A general necessary and sufficient condition for controllability of networks of linear systems

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Outline

1. A necessary condition for controllability of a network
2. Known results for LTI networks with static linear couplings
3. A necessary and sufficient condition for LTI networks
4. A little counting game
A necessary condition for controllability of a network

Known results for LTI networks with static linear couplings

A necessary and sufficient condition for LTI networks

A little counting game

\[ \text{Controllability: For all } x_s = \begin{pmatrix} x^{(1)}(1) \\ x^{(1)}(2) \\ x^{(2)}(1) \\ x^{(2)}(2) \end{pmatrix}, \quad x_f = \begin{pmatrix} x^{(1)}(1) \\ x^{(1)}(2) \\ x^{(2)}(1) \\ x^{(2)}(2) \end{pmatrix} \text{ and } t_s \]

there exist \( t_f \) and \( u_1|_{[t_s,t_f]} \) such that \( x(t_f; x_s, t_s, u_1) = x_f \).
A necessary condition for controllability of a network

Known results for LTI networks with static linear couplings

A necessary and sufficient condition for LTI networks

A little counting game

A network and one of its internal signals

Controllability: For all \( x_s = \begin{pmatrix} x^{(1)}_s \\ x^{(2)}_s \end{pmatrix} \), \( x_f = \begin{pmatrix} x^{(1)}_f \\ x^{(2)}_f \end{pmatrix} \) and \( t_s \)

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A necessary condition for controllability of a network

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**A network and one of its node systems**

Controllability: For all \( x_s = \begin{pmatrix} x^{(1)}_s \\ x^{(2)}_s \end{pmatrix} \), \( x_f = \begin{pmatrix} x^{(1)}_f \\ x^{(2)}_f \end{pmatrix} \) and \( t_s \)

there exist \( t_f \) and \( u_1|_{[t_s, t_f]} \) such that \( x(t_f; x_s, t_s, u_1) = x_f \).
A first result

**Theorem:** If an i/o-coupled network of i/s/o systems is controllable then each of the node systems is controllable.

**Remark:** This does *not quite* follow from the fact that controllability is hereditary under behavior projection.
LTI networks with static linear couplings

Node systems \((i = 1, \ldots, N)\)

\[
\dot{x}^{(i)} = \alpha^{(i)} x^{(i)} + \beta^{(i)} v^{(i)}, \quad x^{(i)} \in \mathbb{R}^{n_i}, \ v^{(i)} \in \mathbb{R}^{m_i}
\]

\[
w^{(i)} = \gamma^{(i)} x^{(i)}, \quad w^{(i)} \in \mathbb{R}^{p_i}
\]

Static linear couplings with external input [and output]

\[
v^{(i)} = \sum_j A_{ij} w^{(j)} + B_i u, \quad u \in \mathbb{R}^{m}
\]

\[
y = \sum_i C_i w^{(i)}, \quad y \in \mathbb{R}^{p}
\]
LTI networks with static linear couplings

Node systems \((i = 1, \ldots, N)\)

\[
\dot{x}^{(i)} = \alpha^{(i)} x^{(i)} + \beta^{(i)} v^{(i)}, \quad x^{(i)} \in \mathbb{R}^{n_i}, \quad v^{(i)} \in \mathbb{R}^{m_i}
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w^{(i)} = \gamma^{(i)} x^{(i)}, \quad w^{(i)} \in \mathbb{R}^{p_i}
\]

Static linear couplings with external input [and output]

\[
v^{(i)} = \sum_j A_{ij} w^{(j)} + B_i u, \quad u \in \mathbb{R}^{m}
\]

Network system (full behavior \(\mathcal{B}_{(x,u,w,v)}\))

\[
\dot{x} = (\alpha + \beta A \gamma) x + \beta B u,
\]

\[
w = \gamma x
\]

\[
v = Aw + Bu
\]
The homogeneous SISO case

Network
\[
\dot{x} = (\alpha + \beta A \gamma) x + \beta Bu
\]

Homogeneous network
\[
\alpha^{(i)} = \alpha^{(0)}, \beta^{(i)} = \beta^{(0)}, \gamma^{(i)} = \gamma^{(0)}, \quad i = 1, \ldots, N
\]
\[
\alpha = I \otimes \alpha^{(0)}, \beta = I \otimes \beta^{(0)}, \gamma = I \otimes \gamma^{(0)}
\]

SISO network: single input single output agents
[not necessarily single integrator]

**Theorem:** [Hara et al., 2007] Let \( \text{rank}(B) < N \). Then the homogeneous SISO network is controllable if and only if
1. The node dynamics is controllable and observable
2. The matrix pair \((A, B)\) is controllable
Network
\[ \dot{x} = (\alpha + \beta A \gamma) x + \beta Bu \]

Left coprime factorization
\[ D(s)^{-1} N(s) = \gamma (sI - \alpha)^{-1} \beta \]

**Theorem**: [Fuhrmann/Helmke, 2014] Let the node systems be controllable and observable. Then the network is controllable if and only if

\[ (D(s) - N(s)A - N(s)B) \]

is left prime.
A necessary condition for controllability of a network
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A little counting game

A behavioral proof of the Fuhrmann-Helmke test

Network system (full behavior $\mathcal{B}_{(x,u,w,v)}$)

$$\dot{x} = (\alpha + \beta A \gamma) x + \beta Bu,$$

$$w = \gamma x$$

$$v = Aw + Bu$$

Node systems (open network), left coprime factorization

$$w = D(s)^{-1} N(s) v = \gamma (sI - \alpha)^{-1} \beta v$$

Rearranging

$$D(s)w = N(s)(Aw + Bu) \iff (D(s) - N(s)A - N(s)B) \begin{pmatrix} w \\ u \end{pmatrix} = 0$$

