Algebraic Systems Theory and Coding Theory
In Memory of our friend Uwe Helmke

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Outline of Talk:

1. Convolutional Codes, Basics
2. MDS Convolutional Codes
3. MDP Convolutional Codes
4. Superregular Matrices
5. Decoding over the Erasure Channel

1. Convolutional Codes, Basics

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Since R is a principal ideal domain, the submodule $\mathcal{C}$ is free and there exists a $n \times k$ matrix $G(z)$ such that:

$$\mathcal{C} = \{G(z)m(z) \mid m(z) \in \mathbb{F}^k[z] = R^k\}$$

We call $G(z)$ a generator matrix of the code $\mathcal{C}$ and one says $\mathcal{C}$ has *rate* k/n.

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Without loss of generality one can assume that a generator matrix is column reduced having column degrees $\delta_1, \ldots, \delta_k$.

Basics

Two $n \times k$ generator matrices $G(z)$ and $\tilde{G}(z)$ define the same code if and only if there is a $k \times k$ unimodular matrix $U(z)$ such that

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Remark. The degree is a code parameter and is also equal to the sum of the column degrees: $\delta = \sum_{i=1}^{k} \delta_i$.

It is a major design problem to construct (n, k, δ) codes, i.e. codes having a rate k/n and degree δ such that the code has "good parameters".

Parity Check Matrix

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Theorem. *If $\mathcal{C}$ is an observable (n,k,δ) code, then there exists an $(n-k)\times n$ parity check matrix $H(z)$ such that $\mathcal{C}$ is equivalently described through*

$$\mathcal{C} = \{v(z) \in \mathbb{F}^n[z] \mid H(z)v(z) = 0.\}$$

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$$G(z) = \begin{pmatrix} (z+1)(z+3) \\ (z+1)(z+4) \\ (z+1)(z+5) \end{pmatrix}$$

defines a rate 1/3 convolutional code of degree $\delta = 2$. The code is NOT observable as $G(z)$ is not right prime.

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$$\tilde{G}(z) = \begin{pmatrix} (z+3) \\ (z+4) \\ (z+5) \end{pmatrix}$$

is right prime and the associated convolutional code has therefore a parity check matric $H(z)$ such that

$$\ker_{\mathbb{F}[z]} = H(z) = \mathrm{im}_{\mathbb{F}[z]} \tilde{G}(z).$$

Historical Remarks

Convolutional Codes were introduced by Elias (1955). For this consider an $[n, k]$ linear block code represented by an $n \times k$ generator matrix G,

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It was the idea of Elias to allow polynomial matrices $G(z)$ in the encoding process. Convolutional codes generalize block codes in a natural way.

Engineering Remarks

- Convolutional codes belong to the most widely implemented codes in (wireless) communications. The field is typically $\mathbb{F}_2$ and the rate and the degree are often small. The degree is small so that the Viterbi decoding algorithm is efficient.

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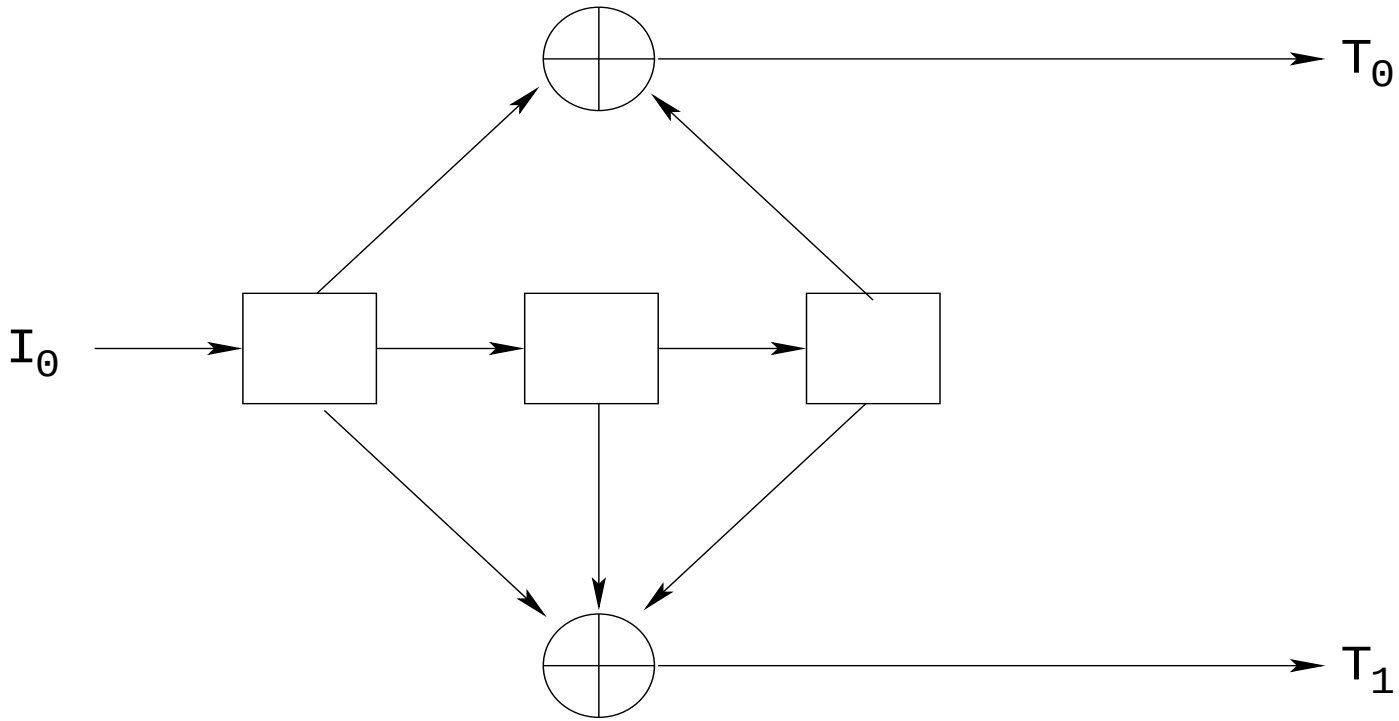
- In collaboration with Tomas and Smarandache [TRS12] we show that in packet switched networks (like e.g. the Internet) convolutional codes over large alphabets have a lot of potentials. See 2015 conference in Banff, Canada.

