INTERPOLATION AND TRANSFER-FUNCTION REALIZATION FOR THE NONCOMMUTATIVE SCHUR-AGLER CLASS

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Abstract. The Schur-Agler class consists of functions over a domain satisfying an appropriate von Neumann inequality. Originally defined over the polydisk, the idea has been extended to general domains in multivariable complex Euclidean space with matrix polynomial defining function as well as to certain multivariable noncommutative-operator domains with a noncommutative linear-pencil defining function. Still more recently there has emerged a free noncommutative function theory (functions of noncommuting matrix variables respecting direct sums and similarity transformations). The purpose of the present paper is to extend the Schur-Agler-class theory to the free noncommutative function setting. This includes the positive-kernel-decomposition characterization of the class, transfer-function realization and Pick interpolation theory. A special class of defining functions is identified for which the associated Schur-Agler class coincides with the contractive-multiplier class on an associated noncommutative reproducing kernel Hilbert space; in this case, solution of the Pick interpolation problem is in terms of the complete positivity of an associated Pick matrix which is explicitly determined from the interpolation data.

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1. Introduction

The goal of this paper is to incorporate classical Nevanlinna-Pick interpolation into the the general setting of free noncommutative function theory as treated in the recent book [52]. To set the results into a broader context, we first review developments in Nevanlinna-Pick interpolation theory, beginning with the classical version, continuing with more elaborate versions involving matrix- and operator-valued interpolants for tangential-type interpolation conditions, then extensions to multivariable settings, and finally the free noncommutative setting.

We are now approaching the centennial of the Nevanlinna-Pick interpolation theorem which characterizes when there is a holomorphic map of the unit disk into its closure satisfying a finite collection of prescribed interpolation conditions:

Theorem 1.1. *(See Pick (1916)* [69] *and Nevanlinna (1919)* [66]*.) Given points $z_1, \dots, z_N$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ and associated preassigned values $\lambda_1, \dots, \lambda_N$ in the complex plane $\mathbb{C}$, there exists a holomorphic function s mapping the unit disk $\mathbb{D}$ into the closed unit disk $\overline{\mathbb{D}}$ and satisfying the interpolation conditions*

$$s(z_i) = \lambda_i \text{ for } i = 1, \dots, N \tag{1.1}$$

if and only if the $N \times N$ so-called Pick matrix $\mathbb{P}$ is positive-definite:

$$\mathbb{P} := \left[\frac{1 - \lambda_i \overline{\lambda_j}}{1 - z_i \overline{z_j}} \right]_{i,j=1,\dots,N} \succeq 0.$$

Much later in the late 1960s, Sarason [76] introduced an operator-theoretic point of view to the problem which led to the Commutant Lifting approach to a variety of more general matrix- and operator-valued interpolation and moment problems. We mention in particular the Fundamental Matrix Inequality approach (based on manipulation of positive operator-valued kernels) of Potapov (see [55] and the references there), the detailed application of the Commutant Lifting approach in the books of Foias-Frazho [43] and Gohberg-Foias-Frazho-Kaashoek [44], as well as the state-space approach of Ball-Gohberg-Rodman [20] for rational matrix functions. Much of this work was stimulated by the connections with and needs of H^∞-control, as also exposed in the books [43, 20, 44] which emphasized the connection between holomorphic functions and transfer functions of input/state/output linear systems. The following is a sample theorem from this era. For $\mathcal{X}$ and $\mathcal{X}^*$ any Hilbert spaces, we let $\mathcal{L}(\mathcal{X}, \mathcal{X}^*)$ denote the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{X}^*$. Let us use the notation $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ for the $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued *Schur class* consisting of holomorphic functions S mapping the unit disk $\mathbb{D}$ into the unit ball $\overline{\mathcal{B}}\mathcal{L}(\mathcal{U}, \mathcal{Y})$ of the space of operators $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ from $\mathcal{U}$ into $\mathcal{Y}$.

Theorem 1.2. *Assume that we are given a subset Ω of the unit disk $\mathbb{D} \subset \mathbb{C}$, three coefficient Hilbert spaces $\mathcal{E}, \mathcal{U}, \mathcal{Y}$, and functions $a \colon \Omega \to \mathcal{L}(\mathcal{Y}, \mathcal{E})$ and $b \colon : \Omega \to \mathcal{L}(\mathcal{U}, \mathcal{E})$. Then the following conditions are equivalent:*

(1) *There exists a Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ such that S satisfies the set of left-tangential interpolation conditions:*

$$a(z)S(z) = b(z) \text{ for each } z \in \Omega. \tag{1.2}$$

(2) *The generalized de Branges-Rovnyak kernel*

$$K_{a,b}^{\mathrm{dBR}}(z, w) := \frac{a(z)a(w)^* - b(z)b(w)^*}{1 - z\overline{w}}$$

is a positive kernel on Ω (written as $K_{a,b} \succeq 0$), i.e., for each finite set of points $\{z_1, \dots, z_N\}$ in Ω, the $N \times N$ block matrix

$$\left[\frac{a(z_i)a(z_j)^* - b(z_i)b(z_j)^*}{1 - z_i \overline{z_j}} \right]_{i,j=1,\dots,N}$$

is a positive semidefinite matrix.

(3) *There is an auxiliary Hilbert space $\mathcal{X}$ and a contractive (or even unitary) colligation matrix*

$$\mathbf{U} := \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

so that the $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function S given by

$$S(z) = D + zC(I - zA)^{-1}B \tag{1.3}$$

satisfies the interpolation conditions (1.2) *on* Ω.

We note that the form (1.3) for a holomorphic function $S(z)$ on the disk is called a *transfer-function realization for* S due to the following connection with the associated input/state/output linear system

$$\mathbf{U}\colon \left\{ \begin{array}{ccc} x(n+1) & = & Ax(n) + Bu(n) \\ y(n) & = & Cx(n) + Du(n): \end{array} \right.$$

if one runs the system with an input string $\{u(n)\}_{n \geq 0}$ *and initial condition* $x(0) = 0$, *then the output string* $\{y(n)\}_{n \geq 0}$ *recursively generated by the system equations is given by*

$$\sum_{n=0}^{\infty} y(n) z^n = S(z) \cdot \sum_{n=0}^{\infty} u(n) z^n$$

where $S(z) = D + \sum_{n=1}^{\infty} CA^{n-1}Bz^n = D + zC(I - zA)^{-1}B$ *is as in* (1.3). Note also that equivalence (1) $\Leftrightarrow$ (2) for the special case where Ω is a finite set $\{z_1, \dots, z_N\}$, $\mathcal{E} = \mathcal{U} = \mathcal{Y} = \mathbb{C}$, $a(z_i) = 1$, $b(z_i) = \lambda_i$ in Theorem 1.2 amounts to the content of Theorem 1.1. Another salient special case is the equivalence (1) $\Leftrightarrow$ (3) for the special case $\Omega = \mathbb{D}$, $\mathcal{E} = \mathcal{Y}$, $a(z) = I_{\mathcal{Y}}$, $b(z) = S(z)$: then the content of Theorem 1.2 is the realization theorem for the $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued Schur class: any holomorphic function $S\colon \mathbb{D} \mapsto \overline{\mathcal{B}}\mathcal{L}(\mathcal{U}, \mathcal{Y})$ can be realized as the transfer function of a conservative input/state/output linear system (i.e., as in (1.3) with $\mathbf{U}$ unitary).

The extension to multivariable domains has several new ideas. First of all, we use a $s \times r$-matrix polynomial (or a holomorphic operator-valued function in possible generalizations) $Q(z)$ in d variables to define a domain $\mathbb{D}_Q$ by

$$\mathbb{D}_Q = \{z = (z_1, \dots, z_d) \in \mathbb{C}^d \colon \|Q(z)\|_{\mathbb{C}^{s \times r}} < 1\}.$$

Secondly, to get a theory parallel to the classical case, it is necessary to replace the Schur class of the domain $\mathbb{D}_Q$ (consisting of holomorphic functions $S\colon \mathbb{D}_Q \to \overline{\mathcal{B}}\mathcal{L}(\mathcal{U}, \mathcal{Y})$) with functions having matrix or operator rather than scalar arguments. Let $\mathcal{K}$ be any fixed auxiliary infinite-dimensional separable Hilbert space. For T a commutative d-tuple $T = (T_1, \dots, T_d)$ of operators on $\mathcal{K}$ with Taylor spectrum inside the region $\mathbb{D}_Q$ and S a holomorphic $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function on $\mathbb{D}_Q$, it is possible to use the Taylor functional calculus (using Vasilescu's adaptation of the Bochner-Martinelli kernel—see [9, 17] for details) to make sense of the function S applied to the commutative operator-tuple T to get an operator $S(T) \in \mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K})$. We define the *Schur-Agler class* $\mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ to consist of those holomorphic $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions on $\mathbb{D}_Q$ such that $\|S(T)\|_{\mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K})} \leq 1$ for all commutative operator d-tuples $T = (T_1, \dots, T_d)$ such that $\|Q(T)\| < 1$. A result from [9] guarantees that T has Taylor spectrum inside $\mathbb{D}_Q$ whenever $\|Q(T)\| < 1$, so the definition makes sense. We then may state our Q-analogue of Theorem 1.2 as follows.

Theorem 1.3. *Assume that Q is an $s \times r$-matrix-valued polynomial defining a domain $\mathbb{D}_Q \subset \mathbb{C}^d$ as above. Assume that we are given a subset Ω of $\mathbb{D}_Q$, three coefficient Hilbert spaces $\mathcal{E}, \mathcal{U}, \mathcal{Y}$, and functions $a \colon \Omega \to \mathcal{L}(\mathcal{Y}, \mathcal{E})$ and $b \colon : \Omega \to \mathcal{L}(\mathcal{U}, \mathcal{E})$. Then the following conditions are equivalent:*

(1) *There exists a Schur-Agler-class function $S \in \mathcal{S}_Q(\mathcal{U}, \mathcal{Y})$ such that S satisfies the set of left-tangential interpolation conditions:*

$$a(z)S(z) = b(z) \text{ for each } z \in \Omega. \tag{1.4}$$

(2) *There is an auxiliary Hilbert space $\mathcal{X}$ so that the kernel on Ω given by*

$$K_{a,b}(z,w) := a(z)a(w)^* - b(z)b(w)^*$$

has a factorization

$$K_{a,b}(z,w) = H(z)\left((I_s - Q(z)Q(w)^*) \otimes I_{\mathcal{X}}\right) H(w)^* \tag{1.5}$$

for some operator-valued function $H \colon \Omega \to \mathcal{L}(\mathbb{C}^r \otimes \mathcal{X}, \mathcal{E})$.

(3) *There is an auxiliary Hilbert space $\mathcal{X}$ and a contractive (or even unitary) colligation matrix*

$$\mathbf{U} := \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X}^s \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^r \\ \mathcal{Y} \end{bmatrix}$$

so that the $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function S given by

$$S(z) = D + C(I - (Q(z) \otimes I_{\mathcal{X}})A)^{-1}(Q(z) \otimes I_{\mathcal{X}})B \tag{1.6}$$

satisfies the interpolation conditions (1.4).

This theorem follows by pasting together various pieces from the results of [9, 17, 18] (see also [60] for a somewhat different setting and point of view). The seminal work of Agler [1, 2] handled the special case where Q is taken to be $Q_{\text{diag}}(z) := \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_d \end{bmatrix}$, so $\mathbb{D}_Q$ becomes the polydisk

$$\mathbb{D}^d = \{z = (z_1, \dots, z_d) \in \mathbb{C}^d \colon |z_k| < 1 \text{ for } k = 1, \dots, d\}.$$

In this setting, one can work with global power series representations rather than the more involved Taylor functional calculus. For a thorough current (as of 2009) update of the ideas from the Agler unpublished manuscript [2], we recommend the paper of Jury-Knese-McCullough [50]. Additional work on the polydisk case was obtained in [3, 25]. Another important special case is the case where Q is taken to be $Q_{\text{row}}(z) = \begin{bmatrix} z_1 & \cdots & z_d \end{bmatrix}$. Various pieces of Theorem 1.3 for this special case appear in [26, 10, 38]. For the case $Q = Q_{\text{row}}$, the Schur-Agler class $\mathcal{SA}_{Q_{\text{row}}}(\mathcal{U}, \mathcal{Y})$ coincides with the contractive multiplier class associated with the Drury-Arveson kernel—we say more about this special situation in Subsection 6.3 below. The case where $Q(z) = \begin{bmatrix} Q_1(z) & & \\ & \ddots & \\ & & Q_N(z) \end{bmatrix}$ where each $Q_k(z)$ has the form $Q_{\text{row}}(z)$ (of various sizes $d_1, \dots, d_N$) was considered by Tomerlin in [80]. Note that $Q_{\text{diag}}(z)$ is the special case of this where each $d_k = 1$. One can also see the ideas of [2] as influencing the test-function approach of Dritschel-McCullough and collaborators, originating in [36, 37] with followup work in [59, 39, 50, 22].

The next development in our story is the extension to noncommutative variables. With motivation from multidimensional system theory, Ball-Groenewald-Malakorn [21] introduced a noncommutative Schur-Agler class defined as follows. We now let $z = (z_1, \dots, z_d)$ be freely noncommuting formal indeterminates. We let $\mathbb{F}_d^+$ be the unital free semigroup (i.e., monoid in the language of algebraists) on d generators, denoted here by the first d natural numbers $\{1, \dots, d\}$. Thus elements of $\mathbb{F}_d^+$ consist of words $\mathfrak{a} = i_N \cdots i_1$ where each letter i_k is in the set $\{1, \dots, d\}$ with multiplication given by concatenation:

$$\mathfrak{a} \cdot \mathfrak{b} = i_N \cdots i_1 j_M \cdots j_1 \text{ if } \mathfrak{a} = i_N \cdots i_1 \text{ and } \mathfrak{b} = j_M \cdots j_1.$$

Furthermore, we consider the empty word $\emptyset$ as an element of $\mathbb{F}_d^+$ and let this serve as the unit element for $\mathbb{F}_d^+$. For $\mathfrak{a} \in \mathbb{F}_d^+$, we define the noncommutative monomial $z^{\mathfrak{a}}$ by

$$z^{\mathfrak{a}} = z_{i_N} \cdots z_{i_1} \text{ if } \mathfrak{a} = i_N \cdots i_1, \quad z^{\emptyset} = 1.$$

For $\mathcal{V}$ a vector space, we let $\mathcal{V}\langle\langle z \rangle\rangle$ be the space of formal power series

$$f(z) = \sum_{\mathfrak{a} \in \mathbb{F}_d^+} f_{\mathfrak{a}} z^{\mathfrak{a}}$$

with coefficients $f_{\mathfrak{a}}$ in $\mathcal{V}$. If all but finitely many of the coefficients $f_{\mathfrak{a}}$ vanish, we refer to $f(z)$ as a **noncommutative polynomial** with the notation $f \in \mathcal{V}\langle z \rangle$. Let us consider the special case where the noncommutative polynomial involves only linear terms

$$Q(z) = \sum_{k=1}^{d} L_k z_k$$

and where $\mathcal{V}$ is taken to be the space $\mathbb{C}^{s \times r}$; the paper [21] assumes some other structure on $Q(z)$ details of which we need not go into here. We define a noncommutative domain $\mathbb{D}_Q \subset \mathcal{L}(\mathcal{K})^d$ consisting of operator tuples $T = (T_1, \dots, T_d) \in \mathcal{L}(\mathcal{K})^d$ (now not necessarily commutative) such that $Q(T) := \sum_{k=1}^{d} L_k \otimes T_k$ has $\|Q(T)\| < 1$. For $\mathfrak{a}$ a word in $\mathbb{F}_d^+$ and for $T = (T_1, \dots, T_d) \in \mathcal{L}(\mathcal{K})^d$, we use the noncommutative functional calculus notation

$$T^{\mathfrak{a}} = T_{i_N} \cdots T_{i_1} \in \mathcal{L}(\mathcal{K}) \text{ if } \mathfrak{a} = i_N \cdots i_1 \in \mathbb{F}_d^+, \quad T^{\emptyset} = I_{\mathcal{K}}.$$

where now the multiplication is operator composition rather than concatenation. Given a formal series $S(z) = \sum_{\mathfrak{a} \in \mathbb{F}_d^+} S_{\mathfrak{a}} z^{\mathfrak{a}}$ with coefficients $S_{\mathfrak{a}} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ and given a operator d-tuple $T = (T_1, \dots, T_d)$, we define

$$S(T) = \sum_{\mathfrak{a} \in \mathbb{F}_d^+} S_{\mathfrak{a}} \otimes T^{\mathfrak{a}} \in \mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K}) \tag{1.7}$$

whenever the series converges in some reasonable sense. Then we define the noncommutative Schur-Agler class $\mathcal{SA}_Q(\mathcal{U},\mathcal{Y})$ to consist of all formal power series $S(z) = \sum_{\mathfrak{a}\in\mathbb{F}_d^+} S_\mathfrak{a} z^\mathfrak{a}$ in $\mathcal{L}(\mathcal{U},\mathcal{Y})\langle\langle z\rangle\rangle$ such that $S(T)$ is defined and $\|S(T)\| \le 1$ for all $T = (T_1,\dots,T_d) \in \mathcal{L}(\mathcal{K})^d$ such that $\|Q(T)\| < 1$. Then the main result from [21] is the following realization theorem for the Schur-Agler class $\mathcal{SA}_Q(\mathcal{U},\mathcal{Y})$ (where $Q(z) = \sum_{k=1}^d L_k z_k$ is a noncommutative linear function having some additional structure not discussed here).

Theorem 1.4. *Suppose that Q is a linear pencil as discussed above and suppose that $S \in \mathcal{L}(\mathcal{U},\mathcal{Y})\langle\langle z\rangle\rangle$ is a given formal power series. Then the following conditions are equivalent.*

1. *S is in the noncommutative Schur-Agler class $\mathcal{SA}_Q(\mathcal{U},\mathcal{Y})$.*
2. *The noncommutative formal kernel $K_{I,S}(z,w) = I - S(z)S(w)^*$ has a formal noncommutative Agler decomposition, i.e.: there exist an auxiliary Hilbert space $\mathcal{X}$ and a formal power series $H(z) \in \mathcal{L}(\mathbb{C}^r \otimes \mathcal{X},\mathcal{Y})\langle\langle z\rangle\rangle$ so that*
$$K_{I,S}(z,w) = H(z)\left((I_{\mathbb{C}^s} - Q(z)Q(w)^*)\otimes I_\mathcal{X}\right)H(w)^*.$$
3. *There is an auxiliary Hilbert space $\mathcal{X}$ and a unitary colligation matrix $\mathbf{U}$*
$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X}^s \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^r \\ \mathcal{Y} \end{bmatrix}$$
so that
$$S(z) = D + C(I - (Q(z)\otimes I_\mathcal{X})A)^{-1}(Q(z)\otimes I_\mathcal{X})B.$$

Theorem 1.4 has a couple of limitations: (1) it is missing an interpolation-theoretic aspect and (2) it is tied noncommutative domains defined by linear pencils, thereby guaranteeing global formal power series representations for noncommutative functions on the associated domain $\mathbb{D}_Q$. While there has been some work on an interpolation theory for such domains $\mathbb{D}_Q$ (see [71, 32, 62, 51, 19, 14]) and thereby addressing the first limitation, the second limitation is more fundamental and is the main inspiration for the present paper.

Motivation comes from a result of Alpay–Kaliuzhnyi-Verbovetskyi [8] which says that that one need only plug in $T = (T_1,\dots,T_d) \in \mathcal{L}(\mathbb{C}^n)^d$ (i.e., a d-tuple of $n\times n$ matrices) sweeping over all $n \in \mathbb{N}$ to determine if a given power series $S \in \mathcal{L}(\mathcal{U},\mathcal{Y})\langle\langle z\rangle\rangle$ is in the noncommutative Schur-Agler class $\mathcal{SA}_Q(\mathcal{U},\mathcal{Y})$ for $Q(z)$ a linear nc function (specifically $Q(z) = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_d \end{bmatrix}$) as in Theorem 1.4. We may thus view S as a function from the disjoint union $(\mathbb{D}_Q)_{\rm nc} := \amalg_{n=1}^\infty (\mathbb{D}_Q)_n$, where we set
$$(\mathbb{D}_Q)_n = \{Z = (Z_1,\dots,Z_d) \in (\mathbb{C}^{n\times n})^d \colon \|Q(Z)\| < 1\},$$

into the space $\mathcal{L}(\mathcal{U},\mathcal{Y})_{\rm nc} = \amalg_{n=1}^{\infty}\mathcal{L}(\mathcal{U},\mathcal{Y})^{n\times n}$ (where we identify $\mathcal{L}(\mathcal{U},\mathcal{Y})^{n\times n}$ with $\mathcal{L}(\mathcal{U}^n,\mathcal{Y}^n)$ when convenient). It is easily checked that the such a function S, when given by a power series representation as in (1.7), satisfies the following axioms:

(A1) S is **graded**, i.e., S maps $(\mathbb{D}_Q)_n$ into $\mathcal{L}(\mathcal{U},\mathcal{Y})^{n\times n}$,

(A2) S **respects direct sums**, i.e., if $Z = (Z_1,\dots,Z_d) \in (\mathbb{D}_Q)_n$ and $W = (W_1,\dots,W_d) \in (\mathbb{D}_Q)_m$ and we set

$$\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} = \left(\begin{bmatrix} Z_1 & 0 \\ 0 & W_1 \end{bmatrix}, \dots, \begin{bmatrix} Z_d & 0 \\ 0 & W_d \end{bmatrix} \right),$$

then

$$S\left(\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} \right) = \begin{bmatrix} S(Z) & 0 \\ 0 & S(W) \end{bmatrix}.$$

(A3) S **respects similarities**, i.e., if α is an invertible $n \times n$ matrix over $\mathbb{C}$, if $Z = (Z_1,\dots,Z_d)$ and $\alpha^{-1}Z\alpha := (\alpha^{-1}Z_1\alpha,\cdots,\alpha^{-1}Z_d\alpha)$ are both in $\mathbb{D}_Q$, then it follows that

$$S(\alpha^{-1}Z\alpha) = \alpha^{-1}S(Z)\alpha.$$

Such an axiom system for an operator-valued function S defined on square-matrix tuples of all possible sizes was introduced by J.L. Taylor [79, Section 6] in connections with representations of the free algebra and the quest for a functional calculus for noncommuting operator tuples. Recent work of Kaliuzhnyi-Verbovetskyi and Vinnikov [52] provides additional insight and completeness into the work of Taylor; in particular, there it is shown that, under mild local boundedness conditions, any function S satisfying the axioms (A1), (A2), (A3) is given locally by a power series representation of the sort in (1.7). Closely related approaches to and results on such a "free noncommutative function theory" can be found in the work of Voiculescu [81, 82] and Helton-Klep-McCullough [45, 46, 47].

