Hilbert function spaces of Dirichlet series with the complete Pick property

Orr Shalit
based on a joint work with John McCarthy

Technion

“Abie’s workshop", March 2017

Hilbert function spaces

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The function $K : X \times X \to \mathbb{C}$

$$K(x, y) = K_y(x)$$

is the reproducing kernel of the space $\mathcal{H}$ (RKHS).

Examples of Hilbert function spaces

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The Hardy space $H^2 = H^2(\mathbb{D})$ is the space on $\mathbb{D}$

$$H^2 = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \langle f, f \rangle := \sum |a_n|^2 < \infty \right\}$$

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Example 2

Let $d \in \mathbb{N} \cup \{\infty\}$. The Drury-Arveson space H^2_d is the Hilbert function space on $\mathbb{B}_d \subset \mathbb{C}^d$ with kernel

$$k(z,w) = k_w(z) = \frac{1}{1 - \langle z, w \rangle} \quad (z, w \in \mathbb{B}_d)$$

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It turns out that:

$$H^2_d = \left\{ f(z) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha z^\alpha : \langle f, f \rangle := \sum |a_\alpha|^2 \frac{\alpha!}{|\alpha|!} < \infty \right\}$$

More examples of Hilbert function spaces

Silly example 1

Let $\mathcal{G}$ be the space of all analytic functions on the disc $\mathbb{D}_{1/2} := \{|z| < 1/2\}$

$$f : \mathbb{D}_{1/2} \to \mathbb{C}$$

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This is **silly**, because clearly $\mathcal{G}$ "$=$" $H^2(\mathbb{D})$. Every $f \in \mathcal{G}$ extends to a function on $\mathbb{D}$, no reason to look just the values on $\{|z| < 1/2\}$.

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Every Hilbert space $\mathcal{H}$ can be considered as a Hilbert function space on itself, i.e. on the set $X = \mathcal{H}$:

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This is **silly**, because $\mathcal{H}$ has no interesting algebraic/function-theoretic properties.

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$$\mathcal{H}_\phi = \{h : \mathbb{C}_+ \to \mathbb{C} : h(s) = \sum \gamma_n n^{-s} \text{ such that } \sum a_n^{-1} |\gamma_n|^2 < \infty\}$$

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Note: In this case, $\mathbb{C}_+$ is the maximal subspace of $\mathbb{C}$ on which all $h \in \mathcal{H}_\phi$ can be defined.

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Note: In this case, $\mathbb{C}_+$ is the maximal subspace of $\mathbb{C}$ on which all $h \in \mathcal{H}_\phi$ can be defined. However, for every $h \in \mathcal{H}_\phi$, $h(+\infty) = \gamma_1$ is well defined.

Generalized kernel functions[1]

[1] Related: **partially multiplicative functionals** Cowen-Macluer; Hartz.

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Definition

A vector $h \in \mathcal{H}$ is said to be a generalized kernel function if

$$\langle fg, h\rangle = \langle f, h\rangle\langle g, h\rangle \quad \text{whenever} \quad f, g \text{ and } fg \text{ are in } \mathcal{H}$$

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We define

$$\widehat{X} = \{h \in \mathcal{H} : h \text{ is a generalized kernel function}\}$$

We have a natural inclusion $b : X \hookrightarrow \widehat{X}$, given by $b(x) = K_x$.

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Proposition

There exists a Hilbert function space $\widehat{\mathcal{H}}$ on $\widehat{X}$ such that the map

$$\hat{f} \mapsto \hat{f} \circ b$$

is a (multiplicative) unitary $\widehat{\mathcal{H}} \to \mathcal{H}$.

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Isomorphism and weak isomorphism of RKHS

Two Hilbert function spaces $\mathcal{H}_1, \mathcal{H}_2$ are said to be isomorphic if there is a bijection $\alpha : X_1 \to X_2$ such that

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Definition

If there exists a multiplicative unitary between two Hilbert function spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, then we say that they are weakly isomorphic.

