Partial State Reachability and Observability of Cascades of Linear Systems

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Networks of Linear Systems
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And connected over a network with **Paul Fuhrmann**.
PART I: Motivation and Problem Formulation

PART II: Unavoidable Abstract Formalism

1. Polynomial Model
2. Shift Realization
3. Reachability and Observability Maps

PART III: Partial State Reachability

1. Toeplitz Operators
2. Polynomial Approach to Partial State Reachability

PART IV: Partial State Observability

PART V: Examples

PART VI: Duality

Conclusions
Part I: Motivation and Problem Formulation

Main idea:
Start from first principles with various definitions of reachability.
Main idea: Cascade of MIMO Systems.

Study of (Partial) Reachability

Elementary cascade: Reachability

Elementary cascade with communication: Reachability

Generalization: Partial Reachability
1. The Problem

Odd numbered systems: \textbf{Node Systems} (to be controlled)
Even numbered systems: \textbf{Connector Systems} (don’t care about their states)

Let each system have a minimal state space representation

\[
\Sigma_i : \quad x_i(t + 1) = A_i x_i(t) + B_i u(t_1) \\
\quad \quad \quad w_i(t) = C_i x(t)
\]

\textbf{Series Connection}

\[
u(t) = u_1(t), \quad \quad \quad w_i(t) = u_{i+1}(t).
\]

\textbf{Direct sum} of the individual state spaces of connectors and nodes, \(x = \text{col}[x_1, x_2, \ldots, x_{2r+1}]\)
gives state space representation

\[
x(t + 1) = Ax(t) + Bu(t) \\
y(t) = Cx(t),
\]
1. The Problem

and system matrices

\[ \mathbf{A} = \begin{bmatrix}
A_1 & & \\
B_2 C_1 & A_2 & \\
& B_3 C_2 & A_3 & \\
& & \ddots & \ddots & \\
& & & B_{2r+1} C_{2r} & A_{2r+1}
\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}
B_1 \\
0 \\
\vdots \\
\vdots \\
0
\end{bmatrix} \]

\[ \mathbf{C} = [0, 0, 0 \cdots 0, C_{2r+1}] . \]

Callier and Nahum (1975): Full reachability of series connection of two systems.

Fuhrmann and Helmke (2013): Full reachability of arbitrary length cascade.

This presentation: Precise characterization of

1. Partial State Reachability.
2. Partial State Observability.
3. Duality.
1. Definitions

Series Connection of Linear Systems over Field $\mathbb{F}$

$$\Sigma_1 \to \Sigma_1 \to \cdots \to \Sigma_r$$

Let $\mathcal{I} = \{i_1, i_2, \ldots, i_k\}$, with $k \leq r$, denote a subset of $\{1, 2, \ldots, r\}$:

$$1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq r$$

**Definition 1**

The $\Sigma_1 \to \cdots \to \Sigma_r$ series connection is $\mathcal{I}$-**partial state reachable** if, for some integer $T \geq 0$, the states of the $\mathcal{I}$-subsystems, i.e., $x_{i_1}(T) \in \mathbb{F}^{n_{i_1}}$, $\ldots$, $x_{i_k}(T) \in \mathbb{F}^{n_{i_k}}$, can be arbitrarily assigned by a judicial choice of the input sequence, $u(0), \ldots, u(T-1)$ to the first system $\Sigma_1$ in the series connection.

**Definition 2**

The $\Sigma_1 \to \cdots \to \Sigma_r$ series connection is $\mathcal{I}$-**partial state observable** if the initial states of the $\mathcal{I}$-subsystems, i.e., $x_{i_1}(0) \in \mathbb{F}^{n_{i_1}}$, $\ldots$, $x_{i_k}(0) \in \mathbb{F}^{n_{i_k}}$, can be unambiguously determined from knowledge of the output sequence, $y(0), y(1), \ldots, y(T)$ of the last system, $\Sigma_r$, and the input sequence $u(0), \ldots, u(T-1)$ to the first system $\Sigma_1$ in the series connection, for some integer $T \geq 0$.