Feed Forward Implementation

$$G(z) = \begin{pmatrix} z^2+1 \\ z^2+z+1 \end{pmatrix}$$

has the implementation:



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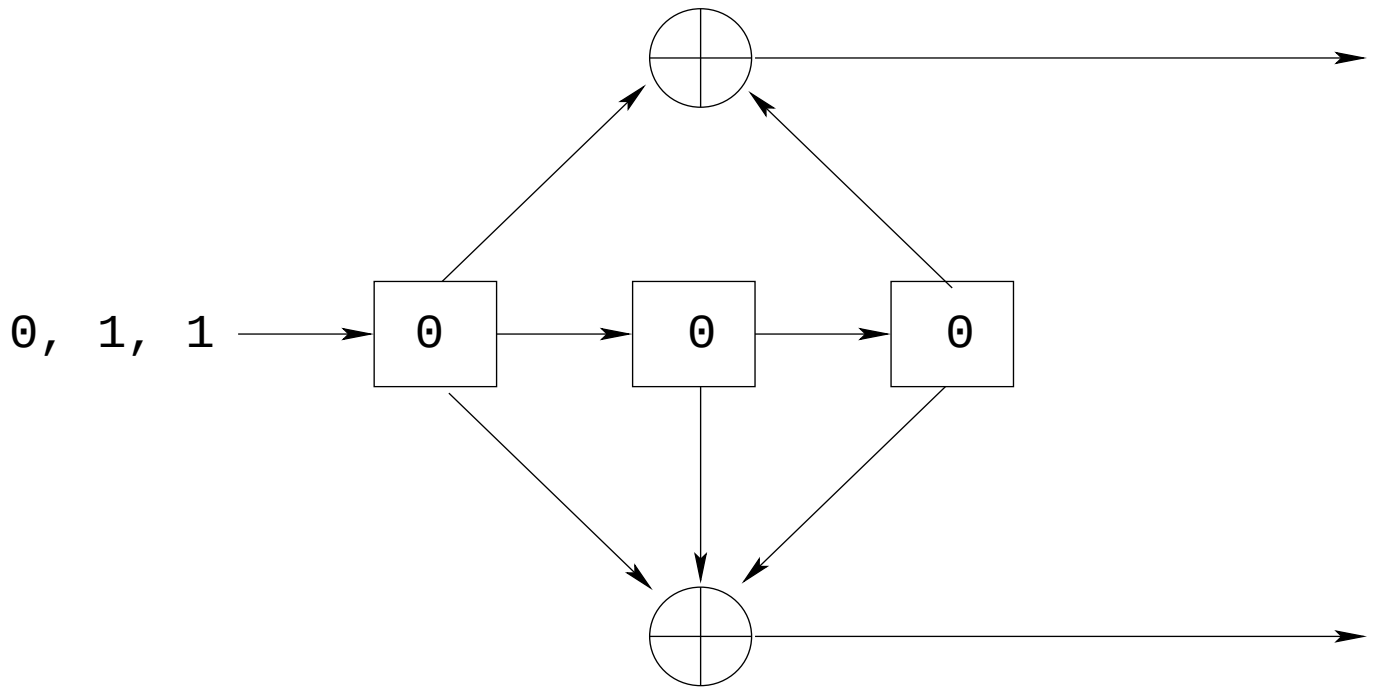

