State/Signal Invariant Properties of Input/State/Output Systems

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Stationary I/S/O System in Continuous Time

A uniformly continuous linear stationary continuous time i/s/o (input/state/output) system is of the form

$$\Sigma_{\text{iso}} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \qquad t \in \mathbb{R}^+. \tag{1}$$

A, B, C, D, are bounded linear operators and $\mathbb{R}^+ = [0, \infty)$.
the input $u(t) \in \mathcal{U}$ = the input space,
the state $x(t) \in \mathcal{X}$ = the state space,
the output $y(t) \in \mathcal{Y}$ = the output space (all Hilbert spaces).
A classical future trajectory = a triple of continuous functions
$\begin{bmatrix} x \\ u \\ y \end{bmatrix} \in \begin{bmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{U}) \\ C(\mathbb{R}^+;\mathcal{Y}) \end{bmatrix}$ satisfying (1).
We get a slghtly more general class by allowing A to be unbounded, but keeping B, C, D bounded. See (CZ91).

- Typical stationary i/s/o systems modelled by partial differential equations are not uniformly continuous.
- Note that equation (1) can be rewritten in the form

$$\Sigma_{\text{iso}} : \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \tag{2}$$

  where S is the bounded block matrix operator $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.
- We get a much more general class of equations by allowing S in (2) bo be unbounded (but still closed) and rewriting (2) in the form

$$\Sigma_{\text{iso}} : \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \quad t \in \mathbb{R}^+. \tag{3}$$

  This class of systems covers "all" the standard models from mathematical physics. We call S the generator of Σ_{iso}.

Passive and Conservative I/S/O Systems

- The above class is too general: In order to say something meaningful we need some additional assumptions.
- One meaningful class of systems are those which are passive or even conservative.
- Conservativity means that the system satisfies certain energy balance equations.
- Passivity means that the system may contain energy sinks but no energy sources, so passive systems satisy some energy balance inequalities.

- A passive or conservative system can store energy in the state variable. The energy at time t is a nonnegative function of the state $x(t)$. In the most important case the energy is a quadratic function of the state $x(t)$, and we may take $\mathcal{X}$ to be a Hilbert space and with the norm $\|\cdot\|_{\mathcal{X}} =$ square root of internal energy. Thus

$$\text{Internal energy at time } t = \|x(t)\|_{\mathcal{X}}^2.$$

- Passive and conservative systems can interchange energy with the surrounding world. The exact form of the power interchange varies, but it is usually a quadratic function of the input $u(t)$ and output $y(t)$.

- In a scattering system the incoming power is a positive quadratic function of the input $u(t)$, and the outgoing power is a positive quadratic function of the output $y(t)$. After normalization, taking $\mathcal{U}$ and $\mathcal{Y}$ to be Hilbert spaces

$$\text{Power entering system at time } t = \|u(t)\|^2_{\mathcal{U}} - \|y(t)\|^2_{\mathcal{Y}}.$$

- In a impedance system the power entering the system is a product of the input $u(t)$ and the output $y(t)$. After normalization and taking $\mathcal{U} = \mathcal{Y}$

$$\text{Power entering system at time } t = 2\Re\langle u(t), y(t)\rangle_{\mathcal{U}}.$$

- In a transmission system power can both enter and leave through both the input or through the output. After normalization and taking $\mathcal{U}$ and $\mathcal{Y}$ to be Kreĭn spaces

$$\text{Power entering system} = [u(t), u(t)]^2_{\mathcal{U}} - [y(t), y(t)]^2_{\mathcal{Y}}.$$

Power Balance Equations

- Scattering system:

$$(\textit{Conservative}):\quad \frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} = \|u(t)\|^2_{\mathcal{U}} - \|y(t)\|^2_{\mathcal{Y}},$$
$$(\textit{Passive}):\quad \frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \le \|u(t)\|^2_{\mathcal{U}} - \|y(t)\|^2_{\mathcal{Y}}.$$