The purpose of the present paper is to extend the Nevanlinna-Pick interpolation and transfer-function realization theory for the Schur-Agler class as presented in the progression of Theorems 1.1, 1.2, 1.3, 1.4 to the general setting of free noncommutative function theory. There has already appeared results in this direction in the work of Agler-McCarthy [6, 7] (see also [15]). Our main result (see Theorem 3.1) presents a more unified setting for their results as well as extending their results to a more natural level of generality (see the comments immediately after Corollary 3.8 below).

The proof strategy for Theorems 1.3, 1.4 as well as Theorem 3.1 has a common skeleton which originates from the seminal 1990 paper of Agler [1]. Indeed, in all these Theorems (as well as in the assorted incremental versions done in [9, 17, 22, 36, 37]), the proof of (1) $\Rightarrow$ (2) involves a cone separation argument; an exception is the work of Paulsen and collaborators [56, 60] where the cone-separation argument is replaced by an operator-algebra approach with an appeal to the Blecher-Ruan-Sinclair characterization of operator algebras. A new feature for the noncommutative setting of Theorem 3.1, first observed by Agler-McCarthy in [7], is that the cone separation argument still applies with a localized weaker version (see condition (1′) in Theorem 3.1) of condition (1) as a hypothesis. The implication (2) ⇒ (3) (in Theorems 1.3, 1.4 as well as in 3.1 and an even larger assortment of closely related versions in the literature) is via what is now called the *lurking isometry argument.* While this idea first appears in [1] for the context of multivariable interpolation, it actually has much earlier manifestations already in the univariate theory: we mention the early work of Livšic [57, 58] where the Characteristic Operator Function was first introduced, the proof of the Nevanlinna-Pick interpolation theorem due to Sz.-Nagy–Koranyi [77], and the Abstract Interpolation Problem framework for interpolation and moment problems due to Katsnelson-Kheifets-Yuditskii [53, 54, 55]. The proof of (3) ⇒ (1) is a straightforward application of a general principle on composing a contractive linear fractional map with a contractive load (we refer [48] for a general formulation). Here we also introduce the notion of **complete Pick kernel** (see [4, 72] as well as the book [5]) for the free noncommutative setting.

The paper is organized as follows. Section 2 collects preliminary material needed for the proof of Theorem 3.1 as well as pushing out the boundary of the free noncommutative function theory. Included here is a review of the basics of free noncommutative function theory from [52] as well as some additional material relevant to the proof of Theorem 3.1: some calculus and open questions concerning **full noncommutative envelopes** and **noncommutative-Zariski closure** of any finite subset Ω of a full noncommutative set Ξ, as well as a review of material from our companion paper [23] concerning **completely positive noncommutative kernels**, a notion needed for the very formulation of condition (2) in Theorem 3.1. Section 3 introduces the noncommutative Schur-Agler class $\mathcal{SA}_Q(\mathcal{U},\mathcal{Y})$ for the free noncommutative setting, poses the **Left-Tangential Interpolation Problem**, and states the main result Theorem 3.1 along with some corollaries and remarks exploring various special cases and consequences. Section 4 then presents the proof of Theorem 3.1 in systematic fashion one step at a time: (1) ⇒ (1′) ⇒ (2) ⇒ (3) ⇒ (1). Section 5 reviews material from [23] concerning noncommutative reproducing kernel Hilbert spaces, identifies a class of kernels k_{Q_0} for which the associated Schur-Agler class $\mathcal{SA}_{Q_0}(\mathcal{U},\mathcal{Y})$ coincides with the contractive multiplier class $\overline{\mathcal{B}}\mathcal{M}(k_{Q_0}\otimes I_{\mathcal{U}}, k_{Q_0}\otimes I_{\mathcal{Y}})$ (multiplication operators mapping the reproducing kernel Hilbert space $\mathcal{H}(k_{Q_0}\otimes I_{\mathcal{U}})$ contractively into $\mathcal{H}(k_{Q_0}\otimes I_{\mathcal{Y}})$), thereby getting noncommutative versions of the Nevanlinna-Pick interpolation and transfer-function realization theory for contractive multipliers on the Drury-Arveson space (see [13, 26, 42]).

2. Preliminaries

We review some preliminary material on noncommutative functions and completely positive noncommutative kernels from [23] which are needed in the sequel. A comprehensive treatise on the topic of noncommutative functions (but with no discussion of noncommutative kernels which are first introduced in [23]) is the book of Kaliuzhnyi-Verbovetskyi and Vinnikov [52]. Henceforth we shall use the abbreviation *nc* for the term *noncommutative*.

2.1. **Noncommutative functions and completely positive noncommutative kernels.** We suppose that we are given a vector space $\mathcal{V}$. Thus $\mathcal{V}$ is equipped with a scalar multiplication by complex numbers which makes $\mathcal{V}$ a bimodule over $\mathbb{C}$. We define the associated nc space $\mathcal{V}_{\rm nc}$ to consist of the disjoint union over all $n \in \mathbb{N}$ of $n \times n$ matrices over $\mathcal{V}$:

$$\mathcal{V}_{\rm nc} = \amalg_{n=1}^{\infty} \mathcal{V}^{n \times n}.$$

A subset Ω of $\mathcal{V}_{\rm nc}$ is said to be a **nc set** if Ω is closed under direct sums:

$$Z = [Z_{ij}]_{i,j=1}^{n} \in \Omega_n,\ W = [W_{ij}]_{i,j=1}^{m} \in \Omega_m \Rightarrow Z \oplus W = \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] \in \Omega_{n+m}.$$

Suppose next that Ω is a subset of $\mathcal{V}_{\rm nc}$), and that $\mathcal{V}_0$ is another vector space (i.e., a bimodule over $\mathbb{C}$). For $\alpha \in \mathbb{C}^{n \times m}$, $V \in \mathcal{V}_0^{m \times k}$, $\beta \in \mathbb{C}^{k \times \ell}$, we can use the module structure of $\mathcal{V}_0$ over $\mathbb{C}$ to make sense of the matrix multiplication $\alpha V \beta \in \mathcal{V}_0^{n \times \ell}$ and similarly $\alpha Z \beta$ makes sense as an element of $\mathcal{V}^{n \times \ell}$ for $Z \in \mathcal{V}^{m \times k}$. Given a function $f \colon \Omega \to \mathcal{V}_{0,nc}$, we say that f is a **noncommutative (nc) function** if

- f is **graded**, i.e., $f \colon \Omega_n \to \mathcal{V}_{0,n} = (\mathcal{V}_0)^{n \times n}$, and
- f **respects intertwinings**:

$$Z \in \Omega_n,\ \widetilde{Z} \in \Omega_m,\ \alpha \in \mathbb{C}^{m \times n} \text{ with } \alpha Z = \widetilde{Z} \alpha \Rightarrow \alpha f(Z) = f(\widetilde{Z}) \alpha. \tag{2.1}$$

An equivalent characterization of nc functions is (see [52, Section I.2.3]): f is a nc function if and only if

- f is a graded,
- f **respects direct sums**, i.e.,

$$Z, W \in \Omega \text{ such that also } \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] \in \Omega \Rightarrow f\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right) = \begin{bmatrix} f(Z) & 0 \\ 0 & f(W) \end{bmatrix}, \tag{2.2}$$

  and
- f **respects similarities**, i.e.: whenever $Z, \widetilde{Z} \in \Omega_n$, $\alpha \in \mathbb{C}^{n \times n}$ with α invertible such that $\widetilde{Z} = \alpha Z \alpha^{-1}$, then

$$f(\widetilde{Z}) = \alpha f(Z) \alpha^{-1}. \tag{2.3}$$

Following [52], we denote the set of all nc functions from Ω into $\mathcal{V}_{0,\rm nc}$ by $\mathcal{T}(\Omega; \mathcal{V}_{0,\rm nc})$; we note that we do not require that the domain Ω for a nc function be a nc set as is done in [52].

We now suppose that we are given two additional vector spaces $\mathcal{V}_0$ and $\mathcal{V}_1$. For K a function from $\Omega \times \Omega$ into the nc space

$$\mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)_{\rm nc} := \amalg_{n,m=1}^{\infty} \mathcal{L}(\mathcal{V}_1^{n \times m}, \mathcal{V}_0^{n \times m}),$$

we say that K is a **nc kernel** if

- K is **graded** in the sense that

$$Z \in \Omega_n,\ W \in \Omega_m \Rightarrow K(Z,W) \in \mathcal{L}(\mathcal{V}_1^{n\times m}, \mathcal{V}_0^{n\times m}) \tag{2.4}$$

and

- K **respects intertwinings** in the following sense:

$$\begin{aligned} &Z \in \Omega_n,\ \widetilde{Z} \in \Omega_{\widetilde{n}},\ \alpha \in \mathbb{C}^{\widetilde{n}\times n} \text{ such that } \alpha Z = \widetilde{Z}\alpha, \\ &W \in \Omega_m,\ \widetilde{W} \in \Omega_{\widetilde{m}},\ \beta \in \mathbb{C}^{\widetilde{m}\times m} \text{ such that } \beta W = \widetilde{W}\beta. \\ &P \in \mathcal{V}_1^{n\times m} \Rightarrow \alpha\, K(Z,W)(P)\, \beta^* = K(\widetilde{Z},\widetilde{W})(\alpha P \beta^*). \end{aligned} \tag{2.5}$$

An equivalent set of conditions is:

- K is graded,
- K **respects direct sums**: for $Z \in \Omega_n$ and $\widetilde{Z} \in \Omega_{\widetilde{n}}$ such that $\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] \in \Omega_{n+m}$, $W \in \Omega_m$ and $\widetilde{W} \in \Omega_{\widetilde{m}}$ such that $\left[\begin{smallmatrix} W & 0 \\ 0 & \widetilde{W} \end{smallmatrix}\right] \in \Omega_{m+\widetilde{m}}$, and $P = \left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{smallmatrix}\right]$ in $\mathcal{V}_1^{(n+m)\times(\widetilde{n}+\widetilde{m})}$, we have

$$K\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & \widetilde{Z} \end{smallmatrix}\right], \left[\begin{smallmatrix} W & 0 \\ 0 & \widetilde{W} \end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{smallmatrix}\right]\right) = \begin{bmatrix} K(Z,W)(P_{11}) & K(Z,\widetilde{W})(P_{12}) \\ K(\widetilde{Z},W)(P_{21}) & K(\widetilde{Z},\widetilde{W})(P_{22}) \end{bmatrix}. \tag{2.6}$$

- K **respects similarities**:

$$\begin{aligned} &Z, \widetilde{Z} \in \Omega_n,\ \alpha \in \mathbb{C}^{n\times n} \text{ invertible with } \widetilde{Z} = \alpha Z \alpha^{-1}, \\ &W, \widetilde{W} \in \Omega_m, \beta \in \mathbb{C}^{m\times m} \text{ invertible with } \widetilde{W} = \beta W \beta^{-1}, \\ &P \in \mathcal{A}^{n\times m} \Rightarrow K(\widetilde{Z},\widetilde{W})(P) = \alpha\, K(Z,W)(\alpha^{-1} P \beta^{-1*})\, \beta^*. \end{aligned} \tag{2.7}$$

We denote the class of all such nc kernels by $\widetilde{\mathcal{T}}^1(\Omega; \mathcal{V}_{1,\mathrm{nc}}, \mathcal{V}_{0,\mathrm{nc}})$.

For the next definition we need to impose an order structure on $\mathcal{V}_0$ and $\mathcal{V}_1$ so that an appropriate notion of positivity is defined for square matrices over $\mathcal{V}_0$ and $\mathcal{V}_1$. Recall (see [68, 40]) that a normed linear space $\mathcal{W}$ is said to be an **operator space** if it is equipped with a system of norms $\|\cdot\|_n$ on $n \times n$ matrices over $\mathcal{W}$ so that there is a map $\varphi\colon \mathcal{W} \to \mathcal{L}(\mathcal{X})$ (where $\mathcal{X}$ is some Hilbert space) so that, for each $n = 1, 2, \dots$, the map

$$\varphi^{(n)} = 1_{\mathbb{C}^{n\times n}} \otimes \varphi\colon \mathcal{W}^{n\times n} \to \mathcal{L}(\mathcal{X})^{n\times n}$$

defined by

$$\mathrm{id}_{\mathbb{C}^{n\times n}} \otimes \varphi\colon [w_{ij}]_{i,j=1,\dots,n} \mapsto [\varphi(w_{ij})]_{i,j=1,\dots,n} \tag{2.8}$$

is a linear isometry. By the theorem of Ruan [40, Theorem 2.3.5], such a situation is characterized by the system of norms satisfying

$$\begin{aligned} &\|X \oplus Y\|_{n+m} = \max\{\|X\|_n, \|Y\|_m\}, \text{ and} \\ &\|\alpha X \beta\|_m \le \|\alpha\| \|X\|_n \|\beta\| \text{ for all } X \in \mathcal{W}^{n\times n},\ \alpha \in \mathbb{C}^{m\times n}, \beta \in \mathbb{C}^{n\times m}. \end{aligned} \tag{2.9}$$

We say that $\mathcal{W}$ is an **operator system** if $\mathcal{W}$ is an operator space such that the image $\varphi(\mathcal{W})$ of $\mathcal{W}$ under the map φ given above is a unital subspace of $\mathcal{L}(\mathcal{X})$ closed under taking adjoints. Note that the set of selfadjoint elements of $\varphi(\mathcal{W})$ is nonempty since $\varphi(\mathcal{W})$ contains the identity element $1_{\mathcal{X}}$. Then the adjoint operation and the notion of positivity in $\mathcal{L}(\mathcal{X})$ pulls back to $\mathcal{W}$ (as well as to square matrices $\mathcal{W}^{n\times n}$ over $\mathcal{W}$). Operator systems also have an abstract characterization: *if $\mathcal{W}$ is a matrix-ordered $*$-vector space with an Archimedean matrix order unit e, then $\mathcal{W}$ is completely order isomorphic (and hence also completely isometrically isomorphic) to a concrete operator system* by a theorem of Choi and Effros (see [68, Theorem 13.1]).

We now specialize the setting for our nc kernels defined by (2.4), (2.5), (2.6), (2.7) by assuming that the vector spaces $\mathcal{V}_1$ and $\mathcal{V}_0$ are both operator systems, now denoted as $\mathcal{S}_1$ and $\mathcal{S}_0$ respectively. Given a nc kernel $K \in \widetilde{\mathcal{T}}^1(\Omega; \mathcal{S}_{0,\rm nc}, \mathcal{S}_{1,\rm nc})$, we say that K is **completely positive** (cp) if in addition, the map

$$[P_{ij}]_{i,j=1,\dots,n} \mapsto \Big[K(Z^{(i)}, Z^{(j)})(P_{ij})\Big]_{i,j=1,\dots,n} \tag{2.10}$$

is a positive map from $\mathcal{S}_1^{N\times N}$ to $\mathcal{S}_0^{N\times N}$ ($N = \sum_{i=1}^n m_i$) for any choice of $Z^{(i)} \in \Omega_{m_i}$, $m_i \in \mathbb{N}$, $i = 1, \dots, n$, n arbitrary. In case Ω is a nc subset of $\mathcal{V}_{\rm nc}$, we can iterate the "respects direct sums" property (2.6) of K to see that

$$K\left(\begin{bmatrix} Z^{(1)} & & \\ & \ddots & \\ & & Z^{(n)} \end{bmatrix}, \begin{bmatrix} Z^{(1)} & & \\ & \ddots & \\ & & Z^{(n)} \end{bmatrix}\right)([P_{ij}]) = \Big[K(Z^{(i)}, Z^{(j)})(P_{ij})\Big]_{i,j=1,\dots,n},$$

for any choice of $Z^{(i)} \in \Omega_{n_i}$, $P_{ij} \in \mathcal{S}_1^{n_i\times n_j}$, $i = 1, \dots, n$. Then the condition (2.10) can be written more simply as

$$Z \in \Omega_n,\ P \succeq 0 \text{ in } \mathcal{S}_1^{n\times n} \Rightarrow K(Z,Z)(P) \succeq 0 \text{ in } \mathcal{S}_0^{n\times n} \text{ for all } n \in \mathbb{N}. \tag{2.11}$$

If $\mathcal{V}_1 = \mathcal{A}_1$ is a C^*-algebra, we can rewrite (2.10) as

$$\Big[K(Z^{(i)}, Z^{(j)})(R_i^* R_j)\Big]_{i,j=1,\dots,n} \succeq 0 \text{ in } \mathcal{V}_0^{N\times N} \tag{2.12}$$

for all $R_i \in \mathcal{A}_1^{m_i\times N}$, $Z^{(i)} \in \Omega_{m_i}$, $i = 1, \dots, n$ with $n, N \in \mathbb{N}$ arbitrary. If $\mathcal{S}_0 = \mathcal{A}_0$ is also a C^*-algebra, (2.12) can turn can be equivalently expressed as

$$\sum_{i,j=1}^n V_i^* K(Z^{(i)}, Z^{(j)})(R_i^* R_j) V_j \succeq 0 \text{ in } \mathcal{V}_0 \tag{2.13}$$

for all $R_i \in \mathcal{A}_1^{N\times m_i}$, $Z^{(i)} \in \Omega_{m_i}$, $V_i \in \mathcal{A}_0^{m_i\times 1}$, $i = 1, \dots, n$ with $n \in \mathbb{N}$ arbitrary. When we restrict to the case $m_i = 1$ for all i, the formulation (2.13) amounts to the notion of complete positivity of a kernel given by Barreto-Bhat-Liebscher-Skeida in [27].

The more concrete setting for nc kernels which we shall be interested in here is as follows. We again take the ambient set of points to be the nc space $\mathcal{V}_{\rm nc}$ associated with a vector space $\mathcal{V}$, while for the operator systems $\mathcal{S}_0$ and $\mathcal{S}_1$ we take

$$\mathcal{S}_0 = \mathcal{L}(\mathcal{E}), \quad \mathcal{S}_1 = \mathcal{A}$$

where $\mathcal{E}$ is a coefficient Hilbert space and $\mathcal{A}$ is a C^*-algebra. Then we have have the following characterization of cp nc kernels for this setting from [23].

Theorem 2.1. *Suppose that Ω be a subset of $\mathcal{V}_{\rm nc}$, $\mathcal{E}$ is a Hilbert space, $\mathcal{A}$ is a C^*-algebra and $K\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{E}))_{\rm nc}$ is a given function. Then the following are equivalent.*

1. *K is a cp nc kernel from $\Omega \times \Omega$ to $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{E}))_{\rm nc}$.*
2. *There is a Hilbert space $\mathcal{H}(K)$ whose elements are nc functions $f\colon \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{E})_{\rm nc}$ such that:*
   (a) *For each $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$, and $y \in \mathcal{E}^m$, the function*
   $$K_{W,v,y}\colon \Omega_n \to \mathcal{L}(\mathcal{A}, \mathcal{E})^{n\times n} \cong \mathcal{L}(\mathcal{A}^n, \mathcal{E}^n)$$
   defined by
   $$K_{W,v,y}(Z)u = K(Z,W)(uv)y \tag{2.14}$$
   for $Z \in \Omega_n$, $u \in \mathcal{A}^n$ belongs to $\mathcal{H}(K)$.
   (b) *The kernel elements $K_{W,v,y}$ as in (2.14) have the reproducing property: for $f \in \mathcal{H}(K)$, $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$,*
   $$\langle f(W)(v^*), y\rangle_{\mathcal{E}^m} = \langle f, K_{W,v,y}\rangle_{\mathcal{H}(K)}. \tag{2.15}$$
   (c) *$\mathcal{H}(K)$ is equipped with a unital $*$-representation σ mapping $\mathcal{A}$ to $\mathcal{L}(\mathcal{H}(K))$ such that*
   $$\left(\sigma(a)f\right)(W)(v^*) = f(W)(v^*a) \tag{2.16}$$
   for $a \in \mathcal{A}$, $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$, with action on kernel elements $K_{W,v,y}$ given by
   $$\sigma(a)\colon K_{W,v,y} = K_{W,av,y}. \tag{2.17}$$
   *We then say that $\mathcal{H}(K)$ is the **noncommutative Reproducing Kernel Hilbert Space** (nc RKHS) associated with the cp nc kernel K.*
3. *K has a **Kolmogorov decomposition**: there is a Hilbert space $\mathcal{X}$ equipped with a unital $*$-representation $\sigma\colon \mathcal{A} \to \mathcal{L}(\mathcal{X})$ together with a nc function $H\colon \Omega \to \mathcal{L}(\mathcal{X}, \mathcal{E})_{\rm nc}$ so that*
   $$K(Z,W)(P) = H(Z)(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P)H(W)^* \tag{2.18}$$
   for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathcal{A}^{n\times m}$.

Remark 2.2. Theorem 3.1 in [23] assumes that Ω is a nc subset of $\mathcal{V}_{\rm nc}$ rather than an arbitrary subset. However, as is explained in Proposition 2.17 below, one can always extend a nc function/nc kernel/nc cp kernel on Ω uniquely to a nc function/nc kernel/nc cp kernel respectively on the nc envelope $[\Omega]_{\rm nc}$. With this fact in hand, one can see that there is no harm done in taking Ω to be an arbitrary subset of $\mathcal{V}_{\rm nc}$ in Theorem 2.1.

Example 2.3. Suppose that $\mathcal{V}$, $\mathcal{V}_0$ and $\mathcal{V}_1$ are complex vector spaces, and $\varphi \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)$ is a linear operator. There are two distinct procedures (at least) for relating φ to our nc function theory.