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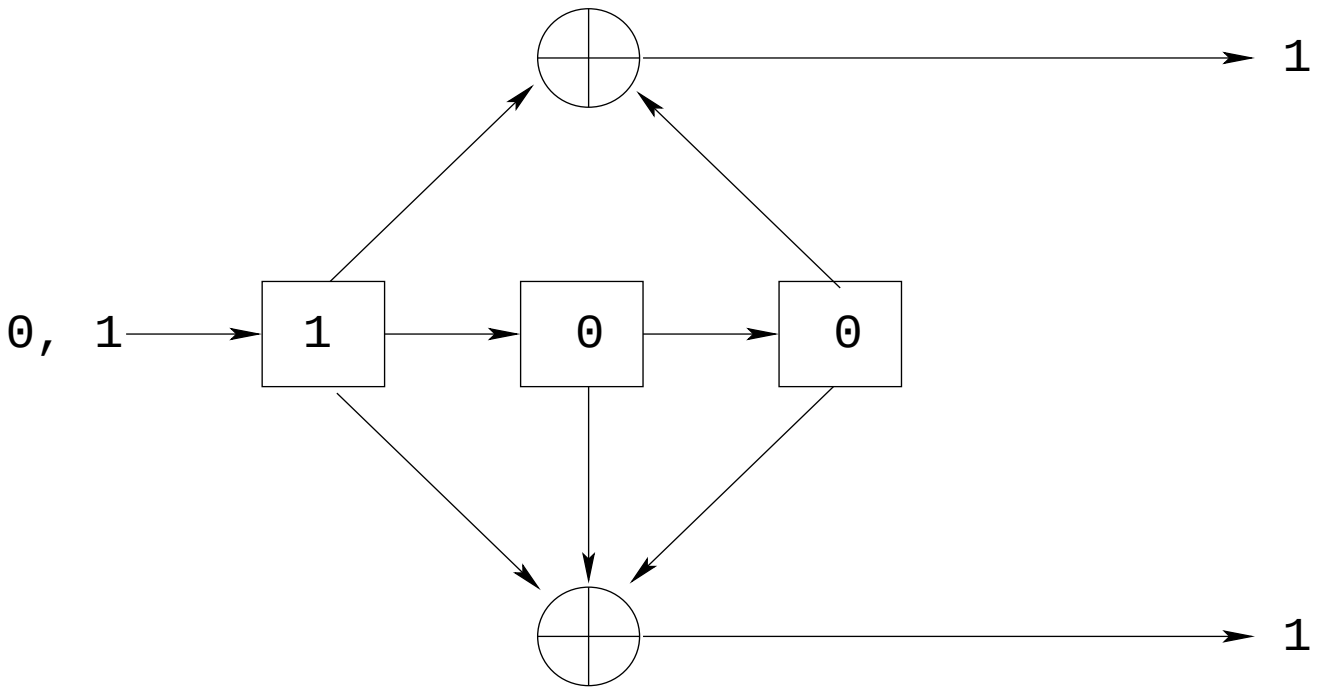

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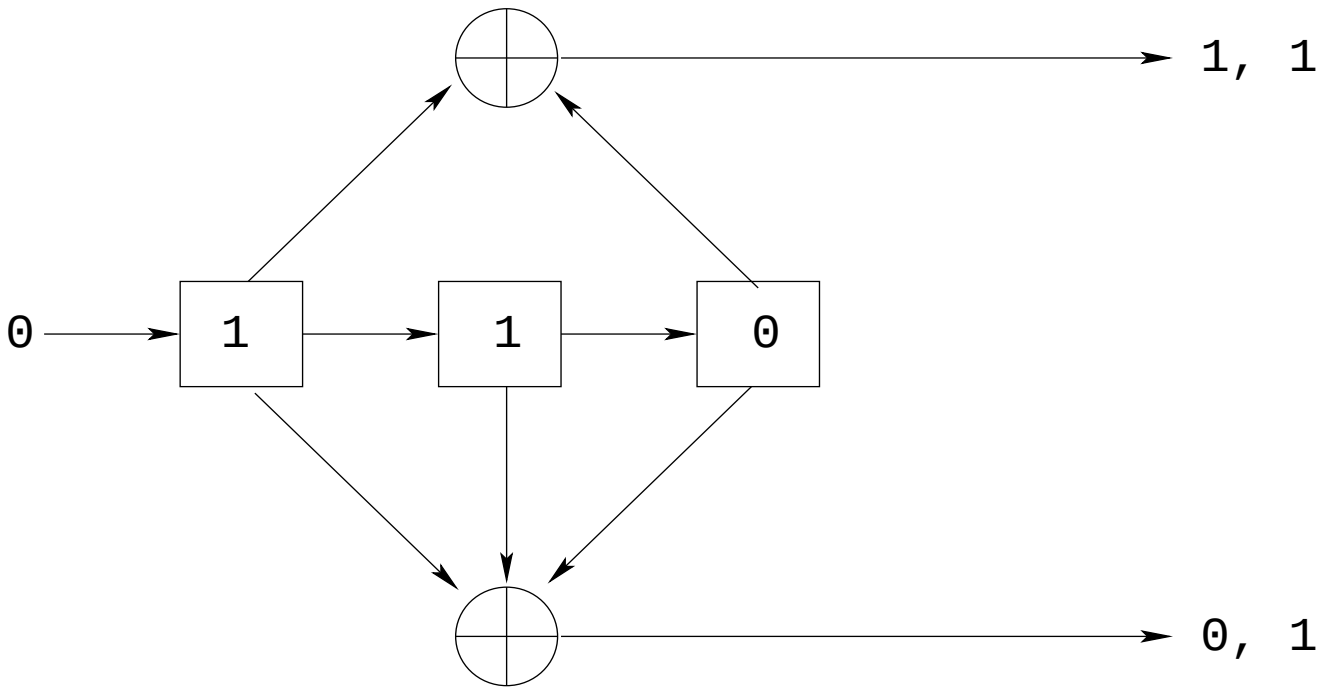