- Impedance system:

$$(\textit{Conservative}):\quad \frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} = 2\Re\langle u(t), y(t)\rangle_{\mathcal{U}},$$
$$(\textit{Passive}):\quad \frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \le 2\Re\langle u(t), y(t)\rangle_{\mathcal{U}}.$$

- Transmission system:

$$(\textit{Conservative}):\quad \frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} = [u(t), u(t)]^2_{\mathcal{U}} - [y(t), y(t)]^2_{\mathcal{Y}},$$
$$(\textit{Passive}):\quad \frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \le [u(t), u(t)]^2_{\mathcal{U}} - [y(t), y(t)]^2_{\mathcal{Y}}.$$

Common Form of Power Balance Equations

$$(\textit{Conservative}):\quad \frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} = \left\langle \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \mathcal{J} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \right\rangle_{\mathcal{U}\oplus\mathcal{Y}},$$

$$(\textit{Passive}):\quad \frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \le \left\langle \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \mathcal{J} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \right\rangle_{\mathcal{U}\oplus\mathcal{Y}},$$

where $\mathcal{J}$ is a signature operator in $\mathcal{U} \oplus \mathcal{Y}$.

- Scattering system: $\mathcal{J} = \mathcal{J}_{\mathrm{sca}} := \begin{bmatrix} 1_{\mathcal{U}} & 0 \\ 0 & -1_{\mathcal{Y}} \end{bmatrix}$,
- Impedance system with $\mathcal{U} = \mathcal{Y}$: $\mathcal{J} = \mathcal{J}_{\mathrm{imp}} := \begin{bmatrix} 0 & 1_{\mathcal{U}} \\ 1_{\mathcal{U}} & 0 \end{bmatrix}$,
- Transmission system: $\mathcal{J} = \mathcal{J}_{\mathrm{tra}} := \begin{bmatrix} \mathcal{J}_{\mathcal{U}} & 0 \\ 0 & -\mathcal{J}_{\mathcal{Y}} \end{bmatrix}$ where $\mathcal{J}_{\mathcal{U}}$ and $\mathcal{J}_{\mathcal{Y}}$ are signature operators in $\mathcal{U}$ respectively $\mathcal{Y}$.

Energy inequalities for passive systems

- Scattering system:

$$\|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|y(s)\|_{\mathcal{Y}}^2\,ds \le \|x(0)\|_{\mathcal{X}}^2 + \int_0^t \|u(s)\|_{\mathcal{U}}^2\,ds, \quad t > 0.$$

- Impedance system:

$$\|x(t)\|_{\mathcal{X}}^2 \le \|x(0)\|_{\mathcal{X}}^2 + 2\int_0^t \Re\langle u(t), y(t)\rangle_{\mathcal{U}}\,ds, \quad t > 0.$$

- Transmission system:

$$\|x(t)\|_{\mathcal{X}}^2 + \int_0^t [y(s), y(s)]_{\mathcal{Y}}^2\,ds \le \|x(0)\|_{\mathcal{X}}^2 + \int_0^t [u(s), u(s)]_{\mathcal{U}}^2\,ds, \quad t > 0.$$

- General $\mathcal{J}$-passive system:

$$\|x(t)\|_{\mathcal{X}}^2 \le \|x(0)\|_{\mathcal{X}}^2 + \int_0^t \left\langle \begin{bmatrix} u(s) \\ y(s) \end{bmatrix}, \mathcal{J} \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} \right\rangle_{\mathcal{U}\oplus\mathcal{Y}} ds, \quad t > 0.$$

- Thus, scattering, impedance, and transmission can all be written in a "common form" by using the signature operator $\mathcal{J}$ in $\mathcal{U} \oplus \mathcal{Y}$.
- However, scattering, impedance, and transmissions systems are have very different continuity and stability properties. From a technical point of view they are very different systems.

Energy inequalities for passive systems

- Scattering systems are well-posed because

$$\|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|y(s)\|_{\mathcal{Y}}^2\,ds \le \|x(0)\|_{\mathcal{X}}^2 + \int_0^t \|u(s)\|_{\mathcal{U}}^2\,ds, \quad t > 0.$$

  The final state $x(t)$ and the L^2-norm of the output y depend continuously on the initial state $x(0)$ and the L^2-norm of the input u.