(a) Define a function $\boldsymbol{\varphi}\colon \mathcal{V}_{\rm nc} \times \mathcal{V}_{\rm nc} \to \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)_{\rm nc}$ by

$$\boldsymbol{\varphi}(Z, W) = {\rm id}_{\mathbb{C}^{n \times m}} \times \varphi\colon [a_{ij}]_{\substack{1 \le i \le n; \\ 1 \le j \le m}} \mapsto [\varphi(a_{ij})]_{\substack{1 \le i \le n; \\ 1 \le j \le m}} \tag{2.19}$$

for $Z \in \Omega_n$ and $W \in \Omega_m$ (so $\boldsymbol{\varphi}(Z, W) \in \mathcal{L}(\mathcal{V}_1^{n \times m}, \mathcal{V}_0^{n \times m})$ for $Z \in \Omega_n$, $W \in \Omega_m$). Thus $\boldsymbol{\varphi}(Z, W) \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)$ depends on Z, W only through the respective sizes n and m: $Z \in \mathcal{V}^{n \times n}$, $W \in \mathcal{V}^{m \times m}$. Thus we really have

$$\boldsymbol{\varphi}(Z, W) = \varphi^{(n,m)} := {\rm id}_{\mathbb{C}^{n \times m}} \otimes \varphi\colon \mathcal{V}_1^{n \times m} \to \mathcal{V}_0^{n \times m}.$$

The computation, for $\alpha \in \mathbb{C}^{n \times N}$, $X \in \mathcal{V}^{N \times M}$, $\beta \in \mathbb{C}^{M \times m}$,

$$\begin{aligned} \alpha \cdot \varphi^{(N,M)}(X) \cdot \beta &= [\alpha_{ij}] \cdot [\varphi(X_{ij})] \cdot [\beta_{ij}] \\ &= [\sum_{k,\ell} \alpha_{ik} \varphi(X_{ij}) \beta_{\ell j}] \\ &= [\varphi(\sum_{k,\ell} \alpha_{ik} X_{k\ell} \beta_{\ell j})] \\ &= \varphi^{(n,m)}(\alpha \cdot X \cdot \beta) \end{aligned} \tag{2.20}$$

shows that $\boldsymbol{\varphi}$ enjoys the **bimodule property**:

$$\alpha \cdot \boldsymbol{\varphi}(Z, W)(X) \cdot \beta = \boldsymbol{\varphi}(\widetilde{Z}, \widetilde{W})(\alpha \cdot X \cdot \beta) \tag{2.21}$$

for $\alpha \in \mathbb{C}^{n \times N}$, $X \in \mathcal{V}_0^{N \times M}$, $\beta \in \mathbb{C}^{M \times m}$, $Z \in \mathcal{V}^{N \times N}$, $W \in \mathcal{V}^{M \times M}$, $\widetilde{Z} \in \mathcal{V}^{n \times n}$, $\widetilde{W} \in \mathcal{V}^{m \times m}$. The "respects intertwining property" (2.5) for the kernel $\boldsymbol{\varphi}$ says that (2.21) holds whenever $Z, W, \widetilde{Z}, \widetilde{W}$ are related via

$$\alpha Z = \widetilde{Z} \alpha, \quad \beta^* W = \widetilde{W} \beta^*. \tag{2.22}$$

We conclude that the bimodule property (2.21) is formally stronger than the "respects intertwining property" in that, for given α, β, the bimodule property does not require one to search for points $Z, \widetilde{Z}, W, \widetilde{W}$ which satisfy the intertwining conditions (2.22). In any case, we conclude that $\boldsymbol{\varphi}$ so defined is a nc kernel. Let us say that a nc kernel of this form is a **nc constant kernel**.

In case $\mathcal{V}_1$ and $\mathcal{V}_0$ are operator systems and φ is a completely positive map in the sense of the operator algebra literature (see [40, 68]), the resulting kernel $\boldsymbol{\varphi}$ is furthermore a completely positive nc kernel; this example is discussed in some detail in [23, Section 3.3],

(b) For this construction we suppose that we have given only two vector spaces $\mathcal{V}$ and $\mathcal{V}_0$ and that φ is a linear map from $\mathcal{V}$ to $\mathcal{V}_0$. Define a map $L\colon \mathcal{V}_{\rm nc} \to \mathcal{V}_{0,{\rm nc}}$ by

$$L_\varphi(Z) = \varphi^{(n,n)}(Z) := [\varphi(Z_{ij})] \text{ for } Z = [Z_{ij}] \in \mathcal{V}^{n \times n}$$

where we use the notation $\varphi^{(n,n)}$ as in Example 2.3 (a) above. From the definition we see that L is graded, i.e., $L\colon \mathcal{V}^{n\times n} \to \mathcal{V}_0^{n\times n}$ for all $n \in \mathbb{N}$. To check the "respects intertwining" property (2.1), we use the bimodule property (2.20) to see that, if $\alpha \in \mathbb{C}^{m\times n}$, $Z \in \mathcal{V}^{n\times n}$, $\widetilde{Z} \in \mathcal{V}^{m\times m}$ are such that $\alpha Z = \widetilde{Z}\alpha$, then

$$\begin{aligned} \alpha \cdot L_\varphi(Z) &= \alpha \cdot \varphi^{(n\times n)}(Z) = \varphi^{(m\times n)}(\alpha \cdot Z) \\ &= \varphi^{(m\times n)}(\widetilde{Z} \cdot \alpha) = \varphi^{(m\times m)}(\widetilde{Z}) \cdot \alpha = L_\varphi(\widetilde{Z}) \cdot \alpha \end{aligned}$$

and it follows that L_φ is a nc function. We shall say that a nc function of this form is a **nc linear map**. In Section 3.2 we shall be particularly interested in the special case where $\mathcal{V}_0 = \mathcal{L}(\mathcal{R}, \mathcal{S})$, the space of bounded linear operators between two Hilbert spaces $\mathcal{R}$ and $\mathcal{S}$.

2.2. **Full noncommutative sets.** We shall be interested in the nc set $\Omega \subset \mathcal{V}_{\rm nc}$ on which our noncommutative functions are defined having some additional structure. In all these examples we suppose that $\mathcal{V}_{\rm nc}$ is the noncommutative set generated by a vector space $\mathcal{V}$.

Definition 2.4. We say that a subset Ξ of $\mathcal{V}_{\rm nc}$ is a **full nc subset** of $\mathcal{V}_{\rm nc}$ if the following conditions hold:

(1) Ξ is **closed under direct sums**: $Z \in \Xi_n$, $W \in \Xi_m \Rightarrow \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] \in \Xi_{n+m}$.
(2) Ξ is **invariant under left injective intertwinings**: $Z \in \Xi_n$, $\widetilde{Z} \in \mathcal{V}^{m\times m}$ such that $\mathcal{I}\widetilde{Z} = Z\mathcal{I}$ for some injective $\mathcal{I} \in \mathbb{C}^{n\times m}$ (so $n \geq m$) $\Rightarrow \widetilde{Z} \in \Xi_m$.

An equivalent version of property (2) in Definition 2.4 is that $\Xi \subset \mathcal{V}_{\rm nc}$ is **closed under restriction to invariant subspaces**: *whenever there is an invertible $\alpha \in \mathbb{C}^{n\times n}$ and a $Z \in \Xi$ of size $n \times n$ such that $\alpha^{-1} Z \alpha = \left[\begin{smallmatrix} \widetilde{Z} & Z_{12} \\ 0 & Z_{22} \end{smallmatrix}\right]$ with $\widetilde{Z}$ of size $m \times m$ ($m \leq n$), then not only is $\left[\begin{smallmatrix} \widetilde{Z} & Z_{12} \\ 0 & Z_{22} \end{smallmatrix}\right]$ in Ξ but also $\widetilde{Z}$ is in Ξ.* Here we view the $\mathbb{C}$-linear space equal to the span of the first m columns of α as an invariant subspace for Z with matrix representation $\widetilde{Z}$ determined by

$$Z\left(\alpha \left[\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right]\right) = \left(\alpha \left[\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right]\right) \widetilde{Z}.$$

Note that here $\left(\alpha \left[\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right]\right)$ is an $n \times m$ matrix over $\mathbb{C}$ while Z and $\widetilde{Z}$ are matrices of respective sizes $n \times n$ and $m \times m$ over $\mathcal{V}$. For a more concrete illustrative example, see Example 2.5 below.

We next suppose that we are given two coefficient Hilbert spaces $\mathcal{S}$ and $\mathcal{R}$ and that $Q\colon \Xi \to \mathcal{L}(\mathcal{R}, \mathcal{S})$ is a nc function. We associate with any such Q the nc set $\mathbb{D}_Q \subset \Xi$ defined by

$$\mathbb{D}_Q = \{Z \in \Xi\colon \|Q(Z)\| < 1\}. \tag{2.23}$$

Here the norm of $Q(Z)$ is taken in $\mathcal{L}(\mathcal{R}, \mathcal{S})^{n\times n} \cong \mathcal{L}(\mathcal{R}^n, \mathcal{S}^n)$ if $Z \in \Xi_n$.

The reader is welcome to have in mind the following examples as illustrative special cases of the general setup.

Example 2.5. We let $\mathcal{V}$ be the vector space $\mathbb{C}^d$ of d-tuples of complex numbers with $\Xi = \mathcal{V}_{\rm nc} := \amalg_{n=1}^\infty (\mathbb{C}^d)^{n\times n}$. We identify $(\mathbb{C}^d)^{n\times n}$ ($n \times n$ matrices with entries from $\mathbb{C}^d$) with $(\mathbb{C}^{n\times n})^d$ (d-tuples of $n \times n$ complex matrices) and hence we may view Ξ as $\amalg_{n=1}^\infty (\mathbb{C}^{n\times n})^d$. Then we write an element $Z \in \Xi_n$ (the elements of $\Xi \cap (\mathbb{C}^{n\times n})^d$) as a d-tuple $Z = (Z_1, \dots, Z_d)$ where each $Z_i \in \mathbb{C}^{n\times n}$. Then an **invariant subspace** for a point $Z \in (\mathbb{C}^{n\times n})^d$ (as in the context of the reformulation of "invariance under left injective intertwinings" (see the discussion immediately after Definition 2.4 above) amounts to a joint invariant subspace for the matrices $Z_1, \dots, Z_n$ in the classical sense.

In the context of this example, we may define a notion of **noncommutative matrix polynomial**, by which we mean a formal expression of the form

$$Q(z) = \sum_{\mathfrak{a} \in \mathbb{F}_d^+} Q_{\mathfrak{a}} z^{\mathfrak{a}} \tag{2.24}$$

where the coefficients $Q_{\mathfrak{a}} \in \mathbb{C}^{s\times r}$ are complex matrices with all but finitely many equal to 0, as in the discussion above leading up to the statement of Theorem 1.4. Such a formal expression $Q(z)$ defines a nc function from $(\mathbb{C}^d)_{\rm nc}$ to $(\mathbb{C}^{s\times r})_{\rm nc}$ if, for $Z = (Z_1, \dots, Z_d) \in (\mathbb{C}^{n\times n})^d$ we define

$$Q(Z) = \sum_{\mathfrak{a} \in \mathbb{F}_d^+} Q_{\mathfrak{a}} \otimes Z^{\mathfrak{a}} \in \mathbb{C}^{rn \times sn} \cong (\mathbb{C}^{r\times s})^{n\times n}$$

as explained in the Introduction. Then the associated Q-disk $\mathbb{D}_Q$ consists of all points $Z = (Z_1, \dots, Z_d) \in (\mathbb{C}^{n\times n})^d$ such that $\|Q(Z)\| < 1$. A set of this form can be thought of as a noncommutative analogue of a semi-algebraic set as defined in real algebraic geometry (see e.g. [28]). We mention some particular cases:

(1) If $Q(z) = \begin{bmatrix} z_1 & \cdots & z_d \end{bmatrix}$ (a $1 \times d$ nc polynomial matrix), then the associated disk $\mathbb{D}_Q$ consists of d-tuples $Z = (Z_1, \dots, Z_d)$ for which $\| \begin{bmatrix} Z_1 & \cdots & Z_d \end{bmatrix} \| < 1$, or equivalently, for which $Z_1 Z_1^* + \cdots + Z_d Z_d^* \prec I_n$. We refer to this set as the **noncommutative ball**.

(2) If $Q(z) = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_d \end{bmatrix}$ (a $d \times d$ nc polynomial matrix), then the associated Q-disk $\mathbb{D}_Q$ consists of d-tuples of $n \times n$ matrices $Z = (Z_1, \dots, Z_d)$ such that $\left\| \begin{bmatrix} Z_1 & & \\ & \ddots & \\ & & Z_d \end{bmatrix} \right\| < 1$, or equivalently, for which $Z_i^* Z_i \prec I_n$ for each $i = 1, \dots, d$. We refer to this set as the **noncommutative polydisk**.

Example 2.6. We next present an infinite-dimensional example. If $\mathcal{C}$ is a C^*-algebra, let $\mathbb{H}^+(\mathcal{C})$ and $\mathbb{H}^-(\mathcal{C})$ be the upper and lower half-planes over $\mathcal{C}$

given by

$$\mathbb{H}^+(\mathcal{C}) = \left\{ a \in \mathcal{C} \colon \operatorname{Im} a = \frac{a - a^*}{2} > 0 \right\},$$
$$\mathbb{H}^-(\mathcal{C}) = \left\{ a \in \mathcal{C} \colon \operatorname{Im} a = \frac{a - a^*}{2} < 0 \right\}.$$

We then define the upper and lower fully matricial half-planes $\mathbb{H}^+(\mathcal{A}_{\rm nc})$ and $\mathbb{H}^-(\mathcal{A}_{\rm nc})$ over a C^*-algebra $\mathcal{A}$ by

$$\mathbb{H}^\pm(\mathcal{A}_{\rm nc}) = \amalg_{n=1}^\infty \mathbb{H}^\pm(\mathcal{A}^{n\times n}).$$

If we let the underlying vector space $\mathcal{V}$ be equal to the C^*-algebra $\mathcal{A}$, let $\Xi = \{a \in \mathcal{A} \colon i1_{\mathcal{A}} + a \text{ invertible}\}$ and then define a nc function $Q_\pm$ from Ξ to $\mathcal{A}_{\rm nc}$ by $Q_+(A) = (A + i1_{\mathcal{A}^{n\times n}})^{-1}(A - i1_{\mathcal{A}^{n\times n}})$ and $Q_-(A) = (A - i1_{\mathcal{A}^{n\times n}})^{-1}(A + i1_{\mathcal{A}^{n\times n}})$ for $A \in \mathcal{A}^{n\times n}$, then we recover $\mathbb{H}^\pm(\mathcal{A}_{\rm nc})$ as $\mathbb{H}^\pm(\mathcal{A}_{\rm nc}) = \mathbb{D}_{Q_\pm}$. This example comes up in free probability (see [70]).

Example 2.7. Let $\mathcal{V} = \mathbb{C}^d$ and $\mathcal{V}_{\rm nc} \cong \amalg_{n=1}^\infty (\mathbb{C}^{n\times n})^d$ as in Example 2.5. Let G be an open subset of $\mathbb{C}_d$ and define a subset Ξ of $(\mathbb{C}^d)_{\rm nc}$ to consist of commutative d-tuples of $n\times n$ matrices $Z = (Z_1, \dots, Z_d)$ (so each $Z_i \in \mathbb{C}^{n\times n}$ and $Z_iZ_j = Z_jZ_i$ for all pairs of indices $i,j = 1, \dots, d$ for $n = 1, 2, \dots$) such that the joint spectrum $\sigma_{\rm joint}(Z)$ (i.e., the set of joint eigenvalues which is the same as the Taylor spectrum of Z for this matrix case) is contained in G. Now it is an easy exercise to verify that Ξ so defined is a full nc subset of $(\mathbb{C}^d)_{\rm nc}$. Suppose next that $\mathcal{R}$ and $\mathcal{S}$ are two coefficient Hilbert spaces and that q is a d-variable holomorphic function on G with values in $\mathcal{L}(\mathcal{R}, \mathcal{S})$ such that $\|q(z)\| < 1$ for all $z \in G$. As the joint spectrum and the Taylor spectrum are the same for commutative matrix tuples, for $Z = (Z_1, \dots, Z_d) \in \Xi_n$, we can define $q(Z) \in \mathcal{L}(\mathcal{R}^n, \mathcal{S}^n)$ by the Taylor functional calculus or the Martinelli-Vasilescu functional calculus (see e.g. [33]). If we define the associated q-disk $\mathbb{D}_q$ by

$$\mathbb{D}_q = \{Z \in \Xi \colon \|q(Z)\| < 1\},$$

then by construction we have $\mathbb{D}_{q,1} = \Xi_{q,1}$. Elementary properties of the Taylor/Martinelli-Vasilescu functional calculus $f \mapsto f(Z)$ are that the "respects direct sums" and "respects similarities" properties are satisfied and thus f so defined is a locally bounded nc function on $\mathbb{D}_q$ (locally bounded referring to a neighborhood of a given point $Z^{(0)} = (Z_1^{(0)}, \dots, Z_d^{(0)}) \in \mathbb{D}_{q,n}$ in the standard Euclidean topology of $(\mathbb{C}^{n\times n})^d$). Conversely, any such locally bounded nc function is analytic (see [52, Theorm 7.4]), and its definition on $\mathbb{D}_{q,n}$ is determined by its definition on $\mathbb{D}_{q,1}$ via the Taylor/Martinelli-Vasilescu functional calculus. Thus the nc function theory can be used to get results for the standard theory of several (commuting) complex variables. We note that domains of the type $\mathbb{D}_q^\infty$ (where one uses commutative-operator d-tuples $Z = (Z_1, \dots, Z_d)$ rather than commutative finite-matrix d-tuples as here) come up in the definition of the commutative Schur-Agler classes in

[9, 17, 60]. We have more to say on this setup in Section 3.5 and Remark 4.5.

2.3. **Noncommutative envelopes.** We fix a noncommutative function Q on the full nc subset Ξ of $\mathcal{V}_{\rm nc}$ as in Subsection 2.2 and suppose that Ω_0 is a subset of $\mathbb{D}_Q$. The following three notions of nc subset generated by Ω_0 will be useful in the sequel.

Definition 2.8. (1) We say that Ω is the **noncommutative envelope of** Ω_0 (or **nc envelope of** Ω_0 for short) (notation: $\Omega = [\Omega_0]_{\rm nc}$) if Ω is equal to the smallest subset of Ξ containing Ω_0 which is closed under direct sums:

$$Z \in \Omega,\, W \in \Omega \Rightarrow \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} \in \Omega. \tag{2.25}$$

Equivalently,

$$[\Omega_0]_{\rm nc} = \bigcap \{\Omega \subset \Xi \colon \Omega \supset \Omega_0,\, \Omega \text{ satisfies (2.25)}\}. \tag{2.26}$$

We say that the set $[\Omega_0]_{\rm nc} \cap \mathbb{D}_Q$ is the **$\mathbb{D}_Q$-relative nc envelope** of Ω_0.

(2) We say that Ω is the **noncommutative similarity-invariant envelope of** Ω_0 (or simply **nc similarity envelope of** Ω_0) (notation: $\Omega = [\Omega_0]_{\rm nc,\, sim}$) if Ω is the smallest subset of Ξ containing Ω_0 which is closed under direct sums (2.25) and under similarity transforms:

$$Z \in \Omega_n,\, \alpha \in \mathbb{C}^{n \times n} \text{ invertible } \Rightarrow \alpha Z \alpha^{-1} \in \Omega_n. \tag{2.27}$$

Equivalently,

$$[\Omega_0]_{\rm nc,\, sim} = \bigcap \{\Omega \subset \Xi \colon \Omega \supset \Omega_0,\, \Omega \text{ satisfies (2.25) and (2.27)}\}. \tag{2.28}$$

We say that the set $[\Omega_0]_{\rm nc,sim} \cap \mathbb{D}_Q$ is the **$\mathbb{D}_Q$-relative nc similarity envelope** of Ω_0.

(3) We say that Ω is the **full nc envelope of** Ω_0 (notation: $\Omega = [\Omega_0]_{\rm full}$) if Ω is the smallest subset of Ξ containing Ω_0 which is closed under direct sums (2.25) and under left injective intertwinings (see Definition 2.4). Equivalently,

$$[\Omega_0]_{\rm full} = \bigcap \{\Omega \subset \Xi \colon \Omega \supset \Omega_0,\, \Omega \text{ is a full nc subset as in Definition 2.4}\}. \tag{2.29}$$

We say that the set $[\Omega_0]_{\rm full} \cap \mathbb{D}_Q$ is the **$\mathbb{D}_Q$-relative full nc envelope** of Ω_0.

We note that the properties (2.25), (2.27), as well as properties (1) and (2) in Definition 2.4 for subsets Ω of Ξ are invariant under intersection. Hence it becomes clear that the intersection of all supersets of Ω_0 satisfying some combination of properties (2.25), (2.27), (1)–(2) again satisfies the same combination, and hence the three respective envelopes exist and are alternatively characterized by (2.26), (2.28), and (2.29) respectively.

Alternatively, these envelopes can be described more constructively via a **bottom-up** (rather than **top-down** as in Definition 2.8) procedure, as explained in the following Proposition.

Proposition 2.9. *Let* Ω_0 *be a subset of a full nc set* $\Xi \subset \mathcal{S}_{\rm nc}$*. Then:*

(1) *The nc-envelope* $[\Omega_0]_{\rm nc}$ *consists of all matrices of the form*

$$Z = \begin{bmatrix} Z^{(1)} & & \\ & \ddots & \\ & & Z^{(N)} \end{bmatrix}$$

where each $Z^{(j)} \in \Omega_0$ *where* $N = 1, 2, \dots$.

(2) *The nc similarity envelope* $[\Omega_0]_{\rm nc,sim}$ *consists of all matrices* Z *in* Ξ *such that there exists a square invertible complex matrix* α *so that* $\alpha Z \alpha^{-1} \in [\Omega_0]_{\rm nc}$.

(3) *The full nc envelope* $[\Omega_0]_{\rm full}$ *consists of all* $\widetilde{Z} \in \Xi$ *for which there is a* $Z \in [\Omega_0]_{\rm nc}$ *and a left injective intertwiner* $\mathcal{I}$ *so that* $\mathcal{I}\widetilde{Z} = Z\mathcal{I}$.