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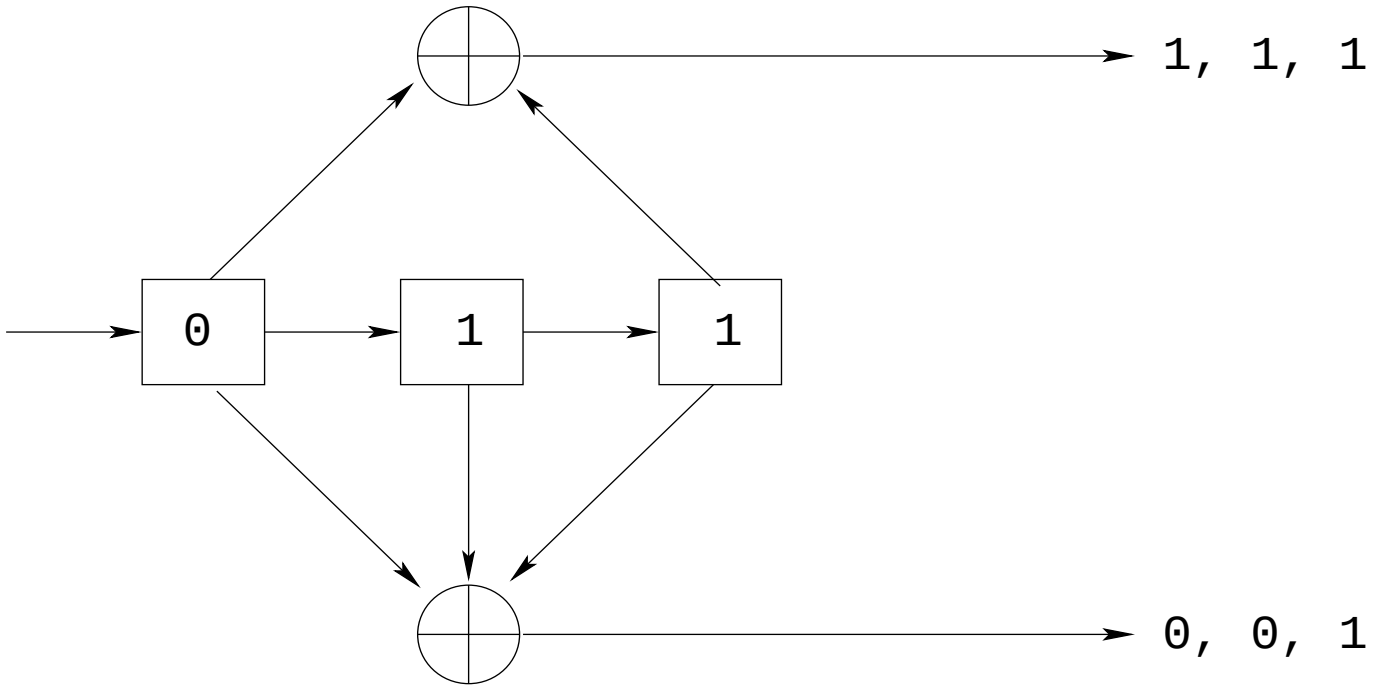

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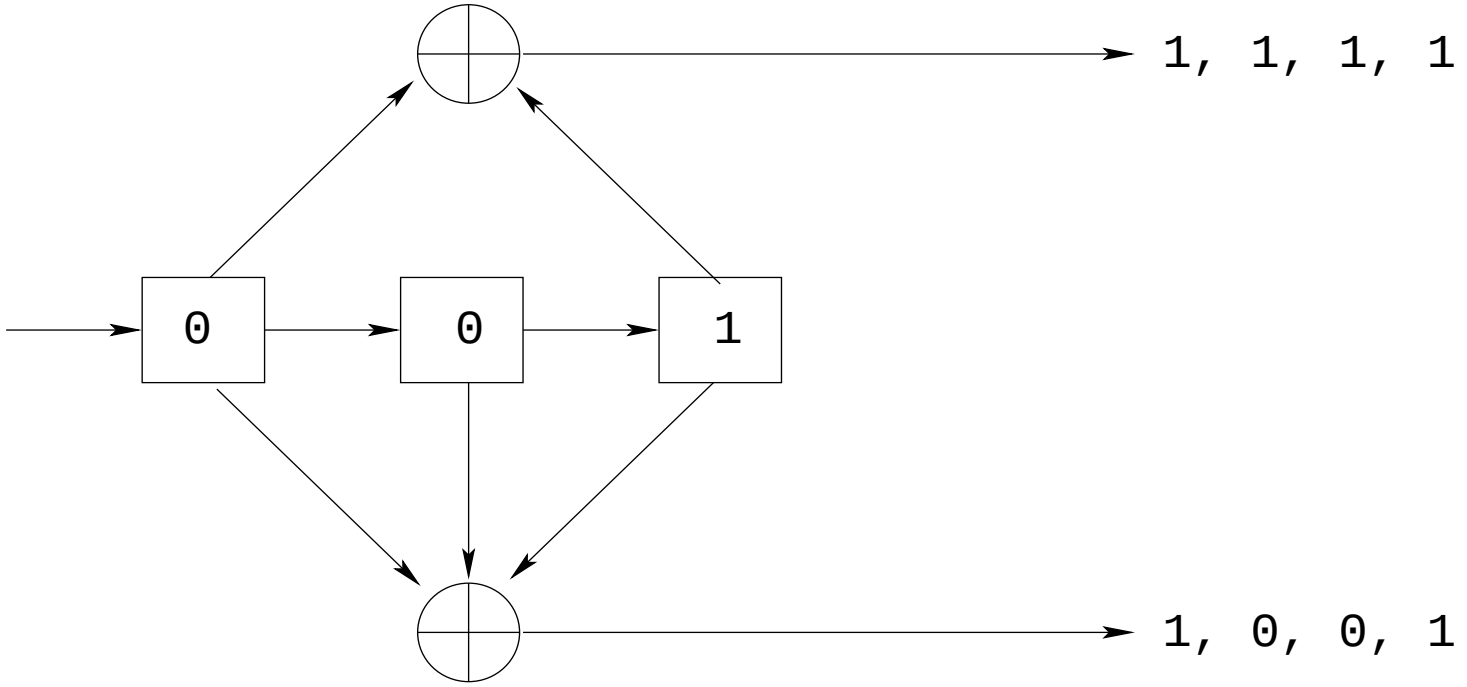