Verriest & Helmke & Fuhrmann, ACC 2016: (1,3)-reachability.
Part II: Abstract Formulation: Polynomial Systems

Main ideas: Rosenbrock 1975
Fuhrmann 1976
Kailath 1980
Fuhrmann and Helmke 2015: The Mathematics of Networks of Linear Systems
2. Abstract Formulation

Notation

\( \mathbb{F}[z] \): Ring of polynomials over \( \mathbb{F} \).

\( \mathbb{F}(\!(z^{-1})\!) \): Ring of truncated Laurent expansions in \( z^{-1} \), \( (f)_-1 = \text{residue of } f \) (multiplier of \( z^{-1} \)).

\( \mathbb{F}[[z^{-1}]] \): Ring of formal power series in \( z^{-1} \) over \( \mathbb{F} \).

\( z^{-1}\mathbb{F}[[z^{-1}]] \): Ring of strictly proper formal power series in \( z^{-1} \) over \( \mathbb{F} \).

Rosenbrock Higher Order Representation

\[
T(\sigma)\xi = U(\sigma)u
\]

\[
y = V(\sigma)\xi + W(\sigma)u.
\]

\( \sigma \): backward shift operator \( \sigma(\zeta(t)) := (\zeta(t + 1)) \) on time sequences \( \xi, u \) and \( y \).

Polynomial matrices: \( T(z) \in \mathbb{F}[z]^{r \times r} \), nonsingular; \( U(z) \in \mathbb{F}[z]^{r \times m} \), \( V(z) \in \mathbb{F}[z]^{p \times r} \) and \( W(z) \in \mathbb{F}[z]^{p \times m} \)

We assume throughout strict properness of the associated transfer matrix

\[
G(z) = V(z)T(z)^{-1}U(z) + W(z)
\]
2. Abstract Formulation

Polynomial Model

Canonical projections onto the strictly proper and polynomial parts, respectively,

\[ \pi_- : \mathbb{F}((z^{-1}))^m \rightarrow z^{-1}\mathbb{F}[[z^{-1}]]^m \]
\[ \pi_+ : \mathbb{F}((z^{-1}))^m \rightarrow \mathbb{F}[z]^m. \]

For a nonsingular polynomial matrix \( T(z) \in \mathbb{F}[z]^{m \times m} \), define a linear projection map

\[ \pi_T : \mathbb{F}[z]^m \rightarrow \mathbb{F}[z]^m, \quad \pi_T f = T \pi_-(T^{-1} f) \]

Polynomial model of \( T(z) \):

\[ X_T := \text{Im} \pi_T = \{ f \in \mathbb{F}[z]^m \mid T(z)^{-1} f(z) \text{ strictly proper} \} \]

\( X_T \) is a finite dimensional \( \mathbb{F} \)-vector space of dimension \( \dim X_T = \deg \det T \).

Shift operator \( S_T : X_T \rightarrow X_T \):

\[ \forall f \in X_T, \quad S_T f = \pi_T(z f) = z f - T(z) \xi_f, \quad \text{where } \xi_f = (T^{-1} f)_1 \]

The polynomial model \( X_T \) then becomes an \( \mathbb{F}[z] \)-module by using the \( S_T \)-induced module structure, i.e.,

\[ \forall p \in \mathbb{F}[z], \forall f \in X_T : \quad \langle p, f \rangle = p \cdot f = \pi_T(p f), \quad . \]
2. Abstract Formulation

Shift Realization

Consider any \( p \times m \) strictly proper rational transfer function

\[
G(z) = V(z)T(z)^{-1}U(z) + W(z) \in \mathbb{F}[z]^{p \times m},
\]

with polynomial matrices \( U, T, V, W \) and \( T(z) \in \mathbb{F}[z]^{r \times r} \) nonsingular.