Isomorphism and weak isomorphism of RKHS

Proposition (McCarthy-S.)

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert function spaces. Then $\mathcal{H}_1$ is weakly isomorphic to $\mathcal{H}_2$ if and only if $\widehat{\mathcal{H}}_1$ and $\widehat{\mathcal{H}}_2$ are isomorphic.

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Let $\mathcal{G}$ be the space of all analytic functions on the disc $\mathbb{D}_{1/2} := \{|z| < 1/2\}$

$$f : \mathbb{D}_{1/2} \to \mathbb{C}$$

with power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\langle f, f \rangle := \sum_n |a_n|^2 < \infty$.

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and these spaces are weakly isomorphic.

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and these spaces are weakly isomorphic. BTW $\widehat{H^2(\mathbb{D})} = H^2(\mathbb{D})$, i.e., $\widehat{\mathbb{D}} = \mathbb{D}$.

Complete Pick spaces

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Problem: Given $z_1, \dots, z_n \in X$ and $w_1, \dots, w_n \in \mathbb{C}$, does there exist $f \in \operatorname{Mult}(\mathcal{H})$ with $||f||_{\operatorname{Mult}(\mathcal{H})} \leq 1$ and

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Examples of complete Pick spaces

Example 1 (Agler-McCarthy, Ball-Trent-Vinnikov, Davidson-Pitts)

For all $d \in \mathbb{N} \cup \{\infty\}$, the Drury-Arveson space H^2_d has the complete Pick property. (Recall that H^2_d is the Hilbert function space on $\mathbb{B}_d$ with kernel

$$k(z, w) = k_w(z) = \frac{1}{1 - \langle z, w \rangle} \quad (z, w \in \mathbb{B}_d)$$

For $d = 1$ this is just the Hardy space $H^2(\mathbb{D})$.)

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$$K_\phi(s, u) = \phi(s + \bar{u}) = \sum a_n n^{-s-\bar{u}}$$

Examples of complete Pick spaces

Example 2 (McCarthy-S.)

Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, $\{a_n\}_{n=1}^\infty$ a sequence of positive numbers such that $\phi(s) = \sum_n a_n n^{-s}$ converges in $\mathbb{C}_+$, and let

$$\mathcal{H}_\phi = \{h : \mathbb{C}_+ \to \mathbb{C} : h(s) = \sum \gamma_n n^{-s} \text{ such that } \sum a_n^{-1} |\gamma_n|^2 < \infty\}$$

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Theorem (McCarthy-S.)

$\mathcal{H}_\phi$ is a complete Pick space if and only if the coefficients of

$$\phi(s)^{-1} = \sum c_n n^{-s}$$

satisfy

$$c_n \leq 0 \text{ for } n \geq 2$$

Universality of the Drury-Arveson space

For $d \in \mathbb{N} \cup \{\infty\}$, the Drury-Arveson space H^2_d is the Hilbert function space on $\mathbb{B}_d$ with kernel $k(z, w) = k_w(z) = \frac{1}{1-\langle z,w \rangle} \quad (z, w \in \mathbb{B}_d)$.

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Theorem (Agler-McCarthy 2000)

Let $\mathcal{H}$ be a complete Pick space with kernel K on a set X. Then (up to normalization), $\mathcal{H}$ can be identified with

$$\mathcal{K}_Y := H^2_d\big|_Y = \{h\big|_Y : h \in H^2_d\}$$

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for some subset $Y \subseteq \mathbb{B}_d$ (possibly $d = \infty$). That is, there exists an injective map $\alpha : X \to \mathbb{B}_d$ such that

$$K(x,y) = k(\alpha(x), \alpha(y)) = \frac{1}{1 - \langle \alpha(x), \alpha(y) \rangle}$$

Varieties and "Zariski" closure

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Every complete Pick space $\mathcal{H}$ on X is isomorphic to

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Proposition

$\mathcal{K}_X = \mathcal{K}_V$ as subspaces of H^2_d, and these Hilbert function spaces are weakly isomorphic.