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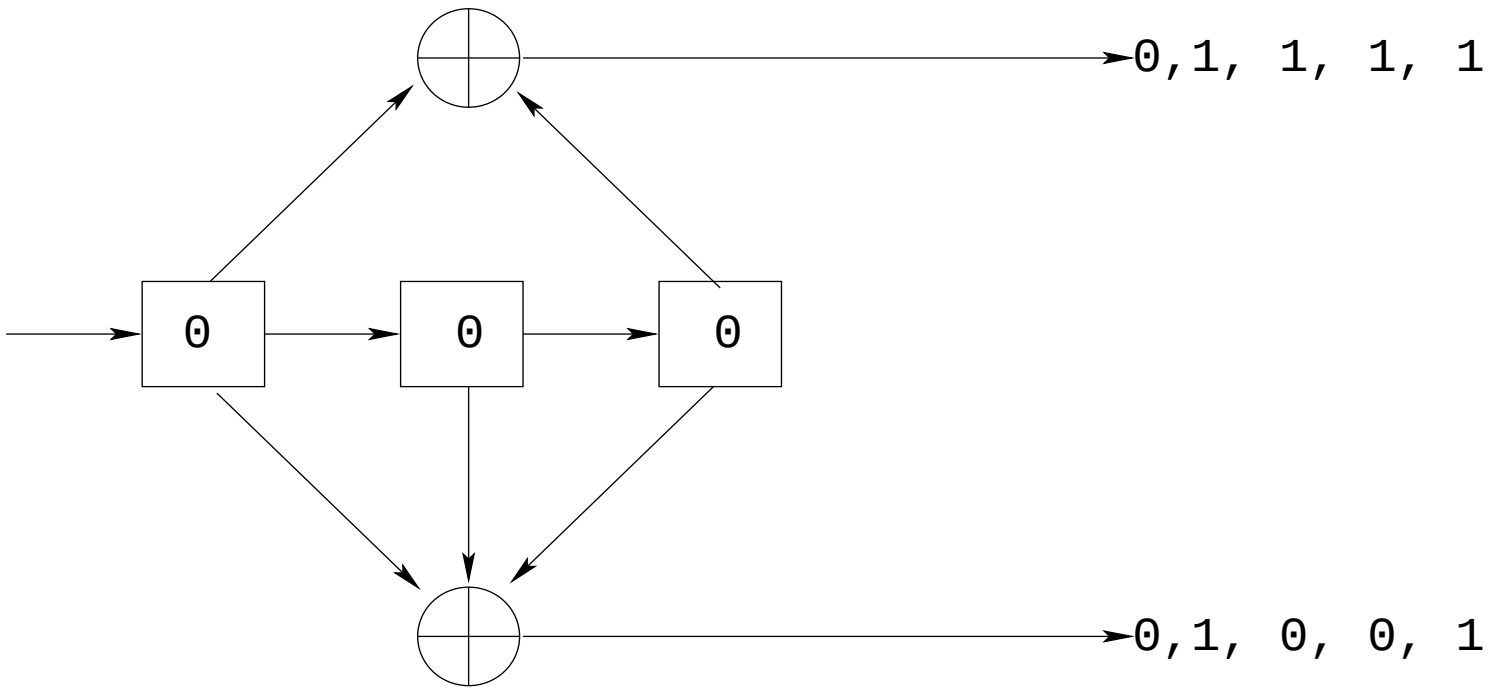

Mathematical Remarks

- A convolutional code $\mathcal{C}$ defines also a rational map

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \mathrm{Grass}(k, \mathbb{F}^n) \\ z & \longmapsto & \mathrm{colsp}_{\mathbb{F}}(G(z)). \end{array}$$

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- $Q^{\delta}_{k,n}$ has the structure of a smooth projective variety [Str87, RR94]. (Compare with work of Curto and Porras).

Multidimensional Convolutional Codes

Multidimensional convolutional codes generalize convolutional codes to polynomial rings in several variables: Let $R = \mathbb{F}[z_1, \ldots, z_m]$.

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Multidimensional convolutional codes have been studied quite a bit in the literature and we refer the reader to [FV94, FV98, VF94, Wei98] and more recent work by Climent, Napp and Pinto. Questions about optimal distances could be answered in the 2D case for general codes it is unknown. Constructions of codes having large distance and efficient decoding are not known.