The shift realization of \( G(z) \) is the system in the state space \( X_T \)

\[
ΣVT^{-1}U+W := \begin{cases} \overline{A}f = ST\ f & f \in X_T \\ \overline{B}\xi = π_T(U\xi), & \xi \in \mathbb{F}^m \\ \overline{C}f = (VT^{-1}f)_1 & f \in X_T, \end{cases}
\]

defined by linear maps \( \overline{A} : X_T \rightarrow X_T, \overline{B} : \mathbb{F}^m \rightarrow X_T, \) and \( \overline{C} : X_T \rightarrow \mathbb{F}^p. \)

Any realization \( (A, B, C) \) can be regarded as the shift realization on the polynomial model space \( X_{zI−A}. \)
In terms of right coprime factorizations $G_i(z) = \frac{N_i(z)}{D_i(z)}$

$$G(z) = G_r(z) \cdots G_1(z) = V(z)T^{-1}(z)U(z)$$

$$T(z) = \begin{bmatrix}
D_1(z) & 0 & \cdots & 0 \\
-N_1(z) & D_2(z) & & \\
& \ddots & \ddots & \\
0 & \cdots & -N_{r-1}(z) & D_r(z)
\end{bmatrix}, \quad U(z) = \begin{bmatrix}
I \\
0 \\
\vdots \\
0
\end{bmatrix}$$

$$V(z) = \begin{bmatrix}
0 & 0 & \cdots & 0 & N_r(z)
\end{bmatrix}, \quad W(z) = 0.$$  

the two polynomial system matrices

$$\begin{bmatrix}
zI - \mathbf{A} & -\mathbf{B} \\
-\mathbf{C} & 0
\end{bmatrix}, \quad \begin{bmatrix}
T(z) & -U(z) \\
V(z) & 0
\end{bmatrix}$$

are Fuhrmann strict system equivalent $\implies$ Their shift realizations are similar $\implies (\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\mathbf{C}}) \sim (\mathbf{A}, \mathbf{B}, \mathbf{C})$
Isomorphism

\[ Z : X_T \rightarrow X_{zI-A}, \quad Zf = \pi_{zI-A}(\overline{B}f), \]

where \( \overline{B} := \text{diag}(B_1, \ldots, B_r). \)

A straightforward computation reveals that

\[ Zf = \begin{bmatrix}
\pi_{zI-A_1}(B_1f_1) \\
\pi_{zI-A_2}(B_2(f_2 + \pi_+(G_1f_1))) \\
\vdots \\
\pi_{zI-A_r}(B_r(f_r + \pi_+(G_{r-1}f_{r-1}) + \cdots \\
+ \cdots + \pi_+(G_{r-1} \cdots G_1f_1)))
\end{bmatrix} \]

Moreover, \( f \in X_T \) if and only if for all \( i = 1, \ldots, r \)

\[ f_i + \pi_+(G_{i-1}f_{i-1}) + \cdots + \pi_+(G_{i-1} \cdots G_1f_1) \in X_{D_i}. \]
2. Abstract Formulation

Observability Maps

\(\mathcal{O} : X_{zI-A} \to z^{-1}\mathbb{F}[[z^{-1}]]^p : \mathcal{O}f := C(zI - A)^{-1}f\)

\(\mathcal{O}_T : X_T \to z^{-1}\mathbb{F}[[z^{-1}]]^p : \mathcal{O}_T f := \pi_-(VT^{-1}f)\)

By the state space isomorphism theorem, isomorphism

\(\mathcal{O} \circ Z = \mathcal{O}_T\)

Reachability Maps

\(\mathcal{R} : \mathbb{F}[z]^m \to X_{zI-A} : \mathcal{R}u = \pi_{zI-A}(Bu(z))\)

\(\mathcal{R}_T : \mathbb{F}[z]^m \to X_T : \mathcal{R}_T u = \pi_T(Uu(z))\).

Again, one obtains \(\mathcal{R} = \mathcal{R}_T \circ Z\).

The shift realization theorem from the polynomial model characterizes observability and reachability in terms of the associated polynomial system matrices.

Explicitly, the polynomial model and hence its state space, is reachable (or observable), if and only if \(T(z)\) and \(U(z)\) are left coprime (or \(T(z)\) and \(V(z)\) are right coprime).
2. Abstract Formulation

**Theorem 1**: Full reachability and Observability of the Series Connection [Fuhrmann-Helmke]

The series connection of the state space model is reachable if and only if the polynomial matrix

\[
\begin{bmatrix}
N_1(z) & D_2(z) & 0 \\
& \ddots & \ddots \\
0 & \cdots & N_{r-1}(z) & D_r(z)
\end{bmatrix}
\]

is left prime.