Consequently, we have the chain of containments

$$\Omega_0 \subset [\Omega_0]_{\rm nc} \subset [\Omega_0]_{\rm nc,sim} \subset [\Omega_0]_{\rm full}. \tag{2.30}$$

Proof. Let $[\Omega_0]^o_{\rm nc}$ be the set of all matrices of the form $Z = \begin{bmatrix} Z^{(1)} & & \\ & \ddots & \\ & & Z^{(N)} \end{bmatrix}$ as in statement (1). Then it is clear that necessarily $[\Omega_0]^o_{\rm nc} \subset [\Omega_0]_{\rm nc}$. Similarly, if $[\Omega_0]^o_{\rm nc,sim}$ and $[\Omega_0]^o_{\rm full}$ are the candidate sets for $[\Omega_0]_{\rm nc,sim}$ and $[\Omega_0]_{\rm full}$ described in statements (2) and (3) of the Proposition, it is clear that $[\Omega_0]^o_{\rm nc,sim} \subset [\Omega_0]_{\rm nc,sim}$ and that $[\Omega_0]^o_{\rm full} \subset [\Omega_0]_{\rm full}$. Thus it remains only to verify the reverse containments.

Suppose that $Z = \begin{bmatrix} Z^{(1)} & & \\ & \ddots & \\ & & Z^{(N)} \end{bmatrix}$ where each $Z^{(j)}$ is in $[\Omega_0]^o_{\rm nc}$ and hence has in turn a direct-sum decomposition $Z^{(j)} = \begin{bmatrix} Z^{(j,1)} & & \\ & \ddots & \\ & & Z^{(j,n_j)} \end{bmatrix}$ with each matrix $Z^{(j,\ell)}$ coming from Ω_0. Then it is clear that

$$Z = \begin{bmatrix} Z^{(1,1)} & & & & & \\ & \ddots & & & & \\ & & Z^{(1,n_1)} & & & \\ & & & \ddots & & \\ & & & & Z^{(N,1)} & & \\ & & & & & \ddots & \\ & & & & & & Z^{(N,n_N)} \end{bmatrix}$$

is the direct sum of $n_1 + \cdots + n_N$ elements from Ω_0. i.e., $Z \in [\Omega_0]^o_{\rm nc}$. Thus $[\Omega_0]^o_{\rm nc}$ is a nc set containing Ω_0 and the reverse containment $[\Omega_0]_{\rm nc} \subset [\Omega_0]^o_{\rm nc}$ follows.

The fact that $[\Omega_0]^o_{\rm nc,sim}$ is a nc set invariant under similarity is the content of Proposition A.1 from [52] and obviously $[\Omega_0]^o_{\rm nc,sim}$ contains Ω_0. Hence the reverse containment $[\Omega_0]_{\rm nc,sim} \subset [\Omega_0]^o_{\rm nc,sim}$ follows.

Finally, since $[\Omega_0]^o_{\rm full}$ contains Ω_0, to show the reverse containment $[\Omega_0]_{\rm full} \subset [\Omega_0]^o_{\rm full}$, it suffices to show that $[\Omega_0]^o_{\rm full}$ is closed under direct sums and left injective intertwinings.

Closure under direct sums: Suppose that $\widetilde{Z}^{(1)}$ and $\widetilde{Z}^{(2)}$ are in $[\Omega_0]^o_{\rm full}$. Then by part (1) of the Proposition already proved, there is a $Z^{(1)} \in [\Omega_0]^o_{\rm nc}$ and a $Z^{(2)} \in [\Omega_0]^o_{\rm nc}$ together with injective $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ so that $\mathcal{I}^{(1)}\widetilde{Z}^{(1)} = Z^{(1)}\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}\widetilde{Z}^{(2)} = Z^{(2)}\mathcal{I}^{(2)}$. Then $\left[\begin{smallmatrix} \mathcal{I}^{(1)} & 0 \\ 0 & \mathcal{I}^{(2)} \end{smallmatrix}\right]$ is also injective and

$$\begin{bmatrix} \mathcal{I}^{(1)} & 0 \\ 0 & \mathcal{I}^{(2)} \end{bmatrix} \begin{bmatrix} \widetilde{Z}^{(1)} & 0 \\ 0 & \widetilde{Z}^{(2)} \end{bmatrix} = \begin{bmatrix} Z^{(1)} & 0 \\ 0 & Z^{(2)} \end{bmatrix} \begin{bmatrix} \mathcal{I}^{(1)} & 0 \\ 0 & \mathcal{I}^{(2)} \end{bmatrix}$$

where $\left[\begin{smallmatrix} Z^{(1)} & 0 \\ 0 & Z^{(2)} \end{smallmatrix}\right]$ is in $[\Omega_0]^o_{\rm nc}$ since both $Z^{(1)}$ and $Z^{(2)}$ are in $[\Omega_0]^o_{\rm nc}$. This enables us to conclude that $\left[\begin{smallmatrix} \widetilde{Z}^{(1)} & 0 \\ 0 & \widetilde{Z}^{(2)} \end{smallmatrix}\right]$ is in $[\Omega]^o_{\rm full}$.

Closure under left injective intertwinings: Suppose that $\mathcal{I}$ is injective, $Z \in [\Omega_0]^o_{\rm full}$ and $\widetilde{Z} \in \Xi$ satisfies $\mathcal{I}\widetilde{Z} = Z\mathcal{I}$. We wish to show that $\widetilde{Z} \in [\Omega_0]^o_{\rm full}$. Toward this end, observe first of all that $Z \in [\Omega_0]^o_{\rm full}$ means that there is an injective $\mathcal{I}_0$ and a $W \in [\Omega_0]^o_{\rm nc}$ so that $\mathcal{I}_0 Z = W\mathcal{I}_0$. Then we see that

$$W\mathcal{I}_0\mathcal{I} = \mathcal{I}_0 Z\mathcal{I} = \mathcal{I}_0\mathcal{I}\widetilde{Z}.$$

As $W \in [\Omega_0]^o_{\rm nc}$ and $\mathcal{I}_0\mathcal{I}$ is again injective, it follows that $\widetilde{Z} \in [\Omega]^o_{\rm full}$ as wanted.

The chain of containments (2.30) is an immediate consequence of the respective envelope characterizations in parts (1), (2), (3) of the Proposition. □

The key property of finitely generated nc subsets is given by the following lemma.

Proposition 2.10. *Let Ω be any one of the three $\mathbb{D}_Q$-relative envelopes $[\Omega_0]_{\rm nc} \cap \mathbb{D}_Q$, $[\Omega_0]_{\rm nc,\,sim} \cap \mathbb{D}_Q$, or $[\Omega_0]_{\rm full} \cap \mathbb{D}_Q$.*

(1) *Suppose that $f \in \mathcal{T}(\Omega; \mathcal{E}_{\rm nc})$ is such that $f|_{\Omega_0} \equiv 0$. Then also $f \equiv 0$. Hence any function $f \in \mathcal{T}(\Omega; \mathcal{E}_{\rm nc})$ is uniquely determined by its restriction $\mathfrak{R}f := f|_{\Omega_0}$ to Ω_0.*

(2) *Suppose also that Ω_0 is a finite subset of $\mathbb{D}_Q$ and that $\mathcal{E}$ is a finite-dimensional Hilbert space. Then the vector space $\mathcal{T}(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc})$ of all $\mathcal{L}(\mathcal{E})$-valued nc functions on Ω has finite dimension:*

$$\dim \mathcal{T}(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc}) < \infty.$$

Proof. From the chain of containments (2.30), we see that it suffices to consider the case where $\Omega = [\Omega_0]_{\rm full}$.

We let $\mathfrak{R}$ be the restriction map $\mathfrak{R}\colon f \mapsto f|_{\Omega_F}$ for $f \in \mathcal{T}(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc})$. Let $\mathcal{F}_{\rm gr}(\Omega_F, \mathcal{L}(\mathcal{E})_{\rm nc})$ be the linear space of all graded $\mathcal{L}(\mathcal{E})_{\rm nc}$-valued functions on Ω_F. Note that

$$\mathfrak{R}\colon \mathcal{T}(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc}) \to \mathcal{F}_{\rm gr}(\Omega_F, \mathcal{L}(\mathcal{E})_{\rm nc}).$$

As $\mathfrak{R}$ is linear, to show that f is uniquely determined by $\mathfrak{R}f$ it suffices to show that $\mathfrak{R}$ is injective, i.e.: $\mathfrak{R}f \equiv 0 \Rightarrow f \equiv 0$. Suppose therefore that $f \in \mathcal{T}(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc})$ vanishes on Ω_0. Let $\mathfrak{Z} = \{Z \in \Omega\colon f(Z) = 0\}$. By assumption $\Omega_0 \subset \mathfrak{Z}$. Since f as a nc function on Ω respects direct sums, it follows that $\mathfrak{Z}$ is closed under direct sums and hence $\mathfrak{Z} \supset [\Omega_0]_{\rm nc}$. Suppose next that $\widetilde{Z}$ is a point in $\mathbb{D}_{Q,m}$ such that there is an injective matrix $\mathcal{I} \in \mathbb{C}^{n\times m}$ and a $Z \in \mathfrak{Z}_n$ so that $\mathcal{I}\widetilde{Z} = Z\mathcal{I}$. Since f respects intertwinings, it follows that $0 = f(Z)\mathcal{I} = \mathcal{I}f(\widetilde{Z})$. As $\mathcal{I}$ is injective, it follows that $f(\widetilde{Z}) = 0$, i.e., $\widetilde{Z} \in \mathfrak{Z}$. Thus $\mathfrak{Z}$ has the invariance properties required for the inclusion $\Omega := [\Omega_0]_{\rm full} \cap \mathbb{D}_Q \subset \mathfrak{Z}$, and hence f vanishes identically on Ω and statement (1) of Proposition 2.10 follows.

We now suppose that Ω_0 is a finite subset and that $\mathcal{E}$ is a finite-dimensional Hilbert space. Enumerate the elements of Ω_0 as $\Omega_0 = \{Z^{(1)}, \dots, Z^{(N)}\}$. Observe that $\mathcal{F}_{\rm gr}(\Omega_0, \mathcal{L}(\mathcal{E})_{\rm nc})$ is finite-dimensional. To see this, say $Z^{(i)} \in \Omega_{F,n_i}$ so that the value $f(Z^{(i)})$ of a graded function f at $Z^{(i)}$ is in $\mathcal{L}(\mathcal{E})^{n_i\times n_i}$. Let $\{E^{(n_i)}_{j,k}\colon 1 \le j,k \le n_i \cdot \dim \mathcal{E}\}$ be a (finite) basis for $\mathcal{L}(\mathcal{E})^{n_i\times n_i}$. For $1 \le i \le N$ and $1 \le j,k \le n_i \cdot \dim\mathcal{E}$ set

$$f_{i;jk}(Z) = \begin{cases} E^{(n_i)}_{jk} & \text{if } Z = Z^{(i)}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the collection $\{f_{i,jk}\colon 1 \le i \le N,\, 1 \le j,k \le n_i \cdot \dim \mathcal{E}\}$ is a finite basis for the linear space $\mathcal{F}_{\rm gr}(\Omega_0, \mathcal{L}(\mathcal{E})_{\rm nc})$. By part (1) we know that $\mathfrak{R}$ is injective. Thus $\mathfrak{R}$ is an injective mapping from the linear space $\mathcal{T}(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc})$ into the finite-dimensional linear space $\mathcal{F}_{\rm gr}(\Omega_0, \mathcal{L}(\mathcal{E})_{\rm nc})$. It now follows from the null-kernel theorem from Linear Algebra that $\dim \mathcal{T}(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc}) < \infty$. □

2.4. **Noncommutative Zariski closed sets.** We include here some material not needed in the sequel, as it may be of independent interest.

Suppose that Ω is a subset of Ξ. We define the nc-Zariski closure $\overline{\Omega}$ of Ω by

$$\overline{\Omega} = \{Z \in \Xi\colon f \in \mathcal{T}(\Xi; \mathbb{C}_{\rm nc}) \text{ with } f|_\Omega = 0 \Rightarrow f(Z) = 0\}. \tag{2.31}$$

We say that Ω is **nc-Zariski closed** if $\Omega = \overline{\Omega}$. If Ω is a subset of $\mathbb{D}_Q$, we say that Ω is $\mathbb{D}_Q$**-relative nc-Zariski closed** if $\Omega = \overline{\Omega} \cap \mathbb{D}_Q$.

The next result gives a relation between nc-Zariski closure and the full nc envelope of a given set Ω.

Proposition 2.11. *For Ω any subset of Ξ, we have the containment*

$$[\Omega]_{\rm full} \subset \overline{\Omega}. \tag{2.32}$$

Proof. To show that $[\Omega]_{\rm full} \subset \overline{\Omega}$, it suffices to show:

$$f \in \mathcal{T}(\Xi; \mathbb{C}),\ f|_{\Omega} = 0,\ Z \in [\Omega]_{\rm full} \Rightarrow f(Z) = 0. \tag{2.33}$$

Note that $f \in \mathcal{T}(\Xi; \mathbb{C}) \Rightarrow f|_{[\Omega]_{\rm full}} \in \mathcal{T}([\Omega]_{\rm full}; \mathbb{C})$. Hence the implication (2.33) follows directly from Proposition 2.10. □

A natural question is whether the containment (2.32) is in fact an equality. For the case where $\mathcal{V} = \mathbb{C}$, $\Xi = \mathcal{V}_{\rm nc} = \mathbb{C}_{\rm nc}$ (e.g. if Q is a single-variable nc scalar polynomial Q), and Ω is taken to be a finite subset, the answer is in the affirmative.

Proposition 2.12. *Suppose that Ω_F is a finite subset of $\mathbb{D}_Q$ where $Q \in \mathcal{T}(\mathbb{C}_{\rm nc}; \mathbb{C}_{\rm nc})$ is a nc scalar-valued function on $\mathbb{C}_{\rm nc}$ (e.g., a nc single-variable polynomial). Then we have the equality:*

$$[\Omega_F]_{\rm full} = \overline{\Omega_F}. \tag{2.34}$$

Proof. The containment $[\Omega_F]_{\rm full} \subset \overline{\Omega_F}$ follows from Proposition 2.11 (without the extra assumption that $\mathcal{V} = \mathbb{C}$).

For the reverse containment, we prove the contrapositive: *if $Z \in \mathbb{D}_Q$ is not in $[\Omega_F]_{\rm full}$, then Z is not in $\overline{\Omega_F}$.* Toward this end, it suffices to produce a nc polynomial p_0 such that $p_0|_{\Omega_F} = 0$ but $p_0(Z) \neq 0$. We first note the formula for the evaluation of a single-variable polynomial p on an $n \times n$ Jordan cell:

$$p\left(\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}\right) = \begin{bmatrix} p(\lambda) & p'(\lambda) & \cdots & \cdots & \frac{1}{(n-1)!}p^{(n-1)}(\lambda) \\ & p(\lambda) & p'(\lambda) & \cdots & \frac{1}{(n-2)!}p^{(n-2)}(\lambda) \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & p'(\lambda) \\ & & & & p(\lambda) \end{bmatrix} \tag{2.35}$$

Consequently, p vanishing on Ω_F is characterized by the condition:

$$p^{(k)}(\lambda) = 0 \text{ for } 0 \leq k < n_\lambda \text{ for all } \lambda \in \sigma(W) \text{ for all } W \in \Omega_F \tag{2.36}$$

where n_λ is the maximum length of a Jordan chain for eigenvalue $\lambda \in \sigma(W)$ for some matrix $W \in \Omega_F$. Since Z is not in $[\Omega_F]_{\rm full}$, it follows that Z is not similar to a matrix of the form $\left[\begin{smallmatrix} \widetilde{Z}_1 & X_{12} \\ 0 & X_{22} \end{smallmatrix}\right]$ with $\widetilde{Z}_1$ equal to a direct sum of matrices in Ω_F. It follows that either (i) Z has an eigenvalue λ_0 distinct from the eigenvalues of all matrices in Ω_F, or (ii) all eigenvalues of Z occur as an eigenvalue of a matrix in Ω_F but there is at least one such eigenvalue λ_0 for which the length of the Jordan chain for the eigenvalue λ_0 for the matrix Z is larger than the length of the Jordan chain with eigenvalue λ_0 for any matrix $W \in \Omega_F$. In case (i), let p_0 be any polynomial satisfying (2.36) but with $p_0(\lambda_0) \neq 0$. In case (ii), let p_0 be any polynomial satisfying (2.36) but with $p_0^{(n_{\lambda_0})}(\lambda_0) \neq 0$. In either case this is a simple Hermite interpolation problem. As a consequence of the functional calculus formula (2.35), we see that the construction yields a polynomial p_0 with $p_0(Z) \neq 0$ while $p_0|_{\Omega} = 0$ as needed. □

Remark 2.13. In addition to the left injective intertwinings introduced in Definition 2.4, there is a symmetric notion of right surjective intertwinings: we say that a subset Ω of $\mathcal{V}_{\rm nc}$ is invariant under **right surjective intertwinings** if: $\widetilde{Z} \in \Xi_n$ such that there is a $Z \in \Omega_m$ together with a surjective $\mathcal{I}' \in \mathbb{C}^{n \times m}$ such that $\widetilde{Z}\mathcal{I}' = \mathcal{I}'Z$, then $\widetilde{Z} \in \Omega_n$. It is easily checked that the Zariski closure $\overline{\Omega}$ of any subset Ω is not only invariant under left injective intertwinings but also under right surjective intertwinings. If it is the case that the Zariski closure always equals the full nc envelope $[\Omega]_{\rm full}$, then it would follow that the full nc envelope $[\Omega]_{\rm full}$ is in fact also invariant under right surjective intertwinings. Whether this is the case in general, we leave as an open question.

2.5. **Tests for complete positivity of nc kernels.** An interesting fact is that, at least in some special cases, there is finite test for complete positivity of a nc kernel with domain equal to a finitely generated nc set.

Proposition 2.14. *Let Ω be the full nc envelope $[\Omega_F]_{\rm full}$ of the finite subset $\Omega_F = \{Z^{(1)}, \dots, Z^{(N)}\}$ of Ξ. Suppose that $K \colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{E}))_{\rm nc}$ is a nc kernel. Then K is a cp nc kernel if and only if*

$$K\left(\bigoplus_{i=1}^N Z^{(i)}, \bigoplus_{i=1}^N Z^{(i)}\right)$$

is a completely positive map.

Proof. Necessity follows from the definition of a cp nc kernel.

For sufficiency, proceed as follows. Let n_i denote the size of $Z^{(i)}$ (so $Z^{(i)} \in \mathcal{V}^{n_i \times n_i}$). For $1 \le i_0 \le N$, let $E^{(i_0)}$ be the $(\sum_{i=1}^N n_i) \times n_{i_0}$ matrix of column-block structure with i-th block column having size $n_i \times n_{i_0}$ such that the i_0-block is equal to the $i_0 \times i_0$ identity matrix I_{i_0} and all other blocks equal to 0:

$$E^{(i_0)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{i_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

From the intertwining relation $\left(\bigoplus_{i=1}^N Z^{(i)}\right) E^{(i_0)} = E^{(i_0)} Z^{(i_0)}$ and the "respects intertwinings" property (2.5) we see that

$$E^{(i_0)*} K\left(\bigoplus_{i=1}^N Z^{(i)}, \bigoplus_{i=1}^N Z^{(i)}\right)([P_{ij}])\, E^{(i_0)} = K(Z^{(i_0)}, Z^{(i_0)})(P_{i_0 i_0}).$$

We conclude that the map $K\left(\bigoplus_{i=1}^N Z^{(i)}, \bigoplus_{i=1}^N Z^{(i)}\right)$ being positive implies that the map $K(Z^{(i_0)}, Z^{(i_0)})$ is a positive map for each i_0, $1 \le i_0 \le$

N. A similar argument gives that $K\left(\bigoplus_{i=1}^N Z^{(i)}, \bigoplus_{i=1}^N Z^{(i)}\right)$ being completely positive implies that $K(Z^{(i_0)}, Z^{(i_0)})$ is completely positive. More generally, $K\left(\bigoplus_{i=1}^N Z^{(i)}, \bigoplus_{i=1}^N Z^{(i)}\right)$ being completely positive implies that $K\left(\bigoplus_1^M \left(\bigoplus_{i=1}^N Z^{(i)}\right), \bigoplus_1^M \left(\bigoplus_{i=1}^N Z^{(i)}\right)\right)$ is completely positive for any $M \in \mathbb{N}$. Invoking a variant of the intertwining argument once again, we see that $K(\bigoplus_{j=1}^L Z^{(i_j)}, \bigoplus_{j=1}^L Z^{(i_j)})$ is a completely positive map, where here $\{Z^{i_1}, \dots, Z^{i_L}\}$ is any subcollection of the set of points $\{Z^{(1)}, \dots, Z^{(N)}\}$ with each allowed to be repeated any number of times up to M times. We conclude that $K(Z,Z)$ is completely positive for any Z in the nc envelope $(\Omega_F)_{\rm nc}$ of Ω_F.

It remains to check that $K(\widetilde{Z}, \widetilde{Z})$ is completely positive for any $\widetilde{Z} \in [\Omega_F]_{\rm full}$. Suppose that such a $\widetilde{Z}$ is in Ω_n. By definition there is an injective matrix $\mathcal{I} \in \mathbb{C}^{n\times m}$ and a $Z \in [\Omega_F]_{\rm nc}$ of size $m \times m$ so that $\mathcal{I}\widetilde{Z} = Z\mathcal{I}$. Use the "respects intertwinings" property (2.5) of the nc kernel K to see that

$$\mathcal{I}K(\widetilde{Z}, \widetilde{Z})(P)\mathcal{I}^* = K(Z,Z)(\mathcal{I}P\mathcal{I}^*) \succeq 0$$

for any $P \succeq 0$ in $\mathcal{A}^{m\times m}$. As $\mathcal{I}$ is injective, we conclude that $K(\widetilde{Z}, \widetilde{Z})(P) \succeq 0$, i.e., $K(\widetilde{Z}, \widetilde{Z})$ is a positive map. A similar argument shows that in fact $K(\widetilde{Z}, \widetilde{Z})$ is completely positive. □

Corollary 2.15. *Suppose that* $\Omega_F = \{Z^{(1)}, \dots, Z^{(N)}\}$ *is a finite subset of* $\mathbb{C}_{\rm nc}$ *and the nc set* $\overline{\Omega_F}$ *is its* $\mathbb{C}_{\rm nc}$*-relative nc-Zariski closure. Suppose that* $K\colon \overline{\Omega_F} \times \overline{\Omega_F} \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{E}))_{\rm nc}$ *is a nc kernel. Then* K *is a cp kernel if and only if*

$$K\left(\bigoplus_{i=1}^N Z^{(i)}, \bigoplus_{i=1}^N Z^{(i)}\right)$$

is a completely positive map.