- Impedance and transmissions systems need not be well-posed (not even causal).
- What would be a good "common setting" if we want to treat all the three cases at the same time? "Well-posed systems" is not good enough.

How do the Supply Rates Differ from each Other?

- The supply rate, i.e., the indefinite quadratic form $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \mapsto \langle \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right], \mathcal{J} \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \rangle_{\mathcal{U}\oplus\mathcal{Y}}$ defines a Kreĭn space inner product in the signal space $\mathcal{W} := \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$

$$\left[\begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} \right]_{\mathcal{W}} = \left\langle \begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \mathcal{J} \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} \right\rangle_{\mathcal{U}\oplus\mathcal{Y}}, \qquad \begin{bmatrix} u_i \\ y_i \end{bmatrix} \in \mathcal{W}.$$

- In the scattering case the input signal is positive and the output signal negative with respect to this inner product:

$$\left[\begin{bmatrix} u \\ 0 \end{bmatrix}, \begin{bmatrix} u \\ 0 \end{bmatrix} \right]_{\mathcal{W}} = \|u\|^2_{\mathcal{U}} > 0, \qquad \left[\begin{bmatrix} 0 \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right]_{\mathcal{W}} = -\|y\|^2_{\mathcal{U}} < 0.$$

How do the Supply Rates Differ from each Other?

- In the impedance case the both the input and the output signals are neutral with respect to this inner product:

$$\left[\begin{bmatrix} u \\ 0 \end{bmatrix}, \begin{bmatrix} u \\ 0 \end{bmatrix}\right]_{\mathcal{W}} = 0, \qquad \left[\begin{bmatrix} 0 \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix}\right]_{\mathcal{W}} = 0.$$

- In the transmission case the both the input and the output signals are indefinite with respect to this inner product:

$$\left[\begin{bmatrix} u \\ 0 \end{bmatrix}, \begin{bmatrix} u \\ 0 \end{bmatrix}\right]_{\mathcal{W}} = \langle u, \mathcal{J}_{\mathcal{U}} u\rangle_{\mathcal{U}} \gtrless 0,$$
$$\left[\begin{bmatrix} 0 \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix}\right]_{\mathcal{W}} = -\langle y, \mathcal{J}_{\mathcal{Y}} y\rangle_{\mathcal{Y}} \gtrless 0.$$

Where do These Systems Come from?

Example. We have a coaxial cable of length 1m. At both ends we measure the voltage and the current. This gives a total of four interaction signals:

u_0 = voltage at left end;
i_0 = current at left end;
u_1 = voltage at right end;
i_1 = current at right end.

Depending on which of these variables we choose as "inputs" and "outputs" we get different types of conservative i/s/o systems:

- Taking u_0 and u_1 as inputs and i_0 and i_1 as outputs (or the other way around) we get an impedance system.
- Taking u_0 and i_0 as inputs and u_1 and i_1 as outputs (or the other way aroung) we get a (non-well-posed and even non-causal) transmission system.
- Taking the "incoming waves" as inputs and the "outgoing waves" as outputs we get a (well-posed) scattering system.

They Come from some State/Signal System!

- **Abstract interpretation:** In the above example we had an "distributed parameter" system with two components:
  1) A state component (the current and voltage distribution inside the coaxial cable),
  2) A set of interactions signals u_0, i_0, u_1, i_1.
- By grouping the interaction signals in different ways into "inputs" and "outputs" we can from the same "circuit" constract i/s/o systems of scattering, impedance, or transmission type.
- What we need in order to develop a "universal theory" for scattering, impedance, and transmission passive i/s/o systems is to develop a theory for distributed parameter systems which have a state and an interaction signal, which has not been split into a dedicated input and a dedicated output. We call this a state/signal system. Think about this as an "infinite-dimensional circuit".