Choose left coprime factorizations \( G_i(z) = \overline{D}_i(z)^{-1}\overline{N}_i(z) \). Observability holds if and only if

\[
\begin{bmatrix}
\overline{D}_1(z) & 0 & \cdots & 0 \\
\overline{N}_2(z) & \overline{D}_2(z) & 0 & 0 \\
0 & \overline{N}_3(z) & & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \overline{D}_{r-1}(z) & \\
0 & \cdots & 0 & \overline{N}_r(z)
\end{bmatrix}
\]

is right coprime.
Part III: Partial State Reachability
### Toeplitz Operators

Given a rational function $G(z) \in \mathbb{F}(z)^{p \times m}$, the algebraic Toeplitz operator is the linear map

$$\mathbb{T}_G : \mathbb{F}[z]^m \to \mathbb{F}[z]^p, \quad \mathbb{T}_G f = \pi_+(Gf).$$

$G(z)$: symbol of $\mathbb{T}_G$.

Let:

$\mathbb{T}_i$: Toeplitz operator with symbol $G_i(z)$

$$R(A_i, B_i) : \mathbb{F}[z]^{m_i} \to \mathbb{F}^{n_1} : R(A_i, B_i)(f) = \pi_{zI-A_i}(B_if)$$

#### Theorem 2.

The series connection $\Sigma_1 \to \ldots \to \Sigma_r$ is $\mathcal{I}$-partial state reachable if and only if the linear operator $\mathbb{T}_G : \mathbb{F}[z]^m \to \mathbb{F}[z]^p$, with rational symbol

$$\mathcal{R}_\mathcal{I} := \begin{bmatrix} R(A_{i_1}, B_{i_1})T_{i_1} \cdots T_1 \\ R(A_{i_2}, B_{i_2})T_{i_2} \cdots T_1 \\ \vdots \\ R(A_{i_k}, B_{i_k})T_{i_r} \cdots T_1 \end{bmatrix}$$

is surjective. - If $i_1 = 1$ the first row is replaced by $R(A_1, B_1)$. 
3. Partial Reachability

Thus, \( \mathcal{I} \)-partial reachability is not equivalent to reachability of the truncated series connection
\[ \Sigma_{i_1} \rightarrow \Sigma_{i_2} \rightarrow \cdots \rightarrow \Sigma_{i_k}. \]

The preceding Theorem is precise but due to the Toeplitz operators difficult to apply in practice.

**Theorem 3:**

The series connection \( \Sigma_1 \rightarrow \ldots \rightarrow \Sigma_r \) is \((1, 3, \ldots, 2k + 1)\)-partial state reachable if and only if the Toeplitz operator \( \mathbb{T}_{G_c} : \mathbb{F}[z]^m \rightarrow \mathbb{F}[z]^p \) of the rational function

\[
G_c = \begin{bmatrix}
G_2 N_1 & D_3 & \cdots & 0 \\
0 & G_4 N_3 & D_5 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & G_{2k} N_{2k-1} & D_{2k+1}
\end{bmatrix}
\]

is surjective.

**Corollary:** The series connection \( \Sigma_1 \rightarrow \Sigma_2 \rightarrow \Sigma_3 \) is \((1, 3)\)-partial state reachable if and only if the Toeplitz operator \( \mathbb{T}_{[G_2 N_1, D_3]} : \mathbb{F}[z]^{m_1 + p_2} \rightarrow \mathbb{F}^{p_2} \) is surjective.
1. Partial Reachability

Duality
Consider the non-degenerate bilinear pairing on $z^{-1}\mathbb{F}[[z^{-1}]]^n \times \mathbb{F}[z]^n$, defined by the residue form

$$[h, f] := (h(z)^\top f(z))_{-1}. $$

The dual space of $\mathbb{F}[z]^n$ with respect to this bilinear form is $z^{-1}\mathbb{F}[[z^{-1}]]^n$.