Proof. The result follows from applying Proposition 2.12 to Proposition 2.14 □

Corollary 2.16. *Let* Ω *and* K *be as in Proposition 2.14 and consider the special case where* $\mathcal{A} = \mathbb{C}$*. Then* K *is a cp nc kernel if and only if*

$$K\left(\bigoplus_{j=1}^{N'} \bigoplus_{i=1}^N Z^{(i)}, \bigoplus_{j=1}^{N'} \bigoplus_{i=1}^N Z^{(i)}\right)(\mathfrak{C}_{N'}) \tag{2.37}$$

is positive where N' *is set equal to the level of* Ω *containing* $\bigoplus_{i=1}^N Z^{(i)}$ *(i.e.,* $\bigoplus_{i=1}^N Z^{(i)} \in \Omega_{N'}$*) and where* $\mathfrak{C}_{N'}$ *(the* ***Choi matrix*** *at level* N'*) is the* $(N')^2 \times (N')^2$ *matrix written out as a block* $N' \times N'$ *matrix with* $N' \times N'$ *matrix entries given by*

$$\mathfrak{C}_{N'} = [E_{i,j}^{N'}]_{i,j=1,\dots,N'}$$

where $E^{N'}_{i,j}$ is the $N' \times N'$ matrix with (i,j)-entry equal to 1 *and all other entries equal to* 0*.*

Proof. Necessity follows from the definition of a cp nc kernel and the fact that $\mathfrak{C}_{N'}$ is a positive map.

For sufficiency, we assume that (2.37) is positive. Since an nc kernel respects direct sums, we have that

$$K\left(\bigoplus_{j=1}^{N'}\bigoplus_{i=1}^{N} Z^{(i)}, \bigoplus_{j=1}^{N'}\bigoplus_{i=1}^{N} Z^{(i)}\right)(\mathfrak{C}_{N'}) = \left[K\left(\bigoplus_{i=1}^{N} Z^{(i)}, \bigoplus_{i=1}^{N} Z^{(i)}\right)\left(E^{N'}_{i,j}\right)\right]_{i,j=1,\dots,N'}$$

By [68, Theorem 3.14], the map $K\left(\bigoplus_{i=1}^{N} Z^{(i)}, \bigoplus_{i=1}^{N} Z^{(i)}\right)$ is completely positive, and the result follows from Proposition 2.14. □

2.6. **Extensions of noncommutative functions and kernels.** The following result clarifies the relation between nc functions, nc kernels, and cp nc kernels defined on a subset Ω versus defined on one of the envelopes $[\Omega]_{\rm nc}$, $[\Omega]_{\rm nc,sim}$, $[\Omega]_{\rm full}$ of Ω.

Proposition 2.17. *Suppose that Ω is a subset (not necessarily a nc subset) of $\mathcal{V}_{\rm nc}$.*

(1) *Any nc function $f\colon \Omega \to \mathcal{V}_{0,\rm nc}$ extends uniquely to a nc function $\widetilde{f}$ on the nc envelope $[\Omega]_{\rm nc}$ and on the nc similarity envelope $[\Omega]_{\rm nc,sim}$ but not necessarily on the full nc envelope $[\Omega]_{\rm full}$.*
(2) *Any nc kernel $K\colon \Omega \to \mathcal{L}(\mathcal{V}_1,\mathcal{V}_0)_{\rm nc}$ extends uniquely to a nc kernel $\widetilde{K}$ on the nc envelope $[\Omega]_{\rm nc}$ and on the nc similarity envelope $[\Omega]_{\rm nc,sim}$, but not necessarily on the full envelope $[\Omega]_{\rm full}$.*
(3) *Any cp nc kernel $K\colon \Omega\times\Omega \to \mathcal{L}(\mathcal{V}_1,\mathcal{V}_0)_{\rm nc}$ extends uniquely to a cp nc kernel on the nc envelope $[\Omega]_{\rm nc}$ and on the nc similarity envelope $[\Omega]_{\rm nc,sim}$, but not necessarily on the full nc envelope $[\Omega]_{\rm full}$.*

Proof of (1). The positive assertions in item (1) essentially follow from Propositions A.1 and A.3 in [52].

To show that the assertion can fail for the full nc envelope, consider the following example. We take $\mathcal{V} = \mathbb{C}^2$ so $\mathcal{V}^{n\times n}$ can be identified with pairs (Z_1, Z_2) of $n \times n$ matrices over $\mathbb{C}$. Take Ω to be the singleton set $\Omega = \{Z^{(0)} = (Z_1^{(0)}, Z_2^{(0)})\}$ where we set

$$Z_1^0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Z_2^{(0)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The only constraint on $\Lambda_0 \in \mathbb{C}^{2\times 2}$ required for the function $f\colon \Omega \to \mathbb{C}^{2\times 2}$ defined by $f(Z^{(0)}) = \Lambda_0$ to be a nc function is that the value Λ_0 be in the double commutant of $Z^{(0)}$, i.e.: $\alpha \in \mathbb{C}^{2\times 2}$ such that $\alpha Z_1^0 = Z_1^0\alpha$ and $\alpha Z_2^0 = Z_2^0\alpha \Rightarrow \alpha\Lambda_0 = \Lambda_0\alpha$. The commutant of $Z_1^{(0)}$ consists of Toeplitz matrices $\{\left[\begin{smallmatrix} a & b \\ 0 & a\end{smallmatrix}\right] : a, b \in \mathbb{C}\}$ while the commutant of $Z_2^{(0)}$ consists of diagonal matrices

$\{\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} : d_1, d_2 \in \mathbb{C}\}$. The intersection of these two commutants consists of scalar multiples of the identity. Hence the value Λ_0 is unconstrained: the function f given by $f(Z^{(0)}) = \Lambda_0$ is a nc function on Ω for any $\Lambda_0 \in \mathbb{C}^{2\times 2}$. If f extends to a nc function on the full nc envelope, then in particular $f(Z^{(0)})$ must have the form

$$\Lambda_0 = f(Z_1^{(0)}, Z_2^{(0)}) = \begin{bmatrix} f(0,0) & * \\ 0 & f(0,1) \end{bmatrix}.$$

Choosing $\Lambda_0 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ for example then leads to a contradiction. For additional information concerning existence and construction of nc-function extensions to full nc envelopes, we refer to Subsection 6.1 below. □

Proof of (2). Given a nc kernel K on $\Omega \times \Omega$, we may define an extension $\widetilde{K}$ to the nc envelope $\Omega_{\rm nc}$ of Ω by

$$\widetilde{K}\left(\begin{bmatrix} Z^{(1)} & & \\ & \ddots & \\ & & Z^{(N)} \end{bmatrix}, \begin{bmatrix} W^{(1)} & & \\ & \ddots & \\ & & W^{(M)} \end{bmatrix}\right)([P_{ij}]) = [K(Z^{(i)}, W^{(j)})(P_{ij})]_{\substack{i=1,\dots,N; \\ j=1,\dots,M}} \tag{2.38}$$

for any $Z^{(1)}, \dots, Z^{(N)}, W^{(1)}, \dots, W^{(M)} \in \Omega$. If it happens that $\begin{bmatrix} Z^{(1)} & & \\ & \ddots & \\ & & Z^{(N)} \end{bmatrix}$ and $\begin{bmatrix} W^{(1)} & & \\ & \ddots & \\ & & W^{(M)} \end{bmatrix}$ are already in Ω, then the formula (2.38) is consistent with how K is already defined by the localized "respects direct sums" condition. Then one can check that $\widetilde{K}$ is a nc kernel on $[\Omega]_{\rm nc} \times [\Omega]_{\rm nc}$ which when restricted to $\Omega \times \Omega$ agrees with K. If $\widetilde{Z}$, $\widetilde{W}$ are in the nc similarity envelope $[\Omega]_{\rm nc,sim}$, then there is an invertible matrix $\alpha = \begin{bmatrix} \alpha_1 & \cdots & \alpha_N \end{bmatrix}$ over $\mathbb{C}$ and points $Z^{(1)}, \dots, Z^{(N)} \in \Omega$ as well as an invertible matrix $\beta = \begin{bmatrix} \beta_1 & \cdots & \beta_M \end{bmatrix}$ over $\mathbb{C}$ and points $W^{(1)}, \dots, W^{(M)}$ in Ω so that

$$\alpha^{-1}\widetilde{Z}\alpha = \begin{bmatrix} Z^{(1)} & & \\ & \ddots & \\ & & Z^{(N)} \end{bmatrix}, \quad \beta^{-1}\widetilde{W}\beta = \begin{bmatrix} W^{(1)} & & \\ & \ddots & \\ & & W^{(M)} \end{bmatrix}.$$

We then define

$$\widetilde{K}(\widetilde{Z}, \widetilde{W})(P) = \alpha \left[K(Z^{(i)}, W^{(j)})([\alpha^{-1} P \beta^{-1*}]_{ij})\right]_{\substack{i=1,\dots,N; \\ j=1,\dots,M}} \beta^*. \tag{2.39}$$

We leave it to the reader to verify that $\widetilde{K}$ given by (2.39) is a well-defined nc kernel on $[\Omega]_{\rm nc,sim} \times [\Omega]_{\rm nc,sim}$ which extends K.

The result concerning lack of extension in general of a nc kernel on $\Omega \times \Omega$ to a nc kernel on $\Omega_{\rm nc,full} \times \Omega_{\rm nc,full}$ will follow from the result for the case of cp nc kernels discussed in the next part. □

Proof of (3). To verify item (3), it suffices to show that the construction of the nc-kernel extension in part (2) leads to a cp kernel on $[\Omega]_{\rm nc} \times [\Omega]_{\rm nc}$

and on $[\Omega]_{\rm nc,sim} \times [\Omega]_{\rm nc,sim}$ where we now assume that the original kernel K was cp on $\Omega \times \Omega$. The fact that the kernel $\widetilde{K}$ given by (2.38) is cp on $[\Omega]_{\rm nc} \times [\Omega]_{\rm nc}$ amounts to the definition of a cp kernel for the setting where Ω is not necessarily a nc set. Similarly, one can verify by inspection that $\widetilde{K}$ given by (2.39) on $[\Omega]_{\rm nc,sim} \times [\Omega]_{\rm nc,sim}$ is cp if the original K is cp.

Finally, if we choose $H \colon \Omega \to \mathcal{L}(\mathcal{E})_{\rm nc}$ to be a nc function on Ω which fails to have a nc-function extension to $[\Omega]_{\rm full}$ (as in the example in the proof of part (1) above), then $K(Z,W)(P) := H(Z)(P \otimes I_{\mathcal{E}})H(W)^*$ is a cp nc kernel on Ω which fails to have a nc-kernel (much less a cp nc-kernel) extension to $[\Omega]_{\rm full}$. This completes the proof of Proposition 2.17. □

In summary, given a nc kernel on a set of the form $\Omega \times \Omega$ with Ω not necessarily a nc set, by Proposition 2.17 we can always consider its extension to the nc envelope $[\Omega]_{\rm nc} \times [\Omega]_{\rm nc}$ or to the nc similarity envelope of $[\Omega]_{\rm nc,sim} \times [\Omega]_{\rm nc,sim}$ but not necessarily to the full nc envelope $[\Omega]_{\rm full} \times [\Omega]_{\rm full}$. However, when there is an extension to the full nc envelope $[\Omega]_{\rm full}$, the extension necessarily is unique. Similar remarks hold for nc functions defined on a set Ω which is not necessarily a nc set. One trivial situation when nc extension to the full nc envelope is possible is when the nc function has the form $\boldsymbol{\varphi}$ as in Example 2.3 (b) on Ω or the nc kernel has the form $\boldsymbol{\varphi}$ as in 2.3 (a) on $\Omega \times \Omega$.

2.7. **Internal tensor product of C^*-correspondences.** We shall need a certain special case of a general C^*-correspondence internal tensor product and related constructions (see [61, 73]). The following theorem summarizes these results in the form needed for the sequel.

Theorem 2.18. *Suppose that $\mathcal{X}$ is a Hilbert space, $\mathcal{E}$ and $\mathcal{F}$ are two auxiliary Hilbert spaces and $\mathcal{X}$ is equipped with a $*$-representation π of $\mathcal{L}(\mathcal{E})$:*

$$\pi \colon \mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{X}).$$

Then the following statements hold:

(1) *Define an inner product $\langle \cdot, \cdot \rangle_0$ on the algebraic tensor product $\mathcal{L}(\mathcal{E}, \mathcal{F}) \otimes_{\rm alg} \mathcal{X}$ by*

$$\langle T \otimes x, S \otimes x' \rangle_0 = \langle \pi(S^*T)x, x' \rangle_{\mathcal{X}} \tag{2.40}$$

for $x, x' \in \mathcal{X}$, $T, S \in \mathcal{L}(\mathcal{E}, \mathcal{F})$. Then $\langle \cdot, \cdot \rangle_0$ is positive semidefinite. Modding out by elements of zero self inner-product gives rise to a pre-Hilbert space $(\mathcal{L}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{X})_0$. We let $\mathcal{L}(\mathcal{E}, \mathcal{F}) \otimes_\pi \mathcal{X}$ denote the Hilbert-space completion of $(\mathcal{L}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{X})_0$ in the $\langle, \cdot, \cdot \rangle_0$ inner product. This Hilbert space completion satisfies the balancing law

$$TE \otimes x = T \otimes \pi(E)x \tag{2.41}$$

for $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, $E \in \mathcal{L}(\mathcal{E})$ and $x \in \mathcal{X}$.

(2) *Furthermore, an operator $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ induces an operator L_T mapping $\mathcal{X}$ into $\mathcal{L}(\mathcal{E}, \mathcal{F}) \otimes_\pi \mathcal{X}$ given by*

$$L_T \colon x \mapsto T \otimes x \tag{2.42}$$

with adjoint action on elementary tensors given by

$$L_T^*\colon S\otimes x \mapsto \pi(T^*S)x, \tag{2.43}$$

such that

$$\|L_T\|_{\mathcal{L}(\mathcal{X},\mathcal{L}(\mathcal{E},\mathcal{F})\otimes_\pi \mathcal{X})} \le \|T\|_{\mathcal{L}(\mathcal{E},\mathcal{F})}$$

with equality in case π is a faithful representation.

Proof. The construction of the space $\mathcal{L}(\mathcal{E},\mathcal{F})\otimes_\pi \mathcal{X}$ is a special case of a more general construction called the *inner tensor product* for C^*-correspondences (also called imprimitivity bimodules). For a proof of the positive-semidefiniteness of the inner product (2.40) in this more general setting, we refer to Proposition 3.16 in [73]. In any case we shall do a different more general version of this computation in the proof of Theorem 2.19 below.

As a nice exercise we go ahead here with verifying explicitly the balance law (2.41):

$$\begin{aligned}
&\langle TE\otimes x - T\otimes \pi(E)x, TE\otimes x - T\otimes\pi(E)x\rangle_0\\
&\quad = \langle \pi(E^*T^*TE)x, x\rangle_{\mathcal{X}} - \langle \pi(E^*T^*T)\pi(E)x, x\rangle_{\mathcal{X}}\\
&\qquad\quad - \langle \pi(T^*TE)x, \pi(E)x\rangle_{\mathcal{X}} + \langle \pi(T^*T)\pi(E)x, \pi(E)x\rangle_{\mathcal{X}}\\
&\quad = 0.
\end{aligned}$$

This completes the proof of statement (1).

To verify the properties of L_T, first note that

$$\begin{aligned}
\|L_Tx\|^2_{\mathcal{L}(\mathcal{E},\mathcal{F})\otimes_\pi\mathcal{X}} &= \|T\otimes x\|^2_{\mathcal{L}(\mathcal{E},\mathcal{F})\otimes_\pi\mathcal{X}} = \langle \pi(T^*T)x, x\rangle_{\mathcal{X}}\\
&\le \|\pi((T^*T)^{1/2})\|^2\|x\|^2 \le \|(T^*T)^{1/2}\|^2\|x\|^2 = \|T\|^2\|x\|^2
\end{aligned}$$

with equality throughout in case π is faithful. Thus L_T is well defined with $\|L_T\| \le \|T\|$ and with equality in case π is faithful. Finally note that

$$\langle L_T^*(S\otimes x), x'\rangle_{\mathcal{X}} = \langle S\otimes x, T\otimes x'\rangle_{\mathcal{L}(\mathcal{E},\mathcal{F})\otimes_\pi\mathcal{X}} = \langle \pi(T^*S)x, x'\rangle_{\mathcal{X}}$$

and the formula (2.43) follows. This completes the proof of statement (2). □

We shall actually need the following extension of Theorem 2.18 to our nc setting.

Theorem 2.19. *Suppose that $\mathcal{X}$, $\mathcal{E}$, $\mathcal{F}$ are Hilbert spaces with $\mathcal{X}$ equipped with a $*$-representation $\pi\colon \mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{X})$ as in Theorem 2.18.Then the following statements hold:*

(1) *Fix a positive integer $k\in\mathbb{N}$ and define an inner product $\langle\cdot,\cdot\rangle_0$ on the disjoint union of algebraic tensor product spaces*

$$(\mathcal{L}(\mathcal{E},\mathcal{F}^k)\otimes_{\rm alg}\mathcal{X})_{\rm nc} := \amalg_{n=1}^\infty \mathcal{L}(\mathcal{E}^n\otimes\mathcal{F}^k)\otimes_{\rm alg}\mathcal{X}^n$$

by

$$\langle T\otimes x, T'\otimes x'\rangle_0 = \langle (\mathrm{id}_{\mathbb{C}^{n'\times n}}\otimes\pi)(T'^*T)x, x'\rangle_{\mathcal{X}^{n'}} \tag{2.44}$$

for $T \in \mathcal{L}(\mathcal{E}^n, \mathcal{F}^k)$, $T' \in \mathcal{L}(\mathcal{E}^{n'}, \mathcal{F}^k)$, $x \in \mathcal{X}^n$, $x' \in \mathcal{X}^{n'}$. *Then the inner-product* $\langle \cdot, \cdot \rangle_0$ *is positive-semidefinite on* $(\mathcal{L}(\mathcal{E}, \mathcal{F}^k) \otimes_{\rm alg} \mathcal{X})_{\rm nc}$.

We let $(\mathcal{L}(\mathcal{E}, \mathcal{F}^k) \otimes_\pi \mathcal{X})_{\rm nc}$ *denote the Hilbertian completion of* $(\mathcal{L}(\mathcal{E}, \mathcal{F}^k) \otimes_{\rm alg} \mathcal{X})_{\rm nc}$ *in the* 0*-inner product (the completion of the positive-definite inner product obtained by identifying elements of self inner-product equal to zero with the zero element of the space). Then elements of* $(\mathcal{L}(\mathcal{E}, \mathcal{F}^k) \otimes_{\rm alg} \mathcal{X})_{\rm nc}$ *satisfy the balancing law*

$$TS \otimes x = T \otimes (\mathrm{id}_{\mathbb{C}^{n \times n'}} \otimes \pi)(S)x \tag{2.45}$$

for $T \in \mathcal{L}(\mathcal{E}^n, \mathcal{F}^k)$, $S \in \mathcal{L}(\mathcal{E}^{n'}, \mathcal{E}^n)$, $x \in \mathcal{X}^{n'}$.

(2) *Suppose that* T *is an operator in* $\mathcal{L}(\mathcal{E}^n, \mathcal{F}^k)$. *Then*

$$L_T \colon x \mapsto T \otimes x \tag{2.46}$$

defines a bounded linear operator from $\mathcal{X}_{\rm nc}$ *into* $(\mathcal{L}(\mathcal{E}, \mathcal{F}^k) \otimes_\pi \mathcal{X})_{\rm nc}$ *with adjoint action on elementary tensors given by*

$$L_T^* \colon T' \otimes x' \mapsto (\mathrm{id}_{\mathbb{C}^{n \times n'}} \otimes \pi)(T^* T')x' \tag{2.47}$$

for $T' \in \mathcal{L}(\mathcal{E}^{n'}, \mathcal{F}^k)$, $x' \in \mathcal{X}^{n'}$.

(3) *The order-*k *tensor product space* $(\mathcal{L}(\mathcal{E}, \mathcal{F}^k) \otimes_\pi \mathcal{X})_{\rm nc}$ *can be identified with the* k*-fold orthogonal direct sum of the order-1 tensor product space* $(\mathcal{L}(\mathcal{E}, \mathcal{F}) \otimes_\pi \mathcal{X})_{\rm nc}$ *via an identification map*

$$\iota_k \colon (\mathcal{L}(\mathcal{E}, \mathcal{F}^k) \otimes_\pi \mathcal{X})_{\rm nc} \mapsto \bigoplus_1^k (\mathcal{L}(\mathcal{E}, \mathcal{F}) \otimes_\pi \mathcal{X})_{\rm nc}$$

with action on elementary tensors given as follows. For T *an operator in* $\mathcal{L}(\mathcal{E}^n, \mathcal{F}^k)$, *we may decompose* T *as a block-column operator matrix* $T = \begin{bmatrix} T_1 \\ \vdots \\ T_k \end{bmatrix}$ *where each* $T_i \in \mathcal{L}(\mathcal{E}^n, \mathcal{F})$ *for* $i = 1, \dots, k$. *Then for* $T \in \mathcal{L}(\mathcal{E}^n, \mathcal{F}^k)$ *and* $x \in \mathcal{X}^n$ *so that* $T \otimes x$ *is an elementary tensor in* $(\mathcal{L}(\mathcal{E}, \mathcal{F}^k) \otimes_\pi \mathcal{X})_{\rm nc}$, *we define*

$$\iota_k \colon T \otimes x = \begin{bmatrix} T_1 \\ \vdots \\ T_k \end{bmatrix} \otimes x \mapsto \begin{bmatrix} T_1 \otimes x \\ \vdots \\ T_k \otimes x \end{bmatrix} \in (\mathcal{L}(\mathcal{E}, \mathcal{F}) \otimes_\pi \mathcal{X})_{\rm nc})^k \tag{2.48}$$

Proof. To prove that the 0-inner product (2.44) is positive semidefinite, choose any positive integers n_j ($j = 1, \dots, n$) along with operators $T_j \in \mathcal{L}(\mathcal{E}^{n_j} \otimes \mathcal{F}_k)$ and vectors $x_j \in \mathcal{X}^{n_j}$, and compute

$$\begin{aligned} \langle \sum_{j=1}^n T_j \otimes x_j, \sum_{i=1}^n T_i \otimes x_i \rangle_0 &= \sum_{i,j=1}^n \langle (\mathrm{id}_{\mathbb{C}^{n_i \times n_j}} \otimes \pi)(T_i^* T_j) x_j, x_i \rangle_{\mathcal{X}^{n_i}} \\ &= \left\langle (\mathrm{id}_{\mathbb{C}^{N \times N}} \otimes \pi)\left([T_i^* T_j]\right) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\rangle_{\mathcal{X}^N} \end{aligned}$$

where $N = \sum_{i=1}^n n_i$. The block $n \times n$ matrix $[T_i^* T_j] = \begin{bmatrix} T_1^* \\ \vdots \\ T_n^* \end{bmatrix} \begin{bmatrix} T_1 & \cdots & T_n \end{bmatrix}$ is a positive element of $\mathcal{L}(\mathcal{E}^N)$ and hence can be factored in the form

$$[T_i^* T_j] = A^* A$$

where $A \in \mathcal{L}(\mathcal{E}^N)$. Hence the preceding calculation can be continued as

$$\left\langle (\mathrm{id}_{\mathbb{C}^{N\times N}} \otimes \pi)\left([T_i^* T_j]\right) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\rangle_{\mathcal{X}^N} = \langle (\mathrm{id}_{\mathbb{C}^{N\times N}} \otimes \pi)(A^* A)x, x\rangle_{\mathcal{X}^N}$$

$$\|(\mathrm{id}_{\mathbb{C}^{N\times N}} \otimes \pi)(A)x\|^2 \geq 0.$$

The verification of the balancing law (2.45) proceeds as in the verification of (2.41) in the proof of Theorem 2.18. This completes the proof of statement (1).