Recall the equation describing the i/s/o dynamics in continuous time:

$$\Sigma_{\text{iso}}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \tag{3}$$

1. First s/s formulation: Write $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, and move the output equation into the domain of a new generator F (whose domain is no longer dense in $\mathcal{W}$):

$$\Sigma: \begin{cases} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in \operatorname{dom}(F), \\ \dot{x}(t) = F\left(\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}\right), \end{cases} \qquad t \in \mathbb{R}^+, \tag{4}$$

$$\operatorname{dom}(F) = \left\{ \left[\begin{smallmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{smallmatrix} \right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \;\middle|\; \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \operatorname{dom}(S),\; y_0 = P_{\mathcal{Y}} S \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \right\},$$

$$F \left[\begin{smallmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{smallmatrix} \right] = P_{\mathcal{X}} S \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right].$$

How to Go from an I/S/O System to a S/S system?

$$\Sigma_{\rm iso}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{dom}\,(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \tag{3}$$

2. Second s/s formulation: Use graph Representation of (3), $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, $\mathfrak{K} = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$:

$$\Sigma\colon \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+. \tag{5}$$

where the generating subspace V is the (reordered) graph of S (or of F):

$$\begin{aligned} V &= \left\{ \begin{bmatrix} z_0 \\ x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} \in \mathfrak{K} \,\middle|\, \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathrm{dom}\,(S),\ \left[\begin{smallmatrix} z_0 \\ y_0 \end{smallmatrix}\right] = S \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \right\} \\ &= \left\{ \begin{bmatrix} z_0 \\ x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} \in \mathfrak{K} \,\middle|\, \begin{bmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} \in \mathrm{dom}\,(F),\ z_0 = F \begin{bmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} \right\}. \end{aligned}$$

Classical Future Trajectories

$$\Sigma_{\text{iso}}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \tag{3}$$

$$\Sigma: \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+. \tag{5}$$

- A classical future trajectory of the i/s/o system Σ_{iso} is a triple of continuous functions $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{U}) \end{smallmatrix}\right] C(\mathbb{R}^+;\mathcal{Y})$ which satisfies (3).
- A classical future trajectory of the s/s system Σ is a pair of continuous functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ which satisfies (5).
- $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ is a classical future trajectory of the i/s/o system Σ_{iso} if and only if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical future trajectory of the corresponding s/s system Σ where $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$.

Generalized Future Trajectories

- A generalized future trajectory of the i/s/o system Σ_{iso} is a triple of functions $\begin{bmatrix} x \\ u \\ y \end{bmatrix} \in \begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{U}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{Y}) \end{bmatrix}$ which can be approximated by a sequence $\begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix}$ of classical future trajectories of Σ_{iso}.
- A generalized future trajectory of the s/s system Σ is a pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$ which can be approximated by a sequence $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ of classical future trajectories of Σ.
- $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ is a generalized future trajectory of the i/s/o system Σ_{iso} if and only if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a generalized future trajectory of the corresponding s/s system Σ where $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$.
- This means that "all" those properties of Σ_{iso} which can be described in terms of properties of future trajectories of Σ_{iso} can be extended to the corresponding s/s system Σ!

How to Go from an S/S System to an I/S/O system?

- To go in the opposite direction and construct an i/s/o representation Σ_{iso} of a s/s Σ we need to first decompose $\mathcal{W}$ into a direct sum $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$, where we interpret $\mathcal{U}$ as the input space and $\mathcal{Y}$ as the output space. This is possible if and only if

  $$\text{If } \begin{bmatrix} z \\ 0 \\ w \end{bmatrix} \in V \text{ and } P_{\mathcal{U}}^{\mathcal{Y}} w = 0, \text{ then both } z = 0 \text{ and } P_{\mathcal{Y}}^{\mathcal{U}} w = 0.$$