Dual of the Toeplitz operator is identified with

$$\mathbb{T}_G^* : z^{-1}\mathbb{F}[[z^{-1}]]^{p_2} \rightarrow z^{-1}\mathbb{F}[[z^{-1}]]^{m_1+m_3},$$

$$\mathbb{T}_G^* h = \pi_- \begin{bmatrix} (G_2 N_1)^\top h \\ D_3^\top h \end{bmatrix}. $$

Intertwining
Let $G_i(z) = N_i(z)D_i(z)^{-1}$ be right coprime polynomial factorizations of the transfer functions $G_1(z), G_2(z), G_3(z)$. Let $G_2(z) = \overline{D}_2(z)^{-1}\overline{N}_2(z)$ be a left coprime factorization and let

$$\Delta(z) := \text{g. c. l. d.} (\overline{N}_2(z)N_1(z), \overline{D}_2(z)D_3(z)).$$

Then

$$\overline{N}_2 N_1 = \Delta \overline{N}_{21}, \quad \overline{D}_2 D_3 = \Delta \overline{D}_{23}, \quad \overline{N}_{21}(z), \overline{D}_{23}(z) \text{ left coprime.}$$
Hankel operator
The Hankel operator of $\Gamma$ is defined as the linear map

$$H_{\Gamma} : \mathbb{F}[z]^{p2} \rightarrow z^{-1}\mathbb{F}[[z^{-1}]]^{m1}, \quad H_{\Gamma}f = \pi_-(\Gamma(z)f(z)).$$

Theorem 4.
Let $\delta(z) = \text{g. c. l. d.}(D_3(z), N_2(z)\hat{N}_1(z))$; let $\Delta(z) := \text{g. c. l. d.}(\overline{N}_2(z)N_1(z), \overline{D}_2(z)D_3(z))$.
Let $\Gamma(z) := N_1^\top G_2^\top D_3^{-\top} = (\overline{N}_{21})^\top(\overline{D}_{23})^{-1}$, right coprime MFD.

The following assertions are equivalent:

1. The series connection $\Sigma_1 \rightarrow \Sigma_2 \rightarrow \Sigma_3$ is $(1, 3)$–partial state reachable.
2. The Toeplitz operator $T_{[G_2N_1,D_3]} : \mathbb{F}[z]^{m_1+m_3} \rightarrow \mathbb{F}[z]^{p2}$ is surjective.
3. $$\mathbb{F}[z]^{p2} = \delta\mathbb{F}[z]^{p2} + \pi_+(N_2\hat{N}_1\hat{D}_2^{-1}X_{\hat{D}_2}).$$
4. The restriction $H_{\Gamma}|_{X_{D_3^\top}}$ of the Hankel operator $H_{\Gamma} : \mathbb{F}[z]^{p2} \rightarrow z^{-1}\mathbb{F}[[z^{-1}]]^{m1}$ on the polynomial model $X_{D_3^\top}$ is injective.
5. $$X_{D_3^\top} \cap \overline{D}_{23}^\top\mathbb{F}[z]^{p2} = \{0\}.$$
Example SISO Systems

Choose coprime factorizations $g_i(z) = \frac{n_i(z)}{d_i(z)}$, $i = 1, 2, 3$. Let $\Delta(z)$ denote the g.c.d. of the polynomials $n_1(z)n_2(z)$ and $d_2(z)d_3(z)$. Then there are unique polynomials $n_{12}(z)$ and $d_{23}(z)$ with

$$n_1(z)n_2(z) = \Delta(z)n_{12}(z), \quad d_2(z)d_3(z) = \Delta(z)d_{23}(z).$$

By condition (5.) in Theorem 4, the cascade is \((1, 3)\)-partial state reachable if and only if

$$\deg d_3(z) \leq \deg d_{23}(z),$$

i.e., if and only if

$$\deg \Delta(z) \leq \deg d_2(z).$$

In contrast full reachability is equivalent to coprimeness of $n_1n_2$ and $d_2d_3$. 
Examples:

1. The series connection

\[
\frac{z + 1}{z(z + 2)} \rightarrow \frac{z}{(z + 2)(z + 1)} \rightarrow \frac{1}{z + 1}
\]

cannot be fully reachable, however, it is (1,3)-reachable.