To verify the formula (2.47), observe that, for $x \in \mathcal{X}^n$,

$$\begin{aligned} \langle L_T^*(T' \otimes x'), x\rangle_{\mathcal{X}^n} &= \langle T' \otimes x', T \otimes x\rangle_{(\mathcal{L}(\mathcal{E},\mathcal{F}^k)\otimes_\pi \mathcal{X})_{\mathrm{nc}}} \\ &= \langle (\mathrm{id}_{\mathbb{C}^{n\times n'}} \otimes \pi)(T^* T')x', x\rangle_{\mathcal{X}^n}. \end{aligned}$$

This completes the proof of statement (2).

To verify that the map ι_k given by (2.48) is an isometry, we compute, for $T_i \in \mathcal{L}(\mathcal{E}^n, \mathcal{F})$, $x \in \mathcal{X}^n$, $T_i' \in \mathcal{L}(\mathcal{E}^{n'}, \mathcal{F})$, $x' \in \mathcal{X}^{n'}$ for $i = 1, \ldots, k$,

$$\left\langle \begin{bmatrix} T_1 \\ \vdots \\ T_k \end{bmatrix} \otimes x, \begin{bmatrix} T_1' \\ \vdots \\ T_k' \end{bmatrix} \otimes x' \right\rangle_{(\mathcal{L}(\mathcal{X},\mathcal{F}^k)\otimes\mathcal{X})_{\mathrm{nc}}} = \sum_{i=1}^k \langle (\mathrm{id}_{\mathbb{C}^{n'\times n}} \otimes \pi)(T_i'^* T_i)x, x'\rangle_{\mathcal{X}^{n'}}$$

$$= \left\langle \begin{bmatrix} T_1 \otimes x \\ \vdots \\ T_k \otimes x \end{bmatrix}, \begin{bmatrix} T_1' \otimes x' \\ \vdots \\ T_k' \otimes x' \end{bmatrix} \right\rangle_{((\mathcal{L}(\mathcal{E},\mathcal{F})\otimes\mathcal{X})_{\mathrm{nc}})^k}.$$

We conclude that ι_k can be extended via linearity to an isometry from the space

$$\mathcal{D} = \operatorname{span}\left\{ \begin{bmatrix} T_1 \\ \vdots \\ T_k \end{bmatrix} \otimes x \colon T_i \in \mathcal{L}(\mathcal{E}^n, \mathcal{F}), x \in \mathcal{X}^n, n \in \mathbb{N} \right\} \subset (\mathcal{L}(\mathcal{E}, \mathcal{F}^k) \otimes_\pi \mathcal{X})_{\mathrm{nc}}$$

onto the space

$$\mathcal{R} = \operatorname{span}\left\{ \begin{bmatrix} T_1 \otimes x \\ \vdots \\ T_k \otimes x \end{bmatrix} : T_i \in \mathcal{L}(\mathcal{E}^n, \mathcal{F}), x \in \mathcal{X}^n, n \in \mathbb{N} \right\} \subset ((\mathcal{L}(\mathcal{E}, \mathcal{F}) \otimes_\pi \mathcal{X})_{\mathrm{nc}})^k.$$

It now suffices to verify that the spaces $\mathcal{D}$ and $\mathcal{R}$ are dense in their respective ambient spaces $(\mathcal{L}(\mathcal{E}, \mathcal{F}^k) \otimes_\pi \mathcal{X})_{\mathrm{nc}}$ and $\oplus_{i=1}^k (\mathcal{L}(\mathcal{E}, \mathcal{F}) \otimes_\pi \mathcal{X})_{\mathrm{nc}}$. The fact that $\mathcal{D}$ is dense is clear since a generic elementary tensor is in $\mathcal{D}$. As for $\mathcal{R}$, we note that a more general tensor $\begin{bmatrix} T_1 \otimes x_1 \\ \vdots \\ T_k \otimes x_k \end{bmatrix}$ can be obtained as a linear combination of generic elements of $\mathcal{R}$:

$$\begin{bmatrix} T_1 \otimes x_1 \\ \vdots \\ T_k \otimes x_k \end{bmatrix} = \begin{bmatrix} T_1 \otimes x_1 \\ 0 \otimes x_1 \\ \vdots \\ 0 \otimes x_1 \end{bmatrix} + \begin{bmatrix} 0 \otimes x_2 \\ T_2 \otimes x_2 \\ \vdots \\ 0 \otimes x_2 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \otimes x_k \\ \vdots \\ 0 \otimes x_k \\ T_k \otimes x_k \end{bmatrix}.$$

□

In case the representation π is a multiple of the identity representation, we have the following simplification of the space $(\mathcal{L}(\mathcal{E},\mathcal{F}) \otimes_\pi \mathcal{X})_{\rm nc}$ and of the operators L_T and L_T^*.

Theorem 2.20. *Suppose that $\mathcal{X}$, $\mathcal{E}$, $\mathcal{F}$ and π are as in Theorem 2.18 with $\pi\colon \mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{X})$ a multiple of the identity representation, i.e., there is a Hilbert space $\mathcal{X}_0$ so that $\mathcal{X}$ has the form $\mathcal{X} = \mathcal{E} \otimes \mathcal{X}_0$ and*

$$\pi(E) = E \otimes I_{\mathcal{X}_0} \text{ for } E \in \mathcal{L}(\mathcal{E}).$$

Then the space $\mathcal{L}(\mathcal{E},\mathcal{F}) \otimes_\pi (\mathcal{E} \otimes \mathcal{X}_0)$ can be identified with $\mathcal{F} \otimes \mathcal{X}_0$ with identification map on elementary tensors given by

$$\iota\colon T \otimes (e \otimes x_0) \mapsto Te \otimes x_0 \tag{2.49}$$

for $T \in \mathcal{L}(\mathcal{E},\mathcal{F})$, $e \in \mathcal{E}$, $x_0 \in \mathcal{X}_0$. Moreover

$$\iota \cdot L_T = T \otimes I_{\mathcal{X}_0}. \tag{2.50}$$

Proof. For $T, T' \in \mathcal{L}(\mathcal{E},\mathcal{F})$ and $e \otimes x_0, e' \otimes x_0' \in \mathcal{E} \otimes \mathcal{X}_0$, we compute

$$\begin{aligned} &\langle T \otimes (e \otimes x_0), T' \otimes (e' \otimes x_0') \rangle_{(\mathcal{L}(\mathcal{E},\mathcal{F}) \otimes_\pi (\mathcal{E} \otimes \mathcal{X}_0))_{\rm nc}} \\ &= \langle (T'^* T \otimes I_{\mathcal{X}_0})(e \otimes x_0), (e' \otimes x_0') \rangle_{\mathcal{E} \otimes \mathcal{X}_0} \\ &= \langle Te \otimes x_0, T'e' \otimes x_0' \rangle_{\mathcal{F} \otimes \mathcal{X}_0}. \end{aligned}$$

We conclude that the map ι given by (2.49) defines an isometry from

$$\mathcal{D} = \operatorname{span}_{T,e,x_0}\{T \otimes (e \otimes x_0)\} \subset (\mathcal{L}(\mathcal{E},\mathcal{F}) \otimes_\pi (\mathcal{E} \otimes \mathcal{X}_0))_{\rm nc}$$

onto

$$\mathcal{R} = \operatorname{span}_{T,e,x_0}\{Te \otimes x_0\} \subset \mathcal{F} \otimes \mathcal{X}_0.$$

It is now a matter of checking that $\mathcal{D}$ and $\mathcal{R}$ each is dense in its respective ambient space to see that (2.49) establishes a unitary identification between $(\mathcal{L}(\mathcal{E},\mathcal{F}) \otimes_\pi (\mathcal{E} \otimes \mathcal{X}_0))_{\rm nc}$ and $\mathcal{F} \otimes \mathcal{X}_0$.

Finally, the identity (2.50) follows immediately from the definitions (2.42) and (2.49). □

Remark 2.21. It is well known that in case $\mathcal{E}$ is finite-dimensional, then any representation $\pi\colon \mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{X})$ has the special form

$$\mathcal{X} = \mathcal{E} \otimes \mathcal{X}_0, \quad \pi(E) = E \otimes I_{\mathcal{X}_0}$$

assumed in Theorem 2.20 (see e.g. [12, Corollary 1 page 20],

3. Schur-Agler class interpolation theorems

We fix a full nc subset Ξ of nc envelope $\mathcal{V}_{\rm nc}$ of a vector space $\mathcal{V}$ and we suppose that $Q \in \mathcal{T}(\Xi; \mathcal{L}(\mathcal{R}, \mathcal{S})_{\rm nc})$ is a nc function from Ξ to $\mathcal{L}(\mathcal{R}, \mathcal{S})_{\rm nc}$ as in Subsection 2.2. Then we define the nc disk $\mathbb{D}_Q$ as in (2.23). Let $\mathcal{U}$ and $\mathcal{Y}$ be two additional coefficient Hilbert spaces. We define the **nc Schur-Agler class** $\mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ by

$$\mathcal{SA}_Q(\mathcal{U}, \mathcal{Y}) = \{S \in \mathcal{T}(\mathbb{D}_Q; \mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc})\colon \|S(Z)\| \le 1 \text{ for all } Z \in \mathbb{D}_Q\}. \quad (3.1)$$

We note that the special case where $\Xi = (\mathbb{C}^d)_{\rm nc}$, $\mathcal{R} = \mathbb{C}^r$, $\mathcal{S} = \mathbb{C}^s$ and $Q(z)$ is linear amounts to the setting of [21] discussed in the Introduction, where one can work with a globally defined power series representation for the Schur-Agler-class function S.

We consider the following left-tangential interpolation problem for the class $\mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$. We suppose that $\mathcal{E}$ is another coefficient Hilbert space and that we are given nc functions $a \in \mathcal{T}(\Omega; \mathcal{L}(\mathcal{Y}, \mathcal{E})_{\rm nc})$ and $b \in \mathcal{T}(\Omega; \mathcal{L}(\mathcal{U}, \mathcal{E})_{\rm nc})$. We consider the following problem:

Left-Tangential Interpolation Problem: Given a set of points $\Omega \subset \mathbb{D}_Q$ and nc functions $a \in \mathcal{T}(\Omega'; \mathcal{L}(\mathcal{Y}, \mathcal{E})_{\rm nc})$ and $b \in \mathcal{T}(\Omega'; \mathcal{L}(\mathcal{U}, \mathcal{E})_{\rm nc})$ where we set

$$\Omega' = [\Omega]_{\rm full} \cap \mathbb{D}_Q \quad (3.2)$$

is the $\mathbb{D}_Q$-relative full nc envelope of Ω, find S in the Schur-Agler class $\mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ such that

$$a(Z)S(Z) = b(Z) \text{ for all } Z \in \Omega. \quad (3.3)$$

We are now ready to state the main result concerning interpolation in the Schur-Agler class $\mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$. We note that statement (3) in the following theorem uses the notations and results from Theorem 2.19.

Theorem 3.1. *Suppose that Q and $\mathbb{D}_Q$ are as in (2.23) and that we are given a data set (a set of points $\Omega \subset \mathbb{D}_Q$ and nc functions $a \in \mathcal{T}(\Omega'; \mathcal{L}(\mathcal{Y}, \mathcal{E})_{\rm nc})$ and $b \in \mathcal{T}(\Omega'; \mathcal{L}(\mathcal{U}, \mathcal{E})_{\rm nc})$ with Ω' as in (3.2)) for a Left-Tangential Interpolation Problem as above. Then the following conditions are equivalent.*

(1) *The Left-Tangential Interpolation Problem has a solution, i.e., there exists an $S\colon \mathbb{D}_Q \to \mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc}$ in the nc Schur-Agler class $\mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ satisfying the left-tangential interpolation condition (3.3) on Ω.*

(1′) *The inequality*

$$a(Z)a(Z)^* - b(Z)b(Z)^* \succeq 0 \quad (3.4)$$

holds for each $Z \in \Omega'$.

(2) *The pair (a, b) has a left-tangential nc Agler decomposition over Ω, i.e., there exists a cp nc kernel $\Gamma\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{L}(\mathcal{S}), \mathcal{L}(\mathcal{E}))_{\rm nc}$ (as defined in Subsection 2.6) so that*

$$\begin{aligned} &a(Z)(P \otimes I_{\mathcal{Y}})a(W)^* - b(Z)(P \otimes I_{\mathcal{U}}))b(W)^* \\ &\quad = \Gamma(Z, W)\left((P \otimes I_{\mathcal{S}}) - Q(Z)(P \otimes I_{\mathcal{R}})Q(W)^*\right) \end{aligned} \quad (3.5)$$

for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathbb{C}^{n \times m}$.

(3) *There exists an auxiliary Hilbert space $\mathcal{X}$ equipped with a unitary $*$-representation $\pi \colon \mathcal{L}(\mathcal{S}) \mapsto \mathcal{L}(\mathcal{X})$ and a contractive (even unitary) colligation matrix*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \colon \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} (\mathcal{L}(\mathcal{S}, \mathcal{R}) \otimes_\pi \mathcal{X})_{\rm nc} \\ \mathcal{Y} \end{bmatrix} \tag{3.6}$$

so that the function $S(Z)$ defined by

$$S(Z) = D^{(n)} + C^{(n)}(I - L^*_{Q(Z)^*} A^{(n)})^{-1} L^*_{Q(Z)^*} B^{(n)} \tag{3.7}$$

for $Z \in \Omega_n$ (where here we use the notation (2.47) with $T = Q(Z)^*$) where

$$\begin{bmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{bmatrix} = \begin{bmatrix} I_n \otimes A & I_n \otimes B \\ I_n \otimes C & I_n \otimes D \end{bmatrix} \colon \begin{bmatrix} \mathcal{X}^n \\ \mathcal{U}^n \end{bmatrix} \to \begin{bmatrix} ((\mathcal{L}(\mathcal{S}, \mathcal{R}) \otimes \mathcal{X})_{\rm nc})^n \\ \mathcal{Y}^n \end{bmatrix}. \tag{3.8}$$

satisfies the Left-Tangential Interpolation condition (3.3) on Ω.

Moreover, the implications (2) ⇒ (3) and (3) ⇒ (1) hold under the weaker assumption that a and b are only graded (rather than nc) functions defined only from Ω to $\mathcal{L}(\mathcal{Y}, \mathcal{E})_{\rm nc}$ and $\mathcal{L}(\mathcal{U}, \mathcal{E})_{\rm nc}$ respectively.

We take up the proof of Theorems 3.1 in Section 4. We explore now various immediate corollaries which fall out as special cases.

3.1. **Agler decomposition and transfer-function realization for nc Schur-Agler class.** If we look at the implication (1) ⇔ (3) in Theorem 3.1 for the special case where $\mathcal{E} = \mathcal{Y}$, $\Omega = \mathbb{D}_Q$, $a(Z) = I_{\mathcal{Y}^n}$ for $Z \in \mathbb{D}_{Q,n}$ and we set $S(Z) = b(Z)$, we arrive at the following result.

Corollary 3.2. *Let Q and $\mathbb{D}_Q$ are as in (2.24) and (2.23) and S is a graded function from $\mathbb{D}_Q$ into $\mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc}$. Then the following are equivalent:*

(1) *$S \in \mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$, i.e. S is a contractive nc function on $\mathbb{D}_Q$.*

(2) *S has a nc Agler decomposition over Ω, i.e., there exists a cp nc kernel $\Gamma \colon \Omega \times \Omega \to \mathcal{L}(\mathcal{S}, \mathcal{L}(\mathcal{Y}))_{\rm nc}$ so that*

$$P \otimes I_{\mathcal{Y}} - S(Z)(P \otimes I_{\mathcal{U}})S(W)^* \tag{3.9}$$
$$= \Gamma(Z, W)\left((P \otimes I_{\mathcal{S}}) - Q(Z)(P \otimes I_{\mathcal{R}})Q(W)^*\right). \tag{3.10}$$

(3) *S has a transfer-function realization as in (3.7).*

It has been observed by Agler-McCarthy (see [6]) that Corollary 3.2 can be used to prove nc analogues of the Oka-Weil approximation theory for holomorphic functions in several complex variables. We present here a somewhat more general setting for this type of result.

We let Ξ be a full nc subset of $\mathcal{V}_{\rm nc}$ for some vector space $\mathcal{V}$. We suppose that $\mathbf{A}$ is an algebra of nc functions contained in $\mathcal{T}(\Xi; \mathbb{C}_{\rm nc})$. If $Q = [q_{ij}]$ is a finite matrix with entries q_{ij} in the algebra $\mathbf{A}$, we define a subset $\mathbb{D}_Q \subset \Xi$ as in (2.23):

$$\mathbb{D}_Q = \{Z \in \Xi \colon \|Q(Z)\| < 1\}. \tag{3.11}$$

A set of the form (3.11) is said to be a **basic A-free open set**. Note that the intersection of two basic **A**-free open sets is again a basic **A**-free open set since $\mathbb{D}_{Q_1} \cap \mathbb{D}_{Q_2} = \mathbb{D}_{Q_1 \oplus Q_2}$. Hence one can define a topology on Ξ, hereby called the **A-free topology**, by declaring that the basic **A**-free open sets form a basis for this topology. We can now state our version of a nc Oka-Weil theorem.

Theorem 3.3. ***(nc Oka-Weil Theorem)*** *With notation and definitions as above, suppose that Ω is an **A**-free open subset of Ξ and suppose that f is a nc function in $\mathcal{T}(\Omega; \mathbb{C}_{\rm nc})$ which is **A**-free locally bounded on Ω (i.e., for each $Z \in \Omega$, there is a **A**-free open subset U of Ω containing Z on which f is uniformly bounded ($\|f(Z)\| \le M$ for all $Z \in U$ for some $M < \infty$). Suppose that K is a nc subset of Ω which is compact in the **A**-free topology. Then there exists a sequence $\{q_N\}_{N=1}^{\infty}$ of functions from **A** such that q_N converges to f uniformly on K.*

Proof. Given a point Z in K, there is an **A**-matrix Q_Z with associated **A**-free basic open set $\mathbb{D}_{Q_Z}$ such that $Z \in \mathbb{D}_{Q_Z} \subset \Omega$ and $f|_{\mathbb{D}_{Q_Z}}$ is bounded. Then

$$\mathfrak{U} = \{\mathbb{D}_{Q_Z} \colon Z \in K\}$$

is an open cover of K. As K is compact in the **A**-free topology, there is a finite subcover:

$$K \subset \mathbb{D}_{Q_{Z^{(1)}}} \cup \cdots \cup \mathbb{D}_{Q_{Z^{(N)}}}$$

for some finitely many points $Z^{(1)}, \dots, Z^{(N)}$ in K. We claim that

$$\min_{j=1,\dots,N} \max_{W \in K} \{\|Q_{Z^{(j)}}(W)\|\} < 1. \tag{3.12}$$

If (3.12) fails to hold, then for each $j = 1, \dots, N$ there is a $W^{(j)} \in K$ with

$$\|Q_{Z^{(j)}}(W^{(j)})\| \ge 1. \tag{3.13}$$

Set $W^{(0)} = \begin{bmatrix} W^{(1)} & & \\ & \ddots & \\ & & W^{(N)} \end{bmatrix}$. As we are assuming that K is a nc set, it follows that $W^{(0)} \in K$. As a consequence of (3.13) we also have $\|Q_{Z^{(j)}}(W^{(0)})\| \ge 1$ for all j implying that $W^{(0)}$ fails to be in $\cup_{j=1}^{N} \mathbb{D}_{Q^{(j)}}$ giving us the contradiction that $W^{(0)}$ is not in K after all. We conclude that indeed the claim (3.12) must hold.

We therefore can find an index j_0 so that

$$r := \max_{W \in K} \|Q_{Z^{(j_0)}}(W)\| < 1.$$

To simplify notation, let us write simply Q_0 rather than $Q_{Z^{(j_0)}}$. Choose t with $1 < t < 1/r$. In this way we arrive at a single **A**-matrix Q_0 so that $K \subset \mathbb{D}_{tQ_0}$.