  (The z- and y-components of a vector $\begin{bmatrix} z \\ x \\ u+y \end{bmatrix} \in V$ must be uniquely determined by the x- and u-components.)
- If this condition holds, and if we denote the linear map from $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$ by S (where $\begin{bmatrix} z \\ x \\ u+y \end{bmatrix} \in V$), then S is the generator of an i/s/o system Σ_{iso}, and V has the graph representation

$$V := \left\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ P_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \in \mathrm{dom}\,(S) \text{ and } \begin{bmatrix} z \\ P_{\mathcal{Y}}^{\mathcal{U}} w \end{bmatrix} = S \begin{bmatrix} x \\ P_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \right\}.$$

- If we want to be "very general", then we can even get rid of the condition

$$\text{If } \begin{bmatrix} z \\ 0 \\ w \end{bmatrix} \in V \text{ and } P_{\mathcal{U}}^{\mathcal{Y}} w = 0, \text{ then both } z = 0 \text{ and } P_{\mathcal{Y}}^{\mathcal{U}} w = 0$$

  by allowing S to be multi-valued. In this way we can interpret every decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ as an "generalized i/o decomposition" of $\mathcal{W}$.
- (This is actually a "necessary" extension if you want to treat all possible "impedance systems".)
- (But it can usually be avoided by adding some extra "regularity" conditions.)

Passive State/Signal Systems

Recall the i/s/o "passivity inequality"

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \le \left\langle \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \mathcal{J} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \right\rangle_{\mathcal{U}\oplus\mathcal{Y}}.$$

The appropriate s/s version of this is (replace $\begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$ by w)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \le [w(t), w(t)]_{\mathcal{W}}.$$

Carrying out the differentiation we get

$$-(\dot{x}(t), x(t))_{\mathcal{X}} - (x(t), \dot{x}(t))_{\mathcal{X}} + [w(t), w(t)]_{\mathcal{W}} \ge 0.$$

Here $\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$, since $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical trajectory of Σ:

$$\Sigma\colon \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+. \tag{5}$$

$$-(\dot{x}(t), x(t))_{\mathcal{X}} - (x(t), \dot{x}(t))_{\mathcal{X}} + [w(t), w(t)]_{\mathcal{W}} \geq 0. \qquad (6)$$

Define the following Kreĭn space inner product $[\cdot,\cdot]_{\mathfrak{K}}$ in the node space $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$:

$$\begin{aligned} \left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \right]_{\mathfrak{K}} &:= -(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}, \\ &:= -2\Re(z_1, x_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}, \qquad (7) \\ \begin{bmatrix} z_i \\ x_i \\ w_i \end{bmatrix} &\in \mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}. \end{aligned}$$

Then (6) says that the vector $\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V$ is a nonnegative vector in $\mathfrak{K}$.

Definition

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system, where $\mathcal{X}$ is a Hilbert space and $\mathcal{W}$ is a Kreĭn space, and let $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ be the Kreĭn node space of Σ with inner product (7). Assume $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \Rightarrow z = 0$.

1. Σ is passive if its generating subspace V is a maximal nonnegative subspace of $\mathfrak{K}$.
2. Σ is conservative if $V = V^{[\perp]}$ (i.e., V is a Lagrangian subspace of $\mathfrak{K}$).
3. Σ is passive energy preserving if V is a maximal nonnegative and neutral subpace of $\mathfrak{K}$ (i.e., $V \subset V^{[\perp]}$).
4. Σ is passive co-energy preserving if V is a maximal nonnegative and co-neutral subpace of $\mathfrak{K}$ (i.e., $V^{[\perp]} \subset V$).

Here $V^{[\perp]}$ is the orthogonal companion to V:

$$V^{[\perp]} := \{k \in \mathfrak{K} \mid [k, k^\dagger]_{\mathfrak{K}} = 0 \text{ for all } k^\dagger \in V\}.$$

Definition

Let $\mathcal{W}$ be a Kreĭn (signal) space, and let $\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y}$ be a direct sum decomposition of $\mathcal{W}$ (which we shall interpret as an "input/output" decomposition of $\mathcal{W}$ and denote by $(\mathcal{U}, \mathcal{Y})$):