2. If \( \Sigma_2 \) is a delay system \( g_2(z) = \frac{K}{z^\tau} \), then (1, 3)-partial state reachability of \( \Sigma_1 \rightarrow \Sigma_2 \rightarrow \Sigma_3 \) equivalent to

\[
\text{deg} \Delta(z) \leq \tau.
\]

This shows that partial state reachability imposes a constraint on the delays.

For example, the (non-reachable) cascade

\[
\frac{z^3}{(z + 1)^2(z + 2)^2} \rightarrow \frac{K}{z^\tau} \rightarrow \frac{1}{z(z + 1)}
\]

is (1, 3)-partially state reachable if and only if \( \tau \geq 3 \).
Parallel Connection

... Trivial.

Parallel-Series Connection

**Theorem 5:**
Assume that $D_1$ and $\overline{D}_1$ are left coprime; let $X$, $Y$ be right coprime polynomial matrices satisfying $D_1X + D_2Y = 0$. Let $D = \text{diag}(\overline{D}_2, D_3)$ and

$$
\Gamma = \begin{bmatrix}
\overline{N}_1X \\
G_2N_1Y
\end{bmatrix}.
$$

The parallel-series connection of $\Sigma_1 \rightarrow \Sigma_2 \rightarrow \Sigma_3$ with $\overline{\Sigma}_1 \rightarrow \overline{\Sigma}_2$ is $((1, 3); (1, 2))$–partial state reachable if and only if

$$F[z]^\bullet = D(z)F[z]^\bullet + \text{Tr}(F[z]^\bullet).$$
Part IV: Partial State Observability
4. Partial Observability

1. \((I, J)\)-Partial Observability of \(\Sigma_1 \to \cdots \to \Sigma_r\)

Let \(I, J\) be disjoint subsets of \(\{1, \ldots, r\}\) with \(I \neq \emptyset\)

\(I^c := \{1, \ldots, r\} \setminus I\): complement of \(I\).

**Definition 2’:**

The \(\Sigma_1 \to \cdots \to \Sigma_r\) series connection is \((I, J)\)-**partial state observable** if there exists \(T \geq 0\) such that initial states of the \(I\)-subsystems, i.e., \(x_i(0) \in \mathbb{F}^{n_i}\), \(i \in I\), can be unambiguously determined by:

- Initial states \(x_j(0) \in \mathbb{F}^{n_j}\), \(j \in J\),
- Output sequence \(y(0), y(1), \ldots, y(T)\) of the last system, \(\Sigma_r\),
- Input sequence \(u(0), \ldots, u(T)\) to the first system, \(\Sigma_1\)

For \(J = \emptyset\) we obtain \(I\)-partial state observability in the sense of definition 2,

For \(J = I^c\) i.t.o. dual of partial state reachability in the sense of definition 1.
W.O.L.O.G. focus on the case \( r = 2k + 1 \) and \( \mathcal{I} = \{1, 3, \ldots, 2k + 1\} \).

### Dual Toeplitz operator

Let \( \mathbb{T}_i^- : \) dual Toeplitz operator with symbol \( G_i(z) \).
\[
O_i : \mathbb{F}^{n_i} \to z^{-1}\mathbb{F}[[z^{-1}]]^{p_i} \text{ observability operator } O_i \xi = C_i (zI - A_i)^{-1} \xi.
\]

Let \( \mathcal{O} \) denote the observability operator of the full cascade \( \Sigma_1 \to \Sigma_2 \to \cdots \to \Sigma_{2k+1} \).