Connection to Systems Theory

It follows e.g. from the Hermann-Martin identification that every convolutional code can also be represented by a linear system. In particular for every (n,k,δ) code there exist matrices $A \in \mathbb{F}^{\delta\times\delta}$, $B \in \mathbb{F}^{\delta\times k}$, $C \in \mathbb{F}^{(n-k)\times\delta}$, and $D \in \mathbb{F}^{(n-k)\times k}$.
The rate k/n convolutional code $\mathcal{C}$ is then described by the linear system of (McMillan) degree δ:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t, \\ y_t &= Cx_t + Du_t, \\ v_t &= \begin{pmatrix} y_t \\ u_t \end{pmatrix}, \; x_0 = 0. \end{aligned} \tag{1}$$

2. MDS Convolutional Codes

Definition. If $v(z) = v_0 + v_1 z + \cdots + v_N z^N \in \mathbb{F}^n[z]$ one defines the *weight* of $v(z)$ through:

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If $v(z), \tilde{v}(z) \in \mathbb{F}^n[z]$ one defines the Hamming distance through:

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For a convolutional code $\mathcal{C}$ one defines the *free distance*

$$d_{\mathrm{free}} := \min_{\substack{u,v \in \mathcal{C} \\ u \neq v}} \mathrm{Ham}\left(u(z), v(z)\right). \tag{2}$$

Remark

For fixed values δ, k, n we are interested in the maximum possible value of

$$d_{free}: \quad \mathcal{Q}_{k,n}^{\delta}(\mathbb{F}) \longrightarrow \mathbb{N} = \{1, 2, 3, \ldots\}$$

For $\delta = 0$ we know that the maximum value is given by the Singleton bound:

$$n - k + 1$$

and this value is attained if $|\mathbb{F}| > n$.

Examples:

Example.

$$G(z) = \begin{pmatrix} z^2+1 \\ z^2+z+1 \end{pmatrix}$$

defines a binary rate 1/2 convolutional code of degree $\delta = 2$ and distance $d_{\text{free}} = 5$.

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Example. Let $\mathbb{F} = \mathbb{F}_7$, the prime field of 7 elements.

$$G(z) = \begin{pmatrix} z^3+2z+5 & 0 \\ 5z^3+3 & 1 \\ z^2+5 & 2 \\ 2z+5 & 3 \end{pmatrix},$$

defines a convolutional code of rate $k/n = 2/4$, degree $\delta = 3$ and free distance $d_0 = d_{\text{free}} = 3$.

Generalized Singleton Bound

Lemma ([RS99])**.** *The free distance of an* (n,k,δ)*-code satisfies*

$$d_{\text{free}} \leq (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + 1. \tag{3}$$

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Theorem ([RS99]). *For every* (n,k,δ) *there exist MDS convolutional codes over sufficiciently large fields.*

Remark. The original proof [RS99] was non-constructive and it showed that MDS convolutional codes are Zariski dense in Grothedieck's Quot Scheme $Q^{\delta}_{k,n}$.

Remarks:

- For $\delta = 0$ the upper bound (2) reduces to the block code situation:

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- If $k = 1$ the upper bound (2) reduces to

$$d_{\text{free}} \leq n(\delta + 1).$$

This situation was studied by Justesen [Jus75].

3. MDP Convolutional Codes

Definition. The jth column distance of the code $\mathcal{C}$ is defined as

$$d_j := \min\left\{ \sum_{t=0}^{j} \mathrm{wt}\,(v_t) \mid \sum_{t=0}^{N} v_t z^t \in \mathcal{C}, v_0 \neq 0 \right\}.$$

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One has that $d_0 \leq d_1 \leq d_2 \leq \ldots$.
The Free distance is also equal to:

$$d_{\mathrm{free}} = \lim_{j \to \infty} d_j \tag{4}$$

Bound on Column Distance Indices

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$\mathcal{C}$ is said to have a *maximum distance profile* if

$$d_j = (n-k)(j+1)+1 \text{ for } j = 0,\ldots,L := \left\lfloor \frac{\delta}{k} \right\rfloor + \left\lfloor \frac{\delta}{n-k} \right\rfloor.$$

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Theorem ([HRS05])**.** *For integers* n,k,δ *and for sufficiently large fields the set of maximum distance profile codes forms a Zariski dense subset of Grothendieck's Quot Scheme* $Q^{\delta}_{k,n}$.

Example

Example. Let

$$G(z) = \begin{pmatrix} (z-1) & 1 \\ (z-2) & 1 \\ (2z-3) & 1 \end{pmatrix}$$

be an encoder for a rate $2/3$ convolutional code $\mathcal{C}$ of degree $\delta = 1$, over $\mathbb{F}_5$. The encoder is non-catastrophic. The generalized Singleton bound gives $d_{\text{free}} \leq 3$.

One shows that the code has $d_{\text{free}} = 3$, hence is a MDS code. The code has maximum distance profile as $d_0 = 2$ and $d_1 = d_{\text{free}} = 3$.

Algebraic Characterization

Assume that the parity check matrix is given as $H(z) = \sum_{i=0}^{\nu} H_i z^i$.
For each $j > \nu$ let $H_j = 0$ and define:

$$\mathcal{H}_j = \begin{pmatrix} H_0 & & & \\ H_1 & H_0 & & \\ \vdots & \vdots & \ddots & \\ H_j & H_{j-1} & \cdots & H_0 \end{pmatrix} \in \mathbb{F}^{(j+1)(n-k)\times(j+1)n}. \qquad (5)$$

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Theorem. *([GLRS06, Proposition 2.1]) Let $d \in \mathbb{N}$. The following properties are equivalent.*

1. $d_j^c = d$;
2. *none of the first n columns of $\mathcal{H}_j$ is contained in the span of any other $d-2$ columns and one of the first n columns of $\mathcal{H}_j$ is in the span of some other $d-1$ columns of that matrix.*

4. Superregular Matrices

Definition. Let A be an $n \times n$ lower triangular Toeplitz matrix and let $A^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}$ be the submatrix obtained from A by picking the rows with indices $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$.

A is called *superregular* if every submatrix $A^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}$ is nonsingular for every $1 \leq r \leq n$ and every $i_1, \ldots, i_r, j_1, \ldots, j_r$ with $j_\nu \leq i_\nu$.