Fuhrmann-Helmke test $\iff \mathcal{B}_{(u,w)}$ controllable
A behavioral proof of the Fuhrmann-Helmke test

Network system (full behavior $\mathcal{B}(x,u,w,v)$)

$$\dot{x} = (\alpha + \beta A\gamma)x + \beta Bu,$$
$$w = \gamma x$$
$$v = Aw + Bu$$

Node systems observable $\Rightarrow x$ observable from $(w,v)$ $\Rightarrow$ $x$ observable from $(u,w)$

Observable i/s/o system:
$\mathcal{B}(u,w)$ controllable $\Rightarrow \mathcal{B}(x,u)$ controllable $[\Rightarrow \mathcal{B}(x,u,w,v)$ controllable$]$  

Conversely:
$\mathcal{B}(x,u,w,v)$ controllable $\Rightarrow \mathcal{B}(u,w)$ controllable
A convenient network representation

Network system (full behavior $B(x,u,w,v)$)

\[
\dot{x} = (\alpha + \beta A\gamma)x + \beta Bu,
\]
\[
w = \gamma x
\]
\[
v = Aw + Bu
\]

Node systems (open network)

\[
\begin{pmatrix}
\alpha_{11} & \alpha_{12} \\
0 & \alpha_{22}
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2
\end{pmatrix}
\begin{pmatrix}
\gamma_2, \alpha_{22}
\end{pmatrix}
\text{observable}
\]

\[
l.c.f. \quad D(s)^{-1}N(s) = \gamma_2 (sl - \alpha_{22})^{-1}
\]

Playing with the system variables

\[
\gamma = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 0 & A \end{pmatrix}, \quad \bar{B} = B
\]
kernel representations are magical!
A kernel representation of the network i/o behavior

Node systems (open network)

\[
\begin{pmatrix}
\alpha_{11} & \alpha_{12} \\
0 & \alpha_{22}
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2
\end{pmatrix}
\]

\[
\bar{\gamma} = \begin{pmatrix} I & 0 \end{pmatrix}, \quad w_1 = x_1
\]

\[
\bar{A} = \begin{pmatrix} 0 & A \end{pmatrix}, \quad \bar{B} = B
\]

\((\gamma_2, \alpha_{22})\) observable

l.c.f. \( D(s)^{-1} N(s) = \gamma_2 (sI - \alpha_{22})^{-1} \)

\[
\begin{pmatrix} X(s) & Y(s) \end{pmatrix}
\begin{pmatrix} sI - \alpha_{22} \\
\gamma_2
\end{pmatrix} = I
\]

Kernel representation of the (augmented) network i/o behavior

\[
\mathcal{B}_{(w_1, w_2, u)} = \text{Ker} \begin{pmatrix}
sl - \alpha_{11} & \alpha_{12}(Y(s) + X(s)\beta_2 A) - \beta_1 A - \alpha_{12} X(s)\beta_2 B - \beta_1 B \\
0 & -D(s) + N(s)\beta_2 A \\
-D(s) + N(s)\beta_2 A & N(s)\beta_2 B
\end{pmatrix}
\]
The main result

Node systems (open network)
\[
\begin{pmatrix}
\alpha_{11} & \alpha_{12} \\
0 & \alpha_{22}
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2
\end{pmatrix}
\]

Network (closed loop)
\[
\dot{x} = (\alpha + \beta A \gamma) x + \beta B u,
\]
\[
w = \gamma x
\]
\[
v = A w + B u
\]

l.c.f. \( D(s)^{-1} N(s) = \gamma_2 (sl - \alpha_{22})^{-1} \)

\[
(X(s) \quad Y(s)) \begin{pmatrix} sl - \alpha_{22} \\ \gamma_2 \end{pmatrix} = I
\]

**Theorem:** [Generalized Fuhrmann-Helmke test] The network is controllable if and only if

\[
\begin{pmatrix}
sl - \alpha_{11} & -\alpha_{12} (Y(s) + X(s)\beta_2 A) - \beta_1 A & -\alpha_{12} X(s)\beta_2 B - \beta_1 B \\
0 & -D(s) + N(s)\beta_2 A & N(s)\beta_2 B
\end{pmatrix}
\]

is left prime.
Homogeneous networks

Node system (open network)

\[ \alpha_{11} = I \otimes \alpha_{11}^{(0)} \]

\[
\begin{pmatrix}
    sI - \alpha_{11} & -\alpha_{12} (Y(s) + X(s)\beta_2 A) - \beta_1 A & -\alpha_{12} X(s)\beta_2 B - \beta_1 B \\
    0 & -D(s) + N(s)\beta_2 A & N(s)\beta_2 B
\end{pmatrix}
\]

Counting rank

Column rank \leq N \cdot \text{rank}(sI - \alpha_{11}^{(0)}) + \sum_i p_i + \text{rank}(B)

Number of rows = N \cdot \text{dim}(x_1^{(0)}) + \sum_i p_i
Homogeneous networks

Let $\text{rank}(B) < N$ and $s \in \sigma(\alpha_{11}^{(0)})$.

Counting rank

column rank $\leq N \cdot \text{rank}(sl - \alpha_{11}^{(0)}) + \sum_i p_i + \text{rank}(B)$

$< N \cdot \left(\dim(x_1^{(0)}) - 1\right) + \sum_i p_i + N$

count (number of rows) $= N \cdot \dim(x_1^{(0)}) + \sum_i p_i$

**Theorem:** Let $\text{rank}(B) < N$. If the network is homogeneous and controllable then the node dynamics is observable.
Thank you.