Conformal geometry determines Hilbert space structure

Conclusion:

Every (irreducible and normalized) complete Pick space is "of the form"

$$\mathcal{K}_V := \overline{\text{span}}\{k_\lambda : \lambda \in V\} = H^2_d\big|_V$$

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Theorem (Davidson-Ramsey-S.)

If $V, W \subseteq \mathbb{B}_d$ are varieties, then (up to normalization):
$\mathcal{K}_V$ is isomorphic to $\mathcal{K}_W$ if and only if there exists an automorphism $\alpha \in \text{Aut}(\mathbb{B}_d)$ such that

$$\alpha(V) = W$$

Finding the variety of a space $\mathcal{H}_\phi$

Consider the complete Pick space

$$\mathcal{H}_\phi = \{h : \mathbb{C}_+ \to \mathbb{C} : h(s) = \sum \gamma_n n^{-s}, \sum a_n^{-1} |\gamma_n|^2 < \infty\}$$

with kernel

$$K_\phi(s,u) = \phi(s+\bar{u}) = \sum a_n n^{-s-\bar{u}}$$

(we are assuming that $a_1 = 1$ (normalization) and that $\phi(s)^{-1} = \sum c_n n^{-s}$ satisfies $c_n \leq 0$ for $n \geq 2$.)

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Questions

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3. What does the variety $V = V(I(X)) \subseteq \mathbb{B}_d$ on which $\mathcal{H}_\phi$ "lives" look like?

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$$b_k = \sqrt{-c_{n_k}}$$

and $\alpha : \mathbb{C}_+ \to \mathbb{B}_d$ (where $d = \#\{n \geq 2 : c_n \neq 0\}$)

$$\alpha(s) = (b_1 n_1^{-s}, b_2 n_2^{-s}, \ldots)$$

Then

$$K_\phi(s,u) = \frac{1}{\phi(s+\bar{u})^{-1}} = \frac{1}{1 - \langle \alpha(s), \alpha(u) \rangle}$$

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Theorem (McCarthy-S.)

If $\log n_1, \log n_2, \ldots$ *are linearly independent over* $\mathbb{Q}$*, then*

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BTW: AHMR recently proved $V = V(I(X)) = \widehat{X}$ in general.

Example

For example, $\phi(s) = \frac{P(2)}{P(2)-P(2+s)}$, where $P(s)$ is the **Prime Zeta Function**

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Let $\mathcal{H}_\phi$ be the space on $\mathbb{C}_+$ with kernel

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In this case $\alpha : \mathbb{C}_+ \to \mathbb{B}_\infty$ is given by

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and

$$V = \text{``}\overline{\alpha(\mathbb{C}_+)}\text{''} = \mathbb{B}_\infty.$$

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Mult($\mathcal{H}$) is unitarily equivalent to Mult(H^2_∞), and this isomorphism is given by

$$\text{Mult}(H^2_\infty) \ni f \mapsto f \circ \alpha \in \text{Mult}(\mathcal{H}_\phi).$$

In partiuclar every complete Pick space is a quotient of Mult($\mathcal{H}_\phi$).

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Theorem (McCarthy-S.)

For every commuting row contraction $(T_1, \ldots, T_n)$ and every polynomial Q

$$\|Q(T_1, \ldots, T_n)\| \leq \|Q(b_1 p_1^{-s}, \ldots, b_n p_n^{-s})\|_{\mathrm{Mult}\,\mathcal{H}_\phi}$$

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But the spaces $\mathcal{H}_\phi$ and Mult($\mathcal{H}_\phi$) are spaces of functions of a **single** complex variable.
THEY DON'T HAVE ENOUGH POINTS!!

Thank you!