4. Superregular Matrices

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Remark. For rate $1/2$ codes the construction of supperregular matrices is essentially equivalent to the construction of MDP convolutional codes.

Example. For $n = 3$ and $\mathbb{F} = \mathbb{F}_3$ the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

is superregular. For $n = 5$ and $\mathbb{F} = \mathbb{F}_7$ the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 6 & 1 & 2 & 1 & 0 \\ 4 & 6 & 1 & 2 & 1 \end{bmatrix}$$ is superregular.

Using a computer algebra program one checks that the following matrices are superregular.

$$\begin{bmatrix} 1 & & & & \\ \beta & 1 & & & \\ \beta^3 & \beta & 1 & & \\ \beta & \beta^3 & \beta & 1 & \\ 1 & \beta & \beta^3 & \beta & 1 \end{bmatrix} \in \mathbb{F}_{2^3}^{5\times 5}, \begin{bmatrix} 1 & & & & & \\ \gamma & 1 & & & & \\ \gamma^5 & \gamma & 1 & & & \\ \gamma^5 & \gamma^5 & \gamma & 1 & & \\ \gamma & \gamma^5 & \gamma^5 & \gamma & 1 & \\ 1 & \gamma & \gamma^5 & \gamma^5 & \gamma & 1 \end{bmatrix} \in \mathbb{F}_{2^4}^{6\times 6},$$

where

$$\beta^3+\beta+1=0, \text{ and } \gamma^4+\gamma+1=0,$$

Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega^9 & \omega & 1 & 0 & 0 & 0 & 0 & 0 \\ \omega^{33} & \omega^9 & \omega & 1 & 0 & 0 & 0 & 0 \\ \omega^{33} & \omega^{33} & \omega^9 & \omega & 1 & 0 & 0 & 0 \\ \omega^9 & \omega^{33} & \omega^{33} & \omega^9 & \omega & 1 & 0 & 0 \\ \omega & \omega^9 & \omega^{33} & \omega^{33} & \omega^9 & \omega & 1 & 0 \\ 1 & \omega & \omega^9 & \omega^{33} & \omega^{33} & \omega^9 & \omega & 1 \end{bmatrix} \in \mathbb{F}_{2^6}^{8\times 8}.$$

where

$$\omega^6 + \omega + 1 = 0.$$

Proof of Existence

Example. Let the matrix X be equal to

$$X = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 1 & \\ & & & & 1 & 1 \end{bmatrix}. \tag{5}$$

Then

$$A = X^{n-1} = \begin{bmatrix} 1 & & & & & \\ n & 1 & & & & \\ \binom{n}{2} & n & 1 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ \binom{n}{n-1} & \binom{n}{n-2} & \dots & n & 1 & \\ 1 & \binom{n}{n-1} & \dots & & n & 1 \end{bmatrix} \tag{5}$$

is *totally positive* over the reals and superregular for sufficiently large prime fields.

Consequence

For every size n there exist superregular matrices over sufficiently large field sizes.

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On the side of the presented existence result we do not have good algebraic construction of superregular matrices. - Come up with constructions! Some new results were recently derived in the Master thesis of Isabelle Raemy [Rae15]

5. Decoding over the Erasure Channel:

Definition. The Erasure channel is a communication channel, where symbols either arrive correctly or the receiver knows that they are in error (or did not arrive at all).

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Convolutional codes over any size alphabet have a polynomial time decoding algorithm over the erasure channel [TRS12]

Illustration: Block Codes

Assume G is the generator matrix of an $[n, k]$ MDS block code. We know then that every $k \times k$ fullsize minor of G is invertible.

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The block code $\mathcal{C} \subset \mathbb{F}^n$ has then distance $n-k+1$. If at most $n-k$ erasures happen then by simple linear algebra one can recover all erasures from the correctly transmitted k symbols.

Results:

Theorem. *Let $\mathcal{C}$ be an (n,k,δ) convolutional code with $d^c_{j_0}$ the $j = j_0$-th column distance. If in any sliding window of length $(j_0+1)n$ at most $d^c_{j_0} - 1$ erasures occur then we can recover completely the transmited sequence.*

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The best scenario happens when the convolutional code is MDP. In this case full error correction 'from left to right' is possible as soon as the fraction of errasures is not more than $\frac{n-k}{n}$ in any sliding window of length $(L+1)n$.

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Corollary 1. *Let $\mathcal{C}$ be an (n,k,δ) MDP convolutional code. If in any sliding window of length $(L+1)n$ at most $(L+1)(n-k)$ erasures occur in a transmited sequence then we can completely recover the sequence in polynomial time in δ.*

Forward Backward Decoding

There exist error patterns which CANNOT be decoded from ‘left to right’ which however can be decoded by a ‘forward backward method’

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Consider a $(2,1,50)$ MDP code and the following error pattern:

$$\ldots \mathbf{v}\overbrace{\mathbf{v}\star\ldots\star}^{(A)22}\ \overbrace{\mathbf{v}\mathbf{v}\star\star\mathbf{v}\mathbf{v}\star\star\ldots\mathbf{v}\mathbf{v}\star\star}^{(B)180}\ \Big|\overbrace{\mathbf{v}\mathbf{v}\ldots\mathbf{v}\mathbf{v}}^{(C)202}\Big|$$

$$\Big|\overbrace{\star\star\ldots\star}^{(D)80}\ \overbrace{\mathbf{v}\mathbf{v}\ldots\mathbf{v}}^{(E)62}\ \overbrace{\star\star\ldots\star}^{(F)60}\ \Big|\ \overbrace{\mathbf{v}\mathbf{v}\ldots\mathbf{v}}^{(G)202},$$

Cmplete-MDP Convolutional Codes

Assume that $(n-k)\nu = \delta$, the degree of the code $\mathcal{C}$ and C has a parity check matrix $H(z) = H_0 + H_1 z + \cdots + H_\nu z^\nu$.