A straightforward computation reveals that
\[
\mathcal{O} := \left[ O_1, \mathbb{T}_1^- O_2, \cdots, \mathbb{T}_1^- \cdots \mathbb{T}_{2k}^- O_{2k+1} \right].
\]
Let
\[ O_I := \begin{bmatrix} O_1, & T_1 T_2 O_3, & \cdots, & T_1 \cdots T_{2k} O_{2k+1} \end{bmatrix} \]
denote the submatrix of \( O \) with block columns indexed by \( I \)
\( O_{IC} \) submatrix consisting of all complementary columns of \( O \) that are not in \( O_I \).

Let \( P \) denote a full row rank matrix with
\[ \text{Ker } P = \text{Im } O_{IC}. \]

**Theorem 6:**
The series connection \( \Sigma_1 \rightarrow \Sigma_2 \rightarrow \Sigma_3 \rightarrow \cdots \rightarrow \Sigma_N \) is \((1, 3, \ldots, 2k + 1)\)-partial state observable if and only if for all \( \xi \)
\[ O\xi = 0 \implies \xi_1 = \xi_3 \cdots = \xi_{2k+1} = 0 \quad (\ast). \]
Equivalently,
\[ I - \text{partial state observability} \iff PO_I \text{ injective}. \]
4. Partial Observability

More efficient characterization: Functional Observability (Fuhrmann)

\[ x_{t+1} = Ax_t + Bu_t \]
\[ y_t = Cx_t \]
\[ z_t = \mathcal{K}x_t \]

with \( x, y, u, z \) taking values in \( \mathbb{F}^n, \mathbb{F}^p, \mathbb{F}^m \) and \( \mathbb{F}^k \).

\( z \) is observable from \( (y, u) \) if each pair of solutions \( (x, y, u, z) \), \( (\bar{x}, y, u, \bar{z}) \) satisfies \( \bar{z} = z \).

Equivalently, all initial conditions \( x_0 \) that satisfy \( y_t = 0 \) and \( u_t = 0 \ \forall t \geq 0 \) also satisfy \( z_t = 0 \ \forall t \geq 0 \).

We also say that the system is functionally observable.

**Proposition (Fuhrmann, Helmke)**

Assume that \( \begin{bmatrix} C & \mathcal{K} \end{bmatrix} \), \( \mathcal{A} \) is observable.

Then the output \( z = \mathcal{K}x \) is observable from \( (y, u) \) if and only if \( (C, \mathcal{A}) \) is observable.
If \((A, B, C)\) arises from the cascade system and \(\mathcal{K}\) is of the form

\[
\mathcal{K} = \begin{bmatrix}
I & 0 & \cdots & 0 & 0 \\
0 & 0 & I & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 & I \\
\end{bmatrix},
\]

with identity matrices at columns 1, 3, \ldots, \(2k + 1\), then the series connection is \(\mathcal{I}\)-partial state reachable if and only if \(z = \mathcal{K}x\) is observable from \((y, u)\).

Application to the \(\mathcal{I}\)-partial observability problem.

**Theorem 7:**
Let \(r = 2k + 1\) and \(\mathcal{I} = \{1, 3, \ldots, 2k + 1\}\). Let \((A_{2i}, B_{2i}, C_{2i})\) be minimal with \(B_{2i+1}\) full column rank. Then the series connection is \(\mathcal{I}\)-partial state observable if and only if it is observable.

SURPRISE! One would have expected the dual notion of \(\mathcal{I}\)-partial state observability to be close to complete reachability of the entire cascade. It is not!
Part V: Examples
5. Examples

Example 1: (1)-partial observability in $\Sigma_1 \to \Sigma_2$ with $n_1 = 1$ and $n_2 = 2$, both systems observable.

Here

$$A = \begin{bmatrix}
\alpha \\
b_1 \\
b_2
\end{bmatrix}
\begin{bmatrix}
-a_1 & 1 \\
-b_2 & -a_2 & 0
\end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

The full observability matrix is

$$O(A, C) = \begin{bmatrix}
0 \\
b_1 \\
(\alpha - a_1)b_2 + b_2
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-a_1 & 1 \\
a_1^2 - a_2 & -a_1
\end{bmatrix}.$$

The (1)-partial observability matrix $[0 \ b_1 \ (\alpha - a_1)b_2 + b_2]^\top$ has full rank iff $[b_1 \ b_2]^\top \neq 0$. 
Example 2: (1)-partial observability in $\Sigma_1 \to \Sigma_2$ with $n_1 = 2$ and $n_2 = 1$, both systems observable.