1. $(\mathcal{U}, \mathcal{Y})$ is a scattering representation of $\mathcal{W}$ if $\mathcal{U}$ is a uniformly positive subspace of $\mathcal{W}$, $\mathcal{Y}$ is a uniformly negative subspace of $\mathcal{W}$, and $\mathcal{U}[\perp]\mathcal{Y}$ (i.e., $(\mathcal{U}, \mathcal{Y})$ is a fundamental decomposition of $\mathcal{W}$).
2. $(\mathcal{U}, \mathcal{Y})$ is an impedance representation of $\mathcal{W}$ if both $\mathcal{U}$ and $\mathcal{Y}$ are neutral subspaces of $\mathcal{W}$.
3. $(\mathcal{U}, \mathcal{Y})$ is a transmission representation of $\mathcal{W}$ if $\mathcal{U}[\perp]\mathcal{Y}$ and $\mathcal{U}$ and $\mathcal{Y}$ are Kreĭn spaces with respect to the inner products inherited from $\mathcal{W}$ (i.e., the decomposition is a "regular" decomposition of $\mathcal{W}$).

Theorem

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system and let $\Sigma_{\rm iso} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o representation of Σ (where S is allowed to be multi-valued).

1. *If $(\mathcal{U}, \mathcal{Y})$ is a scattering representation of Σ, then $\Sigma_{\rm iso}$ is scattering passive if and only if Σ is s/s passive.*
2. *If $(\mathcal{U}, \mathcal{Y})$ is an impedance representation of Σ, then $\Sigma_{\rm iso}$ is impedance passive if and only if Σ is s/s passive.*
3. *If $(\mathcal{U}, \mathcal{Y})$ is a transmission representation of Σ, then $\Sigma_{\rm iso}$ is transmission passive if and only if Σ is s/s passive.*

General philosophy: Scattering representations are well-posed (and they always exist). By using a scattering representation we can show that every passive s/s system Σ has "nice properties". We can then use this fact to draw conclusions about impedance and transmission representations of Σ.

State/Signal Invariant Properties

Principle

Let $\Sigma_{\rm iso}$ be an i/s/o system, and let Σ be the corresponding s/s system.

1. *Characteristic data $\mathcal{D}$ of $\Sigma_{\rm iso}$ are state/signal invariant if*

   Σ has the same characteristic data $\mathcal{D}$, and
   every i/s/o representation of Σ has the same data $\mathcal{D}$.

2. *A property $\mathcal{P}$ for $\Sigma_{\rm iso}$ is state/signal invariant if*

   Σ has property $\mathcal{P}$
   $\Leftrightarrow$ at least one i/s/o representation of Σ has property $\mathcal{P}$
   $\Leftrightarrow$ every i/s/o representation of Σ has property $\mathcal{P}$.

3. *A general method or construction $\mathcal{M}$ that can be applied to $\Sigma_{\rm iso}$ is state/signal invariant if*

   appplying $\mathcal{M}$ to $\Sigma_{\rm iso}$
   $\simeq$ applying $\mathcal{M}$ to Σ and "projecting" the result onto $\Sigma_{\rm iso}$.

Connection Between I/S/O Representations

Two "sufficiently regular" i/s/o representations of the same s/s system Σ differ from each other by certain combinations of

- an output feedback,
- a multiplication of the input or output by an invertible operator,
- addition of a feed-through term.

Thus, intuitively, "state/signal invariant" means roughly "invariant under output feedback". If some data or property or method is invariant under output feedback, then one might expect it to be state/signal invariant.

Connection Between I/S/O Representations

Several of the known "transformation formulas" in the literature describe transformations from one i/s/o representation to another:

- Livshits' diagonal transformation describes the transformation from an impedance to a scattering representation of a s/s system (or the other way around).
- The Potapov–Ginzburg transformation is related to the transformation from a transmission to a scattering representation of a s/s system (or the other way around).
- The chain-scattering transformation describes the transformation of a transmission to a scattering representation of a s/s system.