Cmplete-MDP Convolutional Codes

Assume that $(n-k)\nu = \delta$, the degree of the code $\mathcal{C}$ and $\mathcal{C}$ has a parity check matrix $H(z) = H_0 + H_1 z + \cdots + H_\nu z^\nu$.

Definition. A rate $\frac{k}{n}$ convolutional code $\mathcal{C}$ with parity check matrix $H(z)$ is called a *complete-MDP convolutional code* if in the $(L+1)(n-k) \times (\nu+L+1)n$ matrix

$$\begin{bmatrix} H_\nu & \cdots & H_0 & & & \\ & H_\nu & & H_0 & & \\ & & \ddots & & \ddots & \\ & & & H_\nu & \cdots & H_0 \end{bmatrix}$$

every full size minor which is not trivially zero, is nonzero.

Example

The following parity check matrix represents a $(3,1,2)$ complete-MDP convolutional code over $\mathbb{F}_{128}$ with $\alpha^7+\alpha^6+\alpha^3+\alpha+1=0$.

$$H(z)=\begin{bmatrix} \alpha^{76}+\alpha^{77}z & \alpha^{62}+\alpha^{85}z & 1+\alpha^{76}z \\ \alpha^{73}+\alpha^{37}z & \alpha^{76}+\alpha^{77}z & \alpha^{62}+\alpha^{85}z \end{bmatrix}.$$

The partial parity check matrix has all its full size minors that are not trivially zero nonzero. I.E. minors that don't include columns 1, 2 and 3 or 7, 8 and 9, are nonzero.

$$\begin{bmatrix} \alpha^{77} & \alpha^{85} & \alpha^{76} & \alpha^{76} & \alpha^{62} & 1 & & & \\ \alpha^{13} & \alpha^{77} & \alpha^{85} & \alpha^{73} & \alpha^{76} & \alpha^{82} & & & \\ & & & \alpha^{77} & \alpha^{85} & \alpha^{76} & \alpha^{76} & \alpha^{62} & 1 \\ & & & \alpha^{13} & \alpha^{77} & \alpha^{85} & \alpha^{73} & \alpha^{76} & \alpha^{82} \end{bmatrix}.$$

Results

Lemma. *Complete-MDP convolutional codes are MDP convolutional codes.*

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Theorem. *If in a window of size* $(\nu+L+1)n$ *there are not more than* $(L+1)(n-k)$ *erasures and they are distributed in such a way that between position* 1 *and sn and between positions* $(\nu+L+1)n$ *and* $(\nu+L+1)n-s(n-k)$*, for* $s=1,2,\ldots,L+1$*, there are not more than* $s(n-k)$ *erasures then full correction of all symbols in this interval will be possible.*

Research Questions:

- Come up with new constructions of MDP codes, MDS codes and superreugular matrices.

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- Come up with new constructions of MDP codes, MDS codes and superreugular matrices.
- Show that Complete MDP codes exist for all rates and degrees.
- Find concrete constructions for complete MDP codes.
- Develop practical decoding algorithms capable of dealing with codes over very large alphabets (such as $\mathbb{F}_2^{1000}$.)

Concluding Remarks

(1) Convolutional codes generalize linear block codes in a natural way.

(2) Convolutional codes capable of decoding a large number of errors per time interval require a large free distance and a good distance profile.

(3) Very few constructions for codes with large distance are known.

(4) Typically convolutional codes are decoded via the Viterbi decoding algorithm. The complexity of this algorithm grows exponentially with the McMillan degree. New classes of codes coming with more efficient decoding algorithms are needed.

Banff International Research Station
for Mathematical Innovation and Discovery

Mathematical Coding Theory in Multimedia Streaming (15w5150)

Arriving Sunday, October 11 and departing Friday October 16, 2015

Organizers

Heide Gluesing-Luerssen (University of Kentucky)
Ashish Khisti (University of Toronto)
Joachim Rosenthal (University of Zurich)
Emina Soljanin (Bell Labs Research)

Objectives

section*{A statement of the objectives of the workshop and an indication of its relevance, importance, and timeliness}

Emerging multimedia applications require error correction codes that have very different properties from classical codes. Researchers working in this field have made progress in recent years in finding abstractions of such systems that are realistic, yet analytically tractable. However the error correction codes, such as those discussed in Subtopic~2, have been developed from first principles in specific settings. There is no general theory till date for such constructions. On the other hand researchers working on convolutional codes in recent years have made a significant progress on the dynamical systems theory underlying these constructions without a particular emphasis on multimedia systems. A primary objective of the workshop is to bring together researchers working in these two areas to combine their knowledge and develop a general theory for constructing streaming codes that can be implemented in future multimedia systems. We therefore expect about half of the participants to be experts in coding theory, and many of them will be from applied mathematics departments worldwide. The remaining half of the participants will be communication theorists working on engineering systems. Banff therefore provides an ideal venue for this workshop,

Thanks for your attention!

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Roxana Smarandache
Virtudes Tomás

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