By construction, the nc function f is bounded on $\mathbb{D}_{Q_0}$, i.e., there is an $M < \infty$ so that

$$\frac{1}{M} \cdot f \in \mathcal{SA}_{Q_0}(\mathbb{C}_{\rm nc}).$$

Hence, by Theorem 3.1 (specifically Corollary 3.2), f has a Q_0-realization of the form in (3.7) valid on all of $\mathbb{D}_{Q_0}$. By construction Q_0 is a finite matrix over $\mathbf{A}$, say of size $s \times r$, so we may take the spaces $\mathcal{R}$ and $\mathcal{S}$ in Theorem 3.1 to be

$$\mathcal{R} = \mathbb{C}^r, \quad \mathcal{S} = \mathbb{C}^s.$$

As a consequence of Remark 2.21 and Theorem 2.20, the realization formula (3.7) for $f(Z)$ assumes the simpler form

$$\frac{1}{M} \cdot f(Z) = D^{(n)} + C^{(n)}(I - (Q_0(Z) \otimes I_{\mathcal{X}_0})A^{(n)})^{-1}(Q_0(Z) \otimes I_{\mathcal{X}_0})B^{(n)} \quad (3.14)$$

where the contractive (or even unitary) colligation matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ has the form

$$\begin{bmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{bmatrix} : \begin{bmatrix} (\mathbb{C}^s \otimes \mathcal{X}_0)^n \\ \mathbb{C}^n \end{bmatrix} \mapsto \begin{bmatrix} (\mathbb{C}^r \otimes \mathcal{X}_0)^n \\ \mathbb{C}^n \end{bmatrix}$$

(where we assume $Z \in \mathbb{D}_{Q_0,n}$). As $\|Q_0(Z)\| < 1$ for $Z \in \mathbb{D}_{Q_0}$, we may expand out the representation (3.14) for $f(Z)$ as an infinite series

$$\frac{1}{M} \cdot f(Z) = D^{(n)} + \sum_{j=0}^{\infty} C^{(n)} \left((Q_0(Z) \otimes I_{\mathcal{X}_0})A^{(n)} \right)^j (Q_0(Z) \otimes I_{\mathcal{X}_0})B^{(n)}. \quad (3.15)$$

As $K \subset \mathbb{D}_{tQ_0}$, we have that $\|Q_0(Z)\| \le \frac{1}{t} < 1$ for all $Z \in K$, and hence the above series converges uniformly on K. As the matrix entries of Q_0 are all in the algebra $\mathbf{A}$, it follows that each partial sum q_N of the infinite series (3.15) is in $\mathbf{A}$. We conclude that f is the uniform limit of a sequence of functions $\{M \cdot q_N\}$ from $\mathbf{A}$ as wanted, and Theorem 3.3 follows. □

We note that the special case of Theorem 3.3 where one takes $\Xi = (\mathbb{C}^d)_{\rm nc}$ and $\mathbf{A}$ equal to the algebra of nc scalar polynomials as in Example 2.5 amounts to the nc Oka-Weil theorem of Agler-McCarthy [6, Theorem 9.7].

3.2. **Relatively full nc set of interpolation nodes.** The following result is just a reformulation of the equivalence (1) ⇔ (1′) in Theorem 3.1 for the case where $a(Z) = I_n$ (for $Z \in \Omega_n$), $b(Z) = S_0(Z)$, and $\Omega = \Omega'$.

Corollary 3.4. *Suppose that Q and $\mathbb{D}_Q$ are as in Theorem 3.1 and that $\Omega \subset \mathbb{D}_Q$ is a relatively full nc subset of $\mathbb{D}_Q$, i.e.,*

$$\Omega = [\Omega]_{\rm full} \cap \mathbb{D}_Q.$$

Then any nc function S_0 on Ω can be extended to a nc function S on $\mathbb{D}_Q$ without increasing norm, i.e., given a nc function $S_0 \colon \Omega \to \mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc}$, there exists a nc function $S \colon \mathbb{D}_Q \to \mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc}$ such that

$$S|_\Omega = S_0 \text{ and } \sup_{Z \in \mathbb{D}_Q} \|S(Z)\| = \sup_{Z \in \Omega} \|S_0(Z)\|.$$

This corollary motivates defining a nc subset $\mathcal{D}$ of a full nc subset Ξ to be a **(norm-preserving) nc interpolation domain** if $\mathcal{D}$ has the same property as $\mathbb{D}_Q$ in the above corollary, namely: *if Ω is a relatively full nc subset of $\mathcal{D}$, i.e., if*

$$\Omega = [\Omega]_{\rm full} \cap \mathcal{D},$$

and if $S_0 \colon \Omega \to \mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc}$ is any nc function on Ω, then there is a nc function $S \colon \mathcal{D} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc}$ such that

$$S|_\Omega = S_0 \text{ and } \sup_{Z \in \mathcal{D}} \|S(Z)\| = \sup_{Z \in \Omega} \|S_0(Z)\|.$$

The content of Corollary 3.4 is that any Q-disk $\mathbb{D}_Q$ is a nc interpolation domain. We note that one can think of sets of the form $\mathbb{D}_Q$ as the nc version of analytic polyhedra as occurring in several-complex-variable theory which in turn are closely connected with domains of holomorphy (see e.g. [49]). As a first challenge we pose the following problem: *Find an intrinsic characterization of sets of the nc analytic polyhedra $\mathbb{D}_Q$, or more generally, interpolation domains.*

We mention several rather obvious necessary conditions for a domain $\mathcal{D} \subset \mathcal{V}_{\rm nc}$ to be a nc analytic polyhedron $\mathbb{D}_Q$ for some Q, namely:

(1) $\mathcal{D}$ is a nc set.
(2) $\mathcal{D}$ is Ξ-relatively locally closed with respect to similarity in the following sense: if Z is an element of $\mathcal{D}_n$, then there is a number $\delta > 0$ so that, whenever α is an invertible $n \times n$ matrix such that $\|\alpha^{\pm 1} - I_n\| < \delta$, then $\alpha Z \alpha^{-1}$ is also in $\mathcal{D}_n$.
(3) $\mathcal{D}$ is closed under restriction to invariant subspaces in the following strict sense: whenever $Z \in \mathcal{D}_n$, $\widetilde{Z} \in \Xi_m$ and α is an isometric $n \times m$ matrix (so $\alpha^* \alpha = I_m$) such that $\alpha \widetilde{Z} = Z \alpha$, then $\widetilde{Z} \in \mathcal{D}$.

While characterizing nc analytic polyhedra $\mathbb{D}_Q$ may be difficult, as we shall now show the case where $Q = L_\varphi$ is a nc linear map as in Example 2.3 (b) turns out to be tractable. Helton, Klep, and McCullough in [46] introduced such nc domains $\mathbb{D}_L$ in the finite-dimensional setting, and studied various aspects of the associated nc function theory; following these authors, we shall refer to any such set $\mathbb{D}_L$ as a **pencil ball**. Our goal is to characterize intrinsically which nc subsets $\mathcal{D}$ of $\mathcal{V}_{\rm nc}$ can have the form of a pencil ball $\mathbb{D}_L$.

Suppose first that $\mathcal{D} = \mathbb{D}_{L_\varphi}$ for a nc linear map $L_\varphi \colon \mathcal{V}^{n \times m} \to \mathcal{L}(\mathcal{V}_1^{n \times m}, \mathcal{V}_0^{n \times m})$ as in Example 2.3 (a), where we assume that φ is a linear map from the vector space $\mathcal{V}$ into the operator space $\mathcal{L}(\mathcal{R}, \mathcal{S})$ of bounded linear operators between two Hilbert spaces $\mathcal{R}$ and $\mathcal{S}$. To simplify the notation, we write simply L rather than L_φ. Define a seminorm $\| \cdot \|_n$ on $\mathcal{V}^{n \times n}$ by

$$\|Z\|_n = \|L(Z)\|_{\mathcal{L}(\mathcal{R}^n, \mathcal{S}^n)}.$$

Using the bimodule property (2.20) of L, it is easy to check that this system of norms $\{\| \cdot \|\}_n$ satisfies the Ruan axioms:

(1) $\left\| \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix} \right] \right\|_n = \max\{\|Z\|, \|W\|\}$ for $Z \in \mathcal{V}^{n \times n}$, $W \in \mathcal{V}^{m \times m}$
(2) $\|\alpha \cdot Z \cdot \beta\| \le \|\alpha\| \|Z\|_n \|\beta\|$ for $\alpha \in \mathbb{C}^{m \times n}$, $Z \in \mathcal{V}^{n \times n}$, $\beta \in \mathbb{C}^{n \times m}$.

Indeed, these follow easily from the following properties of L:

(1′) $L\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right) = \begin{bmatrix} L(Z) & 0 \\ 0 & L(W) \end{bmatrix}$ (i.e., L respects direct sums),

(2′) $L(\alpha \cdot Z \cdot \beta) = \alpha \cdot L(Z) \cdot \beta$ (the bimodule property (2.20)).

Given a nc subset $\mathcal{D}$, if we can construct a system of norms (or more generally just seminorms) $\| \cdot \|_n$ satisfying the Ruan axioms so that the n-th level $\mathcal{D}_n$ is the unit ball of $\| \cdot \|_n$

$$\mathcal{D}_n = \{Z \in \mathcal{V}^{n \times n} \colon \|Z\|_n < 1\},$$

then by Ruan's Theorem [40, Theorem 2.3.5], there is a completely isometric isomorphism φ from $\mathcal{V}$ into a subspace of $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ which we can without loss of generality take to have the form $\mathcal{L}(\mathcal{R}, \mathcal{S})$; thus $\|Z\|_n = \|L(Z)\|_{\mathcal{L}(\mathcal{R}^n, \mathcal{S}^n))}$ (where $L(Z) = \varphi^{(n \times n)}(Z) := (\mathrm{id}_{\mathbb{C}^{n \times n}} \otimes \varphi)(Z)$ for $Z \in \mathcal{V}^{n \times n}$) (or only a completely coisometric coisomorphism in the seminorm case), and it then follows that $\mathcal{D} = \mathbb{D}_L$. If we fix a level n, it is well known which sets $\mathcal{D}_n$ can be the open unit ball (at least up to boundary points) for some seminorm $\| \cdot \|_n$, i.e., so that there is a seminorm $\| \cdot \|_n$ so that

$$\{Z \in \mathcal{D}_n \colon \|Z\|_n < 1\} \subset \mathcal{D}_n \subset \{Z \in \mathcal{D}_n \colon \|Z\|_n \leq 1\}$$

(see [75, Theorem 1.35]): namely, $\mathcal{D}_n$ should be (i) **convex** ($Z^{(1)}, Z^{(2)} \in \mathcal{D}_n$, $0 < \lambda < 1 \Rightarrow \lambda Z^{(1)} + (1-\lambda) Z^{(2)} \in \mathcal{D}_n$), (ii) **balanced** ($Z \in \mathcal{D}_n$, $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1 \Rightarrow \lambda Z \in \mathcal{D}_n$), and (iii) **absorbing** ($Z \in \mathcal{V}^{n \times n} \Rightarrow \exists\, t > 0 t$ in $\mathbb{R}$ so that $\frac{1}{t} Z \in \mathcal{D}_n$). It remains to understand what additional properties are needed to get the system of norms $\{\| \cdot \|_n\}$ so constructed to also satisfy the Ruan axioms.

There is a notion of convexity for this nc setting which has already been introduced and used in a number of applications in the literature (see e.g. [41] and the references there): *a nc subset $\mathcal{D}$ of $\mathcal{V}_{\mathrm{nc}}$ is said to be **matrix-convex** if*

$$Z \in \mathcal{D}_n,\ \alpha \in \mathbb{C}^{n \times m} \text{ with } \alpha^* \alpha = I_m \Rightarrow \alpha^* \cdot Z \cdot \alpha \in \mathcal{D}_m.$$

We shall need a nc extension of *balanced* defined as follows: *a nc subset $\mathcal{D}$ of $\mathcal{V}_{\mathrm{nc}}$ is said to be **matrix-balanced** if*

$$Z \in \mathcal{D}_n,\ \alpha \in \mathbb{C}^{m \times n} \text{ with } \|\alpha\| \leq 1,\ \beta \in \mathbb{C}^{n \times m} \text{ with } \|\beta\| \leq 1 \Rightarrow \alpha \cdot Z \cdot \beta \in \mathcal{D}_m.$$

Unlike as in the non-quantized setting, $\mathcal{D}$ being matrix-balanced trivially implies that $\mathcal{D}$ is matrix-convex.

We can now state our characterization of which nc sets $\mathcal{D} \subset \mathcal{V}_{\mathrm{nc}}$ have the form $\mathbb{D}_L$ of a pencil ball, at least up to boundary points.

Proposition 3.5. *Given a nc set $\mathcal{D} \subset \mathcal{V}_{\mathrm{nc}}$, for $Z \in \mathcal{V}^{n \times n}$ define $\| \cdot \|_n$ as the Minkowski functional associated with the set $\mathcal{D}_n \subset \mathcal{V}^{n \times n}$:*

$$\|Z\|_n = \inf\{t > 0 \colon t^{-1} Z \in \mathcal{D}_n\} \text{ for } Z \in \mathcal{D}_n.$$

Then the following are equivalent:

(1) *$\{\| \cdot \|_n\}$ is a system of norms satisfying the Ruan axioms, and hence, from the preceding discussion, there is nc linear map $L \colon \mathcal{V}_{\mathrm{nc}} \to$*

$\mathcal{L}(\mathcal{R}, \mathcal{S})_{\rm nc}$ as in Example 2.3 (a) so that $\mathcal{D} = \mathbb{D}_L$ up to boundary, i.e.,

$$\{Z \in \mathcal{V}^{n \times n} \colon \|L(Z)\| < 1\} \subset \mathcal{D}_n \subset \{Z \in \mathcal{V}^{n \times n} \colon \|L(Z)\| \le 1\}.$$

(2) *$\mathcal{D}$ is a matrix-balanced nc subset of $\mathcal{V}_{\rm nc}$ such that each $\mathcal{D}_n$ is absorbing.*

Proof. Suppose that $\{\| \cdot \|_n\}$ satisfies the Ruan axioms. Then each $\| \cdot \|_n$ is absorbing since $\| \cdot \|_n$ is a (finite-valued) seminorm on $\mathcal{D}_n$. By the first Ruan axiom $\left\| \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix} \right] \right\| = \max\{\|Z\|, \|W\|\}$, we see that $\left\| \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix} \right] \right\| < 1 \Leftrightarrow$ both $\|Z\|_n < 1$ and $\|W\|_m < 1$. We conclude that $\mathcal{D}$ is a nc set. By the second Ruan axiom $\|\alpha \cdot Z \cdot \beta\|_m = \|\alpha\| \|Z\|_n \|\beta\|$, we conclude that

$$\|\alpha\| \le 1,\ \|\beta\| \le 1,\ \|Z\|_n \le 1 \Rightarrow \|\alpha \cdot Z \cdot \beta\|_m \le 1,$$

i.e., $\alpha \cdot Z \cdot \beta \in \mathcal{D}_m$ if $\|\alpha\| \le 1$, $\|\beta\| \le 1$, $Z \in \mathcal{D}_n$. Hence $\mathcal{D}$ is matrix-balanced.

Conversely, assume that each $\mathcal{D}_n$ is absorbing and that $\mathcal{D}$ is a matrix-balanced nc set. Define $\| \cdot \|_n$ as the Minkowski functional of $\mathcal{D}_n$:

$$\|Z\|_n = \inf\{t > 0 \colon \frac{1}{t} Z \in \mathcal{D}_n\} \text{ for } Z \in \mathcal{V}^{n \times n}.$$

Then $\|Z\|_n < \infty$ since $\mathcal{D}_n$ is absorbing. Since $\mathcal{D}$ being matrix-balanced implies that each $\mathcal{D}_n$ is also convex and balanced, we see that $\| \cdot \|_n$ is a seminorm on $\mathcal{D}_n$ such that

$$\{Z \in \mathcal{V}^{n \times n} \colon \|Z\|_n < 1\} \subset \mathcal{D}_n \subset \{Z \in \mathcal{V}^{n \times n} \colon \|Z\|_n \le 1\}.$$

Since $\mathcal{D}$ is a nc set, we see that

$$Z \in \mathcal{D}_n, W \in \mathcal{D}_m \Rightarrow \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix} \right] \in \mathcal{D}_{n+m}.$$

Since $\mathcal{D}$ is also matrix-balanced, we see that

$$\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} \in \mathcal{D}_{n+m} \Rightarrow \begin{bmatrix} I_n & 0 \end{bmatrix} \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} I_n \\ 0 \end{bmatrix} = Z \in \mathcal{D}_n \text{ and}$$
$$\begin{bmatrix} 0 & I_m \end{bmatrix} \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} 0 \\ I_m \end{bmatrix} = W \in \mathcal{D}_m$$

Thus we have

$$\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} \in \mathcal{D}_{n+m} \Leftrightarrow Z \in \mathcal{D}_n \text{ and } W \in \mathcal{D}_m.$$

From this property we deduce that the first Ruan axiom holds: $\left\| \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix} \right] \right\| = \max\{\|Z\|, \|W\|\}$.

Finally, we use the matrix-balanced property of $\mathcal{D}$ to deduce

$$\begin{aligned} \|\alpha Z \beta\|_m &= \inf\{t > 0 \colon t^{-1} \alpha Z \beta \in \mathcal{D}_m\} \\ &= \|\alpha\| \cdot \inf\{t > 0 \colon \frac{\alpha}{\|\alpha\|} \cdot (t^{-1} Z) \cdot \frac{\beta}{\|\beta\|} \in \mathcal{D}_m\} \cdot \|\beta\| \\ &\le \|\alpha\| \cdot \inf\{t > 0 \colon t^{-1} Z \in \mathcal{D}_n\} \cdot \|\beta\| \text{ (since } \mathcal{D} \text{ is matrix-balanced)} \\ &= \|\alpha\| \cdot \|Z\|_n \cdot \|\beta\| \end{aligned}$$

and the second Ruan axiom is verified. □

3.3. **The nc corona theorem.** If we look at the equivalence (1) ⇔ (1′) in Theorem 3.1 for the special case where $\Omega = \mathbb{D}_Q$, we arrive at the following.

Corollary 3.6. Suppose that we are given nc functions $a \in \mathcal{T}(\mathbb{D}_Q; \mathcal{L}(\mathcal{Y}, \mathcal{E})_{\rm nc})$ and $b \in \mathcal{T}(\mathbb{D}_Q; \mathcal{L}(\mathcal{U}, \mathcal{E})_{\rm nc})$. Then the following are equivalent:

(1) $a(Z)a(Z)^* - b(Z)b(Z)^* \succeq 0$ for all $Z \in \mathbb{D}_Q$.
(2) There exists a Schur-Agler class function $S \in \mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ so that $a(Z)S(Z) = b(Z)$ for all $Z \in \mathbb{D}_Q$.

We recall that the Carleson corona theorem (see [29]) asserts than an N-tuple of bounded holomorphic functions on the unit disk is not contained in a proper ideal if and only if the functions are jointly bounded below by a positive constant. A special case of Corollary 3.6 yields a free version of this result.

Corollary 3.7. *Let Q and $\mathbb{D}_Q$ be as in* (2.23) *and suppose that we are given N scalar nc functions on $\mathbb{D}_Q$: $\psi_1, \dots, \psi_N$ in $\mathcal{T}(\mathbb{D}_Q; \mathbb{C}_{\rm nc})$. Assume that the family $\{\psi_i \colon i = 1, \dots, N\}$ is uniformly bounded below in the sense that there exist an $\epsilon > 0$ so that*

$$\sum_{i=1}^{N} \psi_i(Z)\psi_i(Z)^* \succeq \epsilon^2 I_{\mathbb{C}^n}$$

for all $Z \in \mathbb{D}_{Q,n}$ for all $n \in \mathbb{N}$. Then there exist uniformly bounded nc functions $\phi_1, \dots \phi_N$ in $\mathcal{T}(\mathbb{D}_Q; \mathbb{C}_{\rm nc})$ so that the corona identity

$$\sum_{i=1}^{N} \psi_i(Z)\phi_i(Z) = I_{\mathbb{C}^n} \text{ for all } Z \in \mathbb{D}_{Q,n} \text{ for all } n \in \mathbb{N}$$

holds. In fact one can choose $\{\phi_i \colon i = 1, \dots, N\}$ so that

$$\sum_{i=1}^{N} \phi_i(Z)^*\phi_i(Z) \preceq (1/\epsilon^2) I_{\mathbb{C}^n}$$

for all $Z \in \mathbb{D}_{Q,n}$ for all $n \in \mathbb{N}$.

Proof. This result amounts to the special case of Corollary 3.6 where $a(Z) = \begin{bmatrix} \psi_1(Z) & \cdots & \psi_N(Z) \end{bmatrix}$ and $b(Z) = \epsilon I_{\mathbb{C}^n}$ for $Z \in \mathbb{D}_{Q,n}$ and one seeks to solve for S of the form $S(Z) = \epsilon \begin{bmatrix} \phi_1(Z) \\ \vdots \\ \phi_N(Z) \end{bmatrix}$. □

We note that Corollary 3.6 with Ξ specialized to $\Xi = \mathbb{C}^d_{\rm nc}$ as in Example 2.5 amounts to (a corrected version of) Theorem 8.1 in [6] (where the special case giving the nc Carleson corona theorem is also noted). By considering the special case $\mathcal{E} = \mathcal{Y}$, $a(Z) = I_{\mathcal{Y}^n}$ for $Z \in \mathbb{D}_{Q,n}$, and hence $S(Z) = b(Z)$, one can see that it is not enough to assume only that $b(Z) = S(Z)$ is graded as in [6] since it is easy to write down contractive graded functions which are not nc functions. As indicated in the statement of Theorem 3.1, the implication (2) $\Rightarrow$ (3) does hold under the weaker assumption that a and b are only graded (not necessarily nc) functions defined only on the subset Ω where the interpolation conditions are specified.

3.4. **Finite set of interpolation nodes.** We next focus on the special case of Theorem 3.1 where the set of interpolation nodes Ω is a finite set. As already observed by Agler and McCarthy in [7], taking a singleton set $\{Z^{(0)}\}0$ as the set of interpolation nodes is equivalent to taking a finite set $\{Z^{(1)}, \dots, Z^{(N)}\}$ since one can use the nc function structure to get an equivalent problem with the singleton set $\{Z^{(0)}\}$ with $Z^{(0)} = \begin{bmatrix} Z^{(1)} & & \\ & \ddots & \\ & & Z^{(N)} \end{bmatrix}$. Hence we focus on the setting where the interpolation node set is a singleton.