Here

$$A = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ \beta & 0 & -\alpha \end{bmatrix}, \quad C = [0 \ 0 \ 1].$$

The full observability matrix is

$$O(A, C) = \begin{bmatrix} 0 & 0 & 1 \\ \beta & 0 & \alpha \\ (\alpha - a_1)\beta & \beta & \alpha^2 \end{bmatrix}.$$

The (1)-partial observability matrix has full rank iff

$$\beta \neq 0.$$
5. Examples

Example 3: (1)-partial observability in $\Sigma_1 \rightarrow \Sigma_2$ with $n_1 = 1$ and $n_2 = 2$, and $\Sigma_2$ not observable.

Here

$$\mathcal{A} = \begin{bmatrix} \alpha & b_1 & a & 0 \\ b_1 & a & 0 & 0 \\ b_2 & 0 & a & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & | & c_1 & c_2 \end{bmatrix}.$$  

The full observability matrix is

$$O(\mathcal{A}, \mathcal{C}) = \begin{bmatrix} 0 & \begin{array}{c} c_1 \\ c_1 b_1 + c_2 b_2 \\ (\alpha + a)(c_1 b_1 + c_2 b_2) \end{array} \\ \begin{array}{c} c_1 a \\ c_1 a^2 \\ c_1 a^2 \end{array} \end{bmatrix}.$$  

The (1)-partial observability matrix

$$\begin{bmatrix} 0 & c_1 b_1 + c_2 b_2 (\alpha + a)(c_1 b_1 + c_2 b_2) \end{bmatrix}^\top$$

has full rank iff

$$h_1 = c_1 b_1 + c_2 b_2 \neq 0.$$
Part VI: Dualities
1. Review: Partials

$\mathcal{K}$: matrix in that selects the states labeled by $\mathcal{I}$ that are to be assigned by the input.

$\mathcal{K}_c$: matrix that selects its complementary state variables labeled by $\mathcal{I}^c$.

$\mathcal{I}$–partial state reachability is equivalent to the full rank property of a “sufficiently large” matrix

$$\mathcal{K} R_N (A, B).$$

$\mathcal{I}$–partial state observability is the full rank property of the “sufficiently large” operator

$$P_N O_N (A, C) \mathcal{K}^\top$$

where $P_N$ is a full row rank matrix that satisfies

$$\text{Ker } P_N = \text{Im } O_N (A, C) \mathcal{K}_c^\top$$

Hence

$$\text{Im } P_N^\top = \text{Ker } \mathcal{K}_c R_N (A^\top, C^\top)$$

Obviously these two problems are not dual.
6. Dualities

2. Algebraic Duals

Partial state reachability and partial state observability each have a dual.

1. The series combination is $\mathcal{I}$-restricted observable if and only if $O_N(A, C)K^\top$ is injective.

2. The series combination is $\mathcal{I}$-restricted reachable if and only if $KR_N(A, B)P_N^\top$ is surjective, for $P_N$ satisfying

$$\text{Im } P_N^\top = \text{Ker } KcR_N(A, B).$$

3. System theoretic meaning

$\mathcal{I}$-restricted observability $\equiv (\mathcal{I}, \mathcal{I}^c)$-partial state observability.

$\mathcal{I}$-restricted reachability $\equiv$ ability of steering from zero to arbitrary states in $\mathcal{I}$, states in $\mathcal{I}^c$ return to 0.
Conclusions
3. Conclusion

1. **Motivated** by control problems for networks of linear discrete–time systems with communication delays on the interconnection channels. ACC-2016

2. Using methods from algebraic system theory, such as polynomial models and the shift operator, derived necessary and sufficient conditions for partial state reachability/observability of the system nodes in a series connection network.


4. Dualities restricted reachability and observability.

5. Extension of analysis to more general networks: MTNS-2016, Automatica-2016
Thank You

Thank You

Thank You