State/Signal Invariant Data and Properties

- sets of classical and generalized past, future, and two-sided trajectories (properly projected)
- existence of classical trajectories for classical initial data
- existence of generalized trajectories for all initial states
- the state component is (or is not) uniquely determined by the initial state and signal
- the reachable subspace
- the unobservable subspace
- strongly invariant subspaces (nonzero inputs allowed)
- unobservably invariant subspaces
- controllability and observability
- minimality
- stabilizability (by state feedback)
- detectability (by output injection)

(continues....)

State/Signal Invariant Data and Properties

- external equivalence of two i/s/o systems (same "i/o behavior" for zero initial states)
- similarity or pseudo-similarity of two i/s/o systems
- intertwinements of two i/s/o systems
- passive systems (in the appropriate sense)
- conservative systems (in the appropriate sense)
- simplicity (= minimality within conservative class)
- with or without minimal losses (talk by Arov)
- energy and co-energy preserving systems (in appropriate sense)
- minimal "passive balanced" systems
- optimality (Arov) = available storage (Willems)
- $*$-optimality (Arov) = required supply (Willems)
- frequency domain data (properly projected)

State/Signal Invariant Transformations and Constructions

- restriction to a strongly invariant subspace
- projection onto a direct complement of an unobservably invariant subspace
- compressions of i/s/o (and s/s) systems
- constructions of simple conservative, passive controllable energy preserving, passive observable co-energy preserving, minimal optimal, minimal $*$-optimal, and minimal passive balanced realizations
- the adjoint of an i/s/o system (with the appropriate interpretation)
- certain symmetries (reality, reciprocity)
- H^2-optimal control (here $\mathcal{W}$ is a Hilbert space)
- H^2-optimal filter [Kalman filter] (here $\mathcal{W}$ is a Hilbert space)
- LQG-balanced realizations (here $\mathcal{W}$ Hilbert is a space)
- frequency domain analysis (properly projected)

NOT State/Signal Invariant Are

- well-posedness
- stability
- existence of trajectories for "all" input functions (requiring the input to be "free")
- are trajectories uniquely determined by initial state and input or not?
- "input normalized" and "output normalized" realizations (the natural s/s counterpart is related to H^2-optimal control and filtering)
- Hankel balanced realizations
- most standard interconnections ("split"-and sum-junctions, parallel and cascade connections)
- (some interconnections are OK)

“Well-posable” I/S/O Systems

- “Well-posedness” and “stability” play important roles in i/s/o systems theory, but unfortunately these properties are not s/s invariant.
- However, we can often say something meaningful about a s/s system Σ as soon as Σ has at least one well-posed or stable i/s/o representation.

Definition

Let Σ be a s/s system.

1. Σ is (state/signal) well-posed if Σ has at least one well-posed i/s/o representation.
2. Σ is (state/signal) stable if Σ has at least one stable i/s/o representation.

Well-Posable and S/S Stabilizable I/S/O Systems

Definition

Let $\Sigma_{\rm iso}$ be an i/s/o system, and let Σ be the corresponding s/s system.

- $\Sigma_{\rm iso}$ is well-posable if Σ is (s/s) well-posed, i.e., if Σ has at least one well-posed i/s/o representation.
- $\Sigma_{\rm iso}$ is s/s stabilizable if Σ is (s/s) stable, i.e., if Σ has at least one stable i/s/o representation.

Note that s/s stabilizable $\neq$ stabilizable.

- s/s stabilizable $\simeq$ stabilizable by output feedback (instead of state feedback).
- s/s stabilizability $\Rightarrow$ both stabilizability and detectability.

Passive I/S/O Systems are S/S Stabilizable

Impedance passive and transmission passive systems need not be stable, and not even well-posed. However,

- impedance passive and transmission passive i/s/o systems are s/s stabilizable (because the corresponding passive s/s system has a stable scattering passive i/s/o representation).
- In many proofs it is possible to replace "well-posed" or "stable" by "well-posable" or "s/s stabilizable".
- Thus for example, impedance passive and transmission passive systems always have "lots" of classical and generalized trajectories (although the input component need not be "free").
- And of course, the study of all s/s invariant properties of an impedance or transmission passive system can be reduced to the corresponding study of a scattering representation of the same s/s system.

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