The second assertion of the theorem coincides with Lemma 5.5. It follows also from this lemma that $\|P^{\mathbb{C}}\|^{\mathbb{C}} = \|P\|$ for all non-trivial projections P in $\mathcal{B}$. But all non-trivial projections P in $\mathcal{B}$ are one-dimensional, and hence $I-P$ is nontrivial as well. Hence $\|I-P^{\mathbb{C}}\|^{\mathbb{C}} = \|I-P\|$, and equality $\|P\| = \|I-P\|$ implies $\|P^{\mathbb{C}}\|^{\mathbb{C}} = \|I-P^{\mathbb{C}}\|^{\mathbb{C}}$. So, if the space $\mathcal{B}$ is suitable, then $\mathcal{B}^{\mathbb{C}}$ is also suitable. The theorem is proved. □

We conclude this section with the following.

Corollary 5.6. *It follows from Remark* 4.4 *and Theorem* 5.4 *that there exist two-dimensional real and complex Banach spaces which are suitable but non-Hilbert ones.*

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