Corollary 3.8. *Let Q and $\mathbb{D}_Q$ be as in Theorem 3.1 and suppose that $Z^{(0)}$ is one particular point in $\mathbb{D}_{Q,n}$ and Λ_0 is a particular operator in $\mathcal{L}(\mathcal{U} \otimes \mathbb{C}^n, \mathcal{Y} \otimes \mathbb{C}^n)$. Then the following are equivalent:*

1. *There exists a function S in the Schur-Agler class $\mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ so that $S(Z^{(0)}) = \Lambda_0$.*
2. *There exists a nc function $S_{\rm full}$ on the $\mathbb{D}_Q$-relative full nc envelope $\{Z^{(0)}\}_{\rm nc, full} \cap \mathbb{D}_Q$ of the singleton set $\{Z^{(0)}\}$ such that $S_{\rm full}(Z^{(0)}) = \Lambda_0$ and $\|S_{\rm full}(Z)\| \le 1$ for all $Z \in \{Z^{(0)}\}_{\rm nc, full} \cap \mathbb{D}_Q$.*

Proof. This amounts to the equivalence (1) $\Leftrightarrow$ (1′) in Theorem 3.1 for the special case where Ω is the singleton set $\Omega = \{Z^{(0)}\}$ with $a(Z^{(0)}) = I_n$ and $b(Z^{(0)}) = \Lambda_0$. □

We note that Corollary 3.8 implies Theorem 1.3 in [7], apart from the added content in [7] that, in the case where $\Xi = (\mathbb{C}^d)_{\rm nc}$ and Q are taken as in Example 2.5, then one can take $S_{\rm full}$ in statement (2) to be a nc polynomial. The formulation in [7] is in terms of the nc-Zariski closure $\overline{\{Z^{(0)}\}}$ rather than in terms of the full nc envelope $\{Z^0\}_{\rm nc, full}$. However a consequence of the containment (2.32) is that the Agler-McCarthy hypothesis with nc-Zariski closure implies the hypothesis here with full nc envelope. Of course, whenever it is the case that the containment (2.32) is actually an equality (as is the case when $\Xi = \mathbb{C}_{\rm nc}$ by Proposition 2.12), then the hypothesis here and the hypothesis in Theorem 1.3 from [7] are the same.

3.5. **Commutative Schur-Agler class.** As our next illustrative special case, we indicate how the commutative results of [9, 17, 60] follow from the general theory for the nc case. Specifically, the Agler-decomposition as well as transfer-function realization and interpolation (at least for the left-tangential case) results of [9, 17, 60] for the commutative Schur-Agler class determined by a matrix polynomial q in d (commuting) complex variables (or more generally an operator-valued holomorphic function on $\mathbb{C}^d$ as in [60]) follow as a corollary of Theorem 3.1. Indeed, an application of Corollary

3.2 to the special case where Ξ and $Q = q$ are as in Example 2.7 gives all these results for the now commutative Schur-Agler class $\mathcal{SA}_q(\mathcal{U}, \mathcal{Y})$ (the specialization of the general noncommutative theory to the commutative setup of Example 2.7). One can then use the observations made in the discussion of 2.7 that the Taylor/Martinelli-Vasilescu functional calculus extends a holomorphic function defined on $\mathbb{D}_{q,1}$ (i.e., d scalar arguments) to a nc function (i.e., a function respecting intertwinings) defined on $\mathbb{D}_q$. Putting all this together, we see that the formally noncommutative Schur-Agler class $\mathcal{SA}_q(\mathcal{U}, \mathcal{Y})$ for this special case is the same as the commutative Schur-Agler class $\mathcal{CSA}_q(\mathcal{U}, \mathcal{Y})$ as defined in [9, 17, 60], and the results of [9, 17, 60] on Agler decomposition, transfer-function realization and interpolation follow as a special case of Theorem 3.1. We give further discussion of this setting in Remark 4.5 below.

3.6. **Unenhanced Agler decompositions.** In Theorem 3.1 suppose that the Ω is a nc subset which is open in the finite topology of $\mathcal{V}_{\rm nc}$ (see [52, page 83]), i.e., that the intersection of Ω_n with any finite-dimensional subspace $\mathcal{O}$ of $\mathcal{V}^{n \times n}$ is open in the Euclidean topology of $\mathcal{O}$ for each $n = 1, 2, \dots$, and impose the standing assumption in Theorem 3.1 that a and b are nc functions on the full nc envelope of Ω. Then statement (2) in the Theorem 3.1 can be weakened to the requirement that the Agler decomposition (3.5) holds only for Z, W both in Ω_n and $P = I_n$ for each $n = 1, 2, \dots$, i.e., the Agler decomposition (3.5) can be weakened to the "unenhanced" form

$$a(Z)a(W)^* - b(Z)b(W)^* = \Gamma(Z, W)(I - Q(Z)Q(W)^*) \tag{3.16}$$

for $Z, W \in \Omega_n$ for $n = 1, 2, \dots$. To see this, let Z, W be any two points in Ω_n. Since Ω is now assumed to be finitely open, there is an $\epsilon > 0$ so that whenever α and β are invertible $n \times n$ matrices with $\|\alpha^{\pm 1} - I_n\| < \epsilon$ and $\|\beta^{\pm 1} - I_n\| < \epsilon$, then it follows that both $\widetilde{Z} := \alpha Z \alpha^{-1}$ and $\widetilde{W} := \beta W \beta^{-1}$ are in Ω_n. Consequently, for $P = \alpha^{-1}\beta^{-1*}$ we have

$$\begin{aligned}
&a(Z)Pa(W)^* - b(Z)Pb(W)^* \\
&= a(\alpha^{-1}\widetilde{Z}\alpha)\alpha^{-1}\beta^{-1*}a((\beta^{-1}\widetilde{W}\beta)^* - b(\alpha^{-1}\widetilde{Z}\alpha)\alpha^{-1}\beta^{-1*}b(\beta^{-1}\widetilde{W}\beta)^* \\
&= \alpha^{-1}a(\widetilde{Z})\alpha \cdot \alpha^{-1}\beta^{-1*} \cdot \beta^* a(\widetilde{W})^*\beta^{-1*} \\
&\qquad - \alpha^{-1}b(\widetilde{Z})\alpha \cdot \alpha^{-1}\beta^{-1*} \cdot \beta^* b(\widetilde{W})^*\beta^{-1*} \\
&\qquad \text{(since } a \text{ and } b \text{ respect intertwinings)} \\
&= \alpha^{-1}\left(a(\widetilde{Z})a(\widetilde{W})^* - b(\widetilde{Z})b(\widetilde{W})^*\right)\beta^{-1*} \\
&= \alpha^{-1}\,\Gamma(\widetilde{Z}, \widetilde{W})\left(I_{\mathcal{S}^n} - Q(\widetilde{Z})Q(\widetilde{W})^*\right)\beta^{-1*} \text{ (by (3.16))} \\
&= \Gamma(Z, W)\left(\alpha^{-1}\beta^{-1*} - \alpha^{-1}Q(\widetilde{Z})Q(\widetilde{W})^*\beta^{-1*}\right) \\
&\qquad \text{(since } \Gamma \text{ respects intertwinings)} \\
&= \Gamma(Z, W)\left(\alpha^{-1}\beta^{-1*} - Q(Z)\alpha^{-1}\beta^{-1*}Q(W)^*\right)
\end{aligned}$$

$$\text{(since } Q \text{ respects intertwinings)}$$
$$= \Gamma(Z,W)(P - Q(Z)PQ(W)^*).$$

Hence, for given $Z, W \in \Omega_n$, (3.5) holds for all $P \in \mathbb{C}^{n\times n}$ in an open set around I_n. But both sides of (3.5) are holomorphic in the entries of P. Hence by the uniqueness of analytic continuation off an open set it follows that (3.5) holds for all $P \in \mathbb{C}^{n\times n}$.

For $Z \in \Omega_n$ and $W \in \Omega_m$ with possibly $n \neq m$, apply the preceding result with $\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] \in (\Omega)_{\text{nc},n+m}$ in place of Z and W and with $\left[\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right] \in \mathbb{C}^{(n+m)\times(n+m)}$ in place of P. Then the resulting identity

$$a\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right)\left[\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right] a\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right)^* - b\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right)\left[\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right] b\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right)^*$$
$$= \Gamma\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right], \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right] - Q\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right)\left[\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right] Q\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right)^*\right)$$

combined with the "respects direct sums" property of a, b, Γ, Q leads to the identity

$$\begin{bmatrix} 0 & a(Z)Pa(W)^* - b(Z)Pa(W)^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Gamma(Z,W)(P - Q(Z)PQ(W)^*) \\ 0 & 0 \end{bmatrix},$$

and hence the identity (3.5) holds for $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathbb{C}^{n\times m}$ with $n \neq m$ as well.

4. Proofs of Schur-Agler class interpolation theorems

We shall prove (1) $\Rightarrow$ (1′) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) in Theorem 3.1.

Proof of (1) $\Rightarrow$ (1′) in Theorem 3.1: Suppose that the left-tangential interpolation condition (3.3) holds on Ω for a Schur-Agler class function $S \in \mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$. It is easily checked that the pointwise product $a \cdot S$ is again a nc function whenever each of a and S is a nc function. By the uniqueness of a nc-function extension from Ω to $\Omega' = \Omega_{\text{nc,full}} \cap \mathbb{D}_Q$ (see Proposition 2.10), the identity $a(Z)S(Z) = b(Z)$ holding on Ω implies that it continues to hold on $\Omega' = \Omega_{\text{nc,full}} \cap \mathbb{D}_Q$. By assumption, $S(Z)$ is contractive for all $Z \in \mathbb{D}_Q$, and hence in particular for all $Z \in \Omega_{\text{nc,full}}$. We conclude that

$$a(Z)a(Z)^* - b(Z)b(Z)^* = a(Z)(I - S(Z)S(Z)^*)a(Z)^* \succeq 0 \text{ for all } Z \in \Omega_{\text{full}} \cap \mathbb{D}_Q$$

and statement (2) of the theorem follows. □

Proof of (1′) $\Rightarrow$ (2) in Theorem 3.1; We subdivide the proof into two cases.

Case 1: Ω and $\dim \mathcal{E}$ finite. For this case we assume that both Ω and $\dim \mathcal{E}$ are finite. We first need a few additional preliminaries.

The finite point set Ω. Recall the underlying framework from Subsection 2.2. We are given a vector space $\mathcal{V}$, a full nc subset $\Xi \subset \mathcal{V}_{\text{nc}}$, a nc function Q from Ξ to $\mathcal{L}(\mathcal{R}, \mathcal{S})_{\text{nc}}$ with associated nc Q-disk $\mathbb{D}_Q \subset \Xi$. For the present Case 1, we are assuming that Ω is a finite subset of $\mathbb{D}_Q$. Therefore the subspace $\mathcal{V}^0$ of $\mathcal{V}$ spanned by all the matrix entries of elements Z of Ω is finite dimensional, say $\dim \mathcal{V}^0 = d$. For this Case 1 part of the proof, it is only vectors in $\mathcal{V}^0$ which come up, so without loss of generality we assume that $\mathcal{V} = \mathcal{V}^0$. By choosing a basis we identify $\mathcal{V}$ with $\mathbb{C}^d$, and thus each point $Z \in \Omega_n$ is identified with an element of $(\mathbb{C}^d)^{n\times n} \cong (\mathbb{C}^{n\times n})^d$. For $Z \in \Omega_n$, we therefore view Z as a d-tuple $Z = (Z_1, \dots, Z_d)$ of complex $n \times n$ matrices ($Z_k \in \mathbb{C}^{n\times n}$ for $k = 1, \dots, d$).

We next define nc functions χ_k defined on all of $(\mathbb{C}^d)_{\rm nc}$ with values in $\mathbb{C}_{\rm nc}$ by

$$\chi_k(Z) = Z_k \text{ if } Z = (Z_1, \dots, Z_d).$$

We stack these into a block row matrix to define a nc function from $\Xi \subset (\mathbb{C}^d)_{\rm nc}$ into $\mathcal{L}(\mathbb{C}^d, \mathbb{C})_{\rm nc}$ by

$$\chi(Z) = \begin{bmatrix} \chi_1(Z) & \cdots & \chi_d(Z) \end{bmatrix} = \begin{bmatrix} Z_1 & \cdots & Z_d \end{bmatrix}.$$

We view each such Z_k as an operator Z_k^r acting on row vectors via right multiplication: thus

$$Z_k^r \colon x^* \mapsto x^* Z_k \text{ for } x^* \in \mathbb{C}^{1\times n}.$$

Thus, for $Z \in (\mathbb{C}^{n\times n})^d$, we identify $\chi(Z)$ with the operator $\chi(Z)^r$ acting from $\mathbb{C}^{1\times n}$ to $\mathbb{C}^{1\times nd}$ by

$$\chi(Z)^r \colon x^* \mapsto \begin{bmatrix} x^* Z_1 & \cdots & x^* Z_d \end{bmatrix} \text{ for } x^* \in \mathbb{C}^{1\times n}.$$

The linear space $\mathfrak{X}$ and its cone $\mathcal{C}$. We let $\mathfrak{X}$ be the linear space of all nc kernels $K \in \widetilde{\mathcal{T}}^1(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc}, \mathbb{C}_{\rm nc})$ with norm given by

$$\|K\|_{\mathfrak{X}} = \max\{\|K(Z,W)\| \colon Z, W \in \Omega\}.$$

We define a subset $\mathcal{C}$ of $\mathfrak{X}$ by

$$\begin{aligned} \mathcal{C} = \{ & K \in \mathfrak{X} \colon \exists \text{ a cp nc kernel } \Gamma \in \widetilde{\mathcal{T}}^1(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc}, \mathcal{L}(\mathcal{S})_{\rm nc}) \text{ so that} \\ & K(Z,W)(P) = \Gamma(Z,W)\left(P \otimes I_{\mathcal{S}} - Q(Z)(P \otimes I_{\mathcal{R}})Q(W)^*\right) \\ & \text{for all } Z \in \Omega_n,\ W \in \Omega_m,\ P \in \mathbb{C}^{n\times m}\}. \end{aligned} \tag{4.1}$$

Key properties of $\mathcal{C}$ are given by the following lemma.

Lemma 4.1. (1) *The subset $\mathcal{C}$ is a closed cone in $\mathfrak{X}$.*

(2) *For $f \in \mathcal{T}(\Omega; \mathcal{E}_{\rm nc})$, define $D_{f,f} \in \widetilde{\mathcal{T}}^1(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc}, \mathbb{C}_{\rm nc})$ by*

$$D_{f,f}(Z,W)(P) = f(Z)Pf(W)^*. \tag{4.2}$$

Then $D_{f,f} \in \mathcal{C}$.

Proof of Lemma 4.1 part (1). One easily verifies from the definitions that $\tau K \in \mathcal{C}$ whenever $K \in \mathcal{C}$ and $\tau > 0$ and that $K_1 + K_2 \in \mathcal{C}$ whenever K_1 and K_2 are in $\mathcal{C}$, i.e., $\mathcal{C}$ is a cone.

It remains to show that $\mathcal{C}$ is closed in the norm topology of $\mathfrak{X}$. Toward this end suppose that $\{K_N \colon N \in \mathbb{N}\}$ is a sequence of elements of $\mathcal{C}$ such that $\|K - K_N\|_{\mathfrak{X}} \to 0$ as $N \to \infty$ for some $K \in \mathfrak{X}$. By definition, for each N there is a cp nc kernel $\Gamma_N \in \widetilde{\mathcal{T}}^1(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc}, \mathcal{L}(\mathcal{S})_{\rm nc})$ so that

$$K_N(Z,W)(P) = \Gamma_N(Z,W)((P \otimes I_{\mathcal{S}}) - Q(Z)(P \otimes I_{\mathcal{R}})Q(W)^*)$$

for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathbb{C}^{n \times m}$ for all $m, n \in \mathbb{N}$. The goal is to produce a cp nc kernel Γ in $\widetilde{\mathcal{T}}^1(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc}, \mathcal{L}(\mathcal{S})_{\rm nc})$ so that K can be expressed in the form (4.1).

Define a number ρ_0 by

$$\rho_0 = \max\{\|Q(Z)\| \colon Z \in \Omega\}. \tag{4.3}$$

As Ω is a finite subset of $\mathbb{D}_Q$, we see that $\rho_0 < 1$. Since $\Gamma_N(Z,Z)$ is a positive map for each N and each $Z \in \Omega$, we get the estimate

$$\begin{aligned} K_N(Z,Z)(I_n) &= \Gamma_N(Z,Z)\left(I_{\mathcal{S}^n} - Q(Z)Q(Z)^*\right) \\ &\geq (1-\rho_0^2)\Gamma_N(Z,Z)(I_{\mathcal{S}^n}). \end{aligned}$$

Consequently, we get

$$\|\Gamma_N(Z,Z)(I_{\mathcal{S}^n})\|_{\mathcal{L}(\mathcal{E}^n)} \leq \frac{1}{1-\rho_0^2}\|K_N(Z,Z)\|_{\mathcal{L}(\mathcal{L}(\mathbb{C}^n),\mathcal{L}(\mathcal{E}^n))}. \tag{4.4}$$

As a consequence of $\|K_N - K\|_{\mathfrak{X}} \to 0$ as $N \to \infty$, it follows that in particular $\|K_N(Z,Z) - K(Z,Z)\|_{\mathcal{L}(\mathcal{L}(\mathbb{C}^n),\mathcal{L}(\mathcal{Y}^n))} \to 0$ as $N \to \infty$. Thus $\|K_N(Z,Z)\|$ is uniformly bounded in $\mathcal{L}(\mathcal{L}(\mathbb{C}^n), \mathcal{L}(\mathcal{Y}^n))$. As a consequence of (4.4) we then see that $\|\Gamma_N(Z,Z)(I_{\mathcal{S}^n})\|_{\mathcal{L}(\mathcal{E}^n)}$ is uniformly bounded in $N = 1, 2, \dots$. Since $\Gamma_N(Z,Z)$ is completely positive, we have

$$\|\Gamma_N(Z,Z)\|_{\mathcal{L}(\mathcal{L}(\mathcal{S}^n),\mathcal{L}(\mathcal{E}^n))} = \|\Gamma_N(Z,Z)(I_{\mathcal{S}^n})\|_{\mathcal{L}(\mathcal{E}^n)}.$$

Hence $\|\Gamma_N(Z,Z)\|_{\mathcal{L}(\mathcal{L}(\mathcal{S}^n),\mathcal{L}(\mathcal{E}^n))}$ is uniformly bounded with respect to $N \in \mathbb{N}$ for $Z \in \Omega$. Moreover, since Γ_N is a cp nc kernel, $\Gamma_N(Z,Z)$ is completely positive as a map from $\mathcal{L}(\mathcal{S})^{n \times n}$ to $\mathcal{L}(\mathcal{E})^{n \times n}$ (where $Z \in \Omega_n$).

Let now Z and W be two points in Ω. Then $\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]$ is a point in the nc envelope $\Omega_{\rm nc}$ of Ω and Γ. By Proposition 2.17, both K and Γ can be extended as nc and cp nc kernels respectively to $\Omega_{\rm nc}$, and hence to the finite set $\Omega \cup \left\{\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right\}$. Then the estimate (4.4) and the analysis there with $\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]$ in place of Z, we see that $\left\|\Gamma_N\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right)\right\|$ is uniformly bounded in N. Let us note that

$$\Gamma_N\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right], \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 & \Gamma_N(Z,W)(P) \\ 0 & 0 \end{smallmatrix}\right].$$

We conclude that $\|\Gamma_N(Z,W)\|$ is uniformly bounded in norm with respect to N for each Z and W in Ω.

Note that $\Gamma_N(Z,W) \in \mathcal{L}(\mathcal{L}(\mathcal{S}^m, \mathcal{S}^n), \mathcal{L}(\mathcal{E}^m, \mathcal{E}^n))$ if $Z \in \Omega_n$ and $W \in \Omega_m$. A key point at this stage is that the Banach space $\mathcal{L}(\mathcal{L}(\mathcal{S}^m, \mathcal{S}^n), \mathcal{L}(\mathcal{E}^m, \mathcal{E}^n))$ has a predual $\mathcal{L}(\mathcal{L}(\mathcal{S}^m, \mathcal{S}^n), \mathcal{L}(\mathcal{E}^m, \mathcal{E}^n))_*$ such that on bounded sets the weak-$*$ topology is the same as the pointwise weak-$*$ topology: a bounded net $\{\Phi_\lambda\}$ converges to Φ means that $\Phi_\lambda(T) \to \Phi(T)$ in the ultraweak (or weak-$*$) topology of $\mathcal{L}(\mathcal{E}^m, \mathcal{E}^n)$ for each fixed $T \in \mathcal{L}(\mathcal{S}^m, \mathcal{S}^n)$ and the topology on the whole space is defined to be the strongest topology which agrees with this topology on bounded subsets. Since the weak and weak-$*$ topologies agree on bounded subsets, this topology is sometimes also called the *BW-topology* (for "bounded-weak topology"). In fact there is a more general result: *if $\mathcal{X}$ and $\mathcal{Z}$ are Banach spaces, then the space $\mathcal{L}(\mathcal{X}, \mathcal{Z}^*)$ is isometrically isomorphic to the dual of the Banach projective tensor-product space $\mathcal{X}\widehat{\otimes}\mathcal{Z}$; moreover a bounded net Φ_λ in $\mathcal{L}(\mathcal{X}, \mathcal{Z}^*)$ converges to $\Phi \in \mathcal{L}(\mathcal{X}, \mathcal{Z}^*)$ in the associated weak-$*$ topology if and only if the $\mathbb{C}$-valued net $(\Phi_\lambda(x))(\mathfrak{z})$ converges to $(\Phi(x))(\mathfrak{z})$ for each fixed $x \in \mathcal{X}$ and $\mathfrak{z} \in \mathcal{Z}$* (see [34, Corollary 2 page 230] as well as [68, pages 84–85] and [78, Section IV.2]). We apply this result with $\mathcal{X} = \mathcal{L}(\mathcal{S}^m, \mathcal{S}^n)$ and $\mathcal{Z}^* = \mathcal{L}(\mathcal{E}^m, \mathcal{E}^n)$ (so $\mathcal{Z}$ can be taken to be the trace-class operators $\mathcal{C}_1(\mathcal{E}^n, \mathcal{E}^m)$ from $\mathcal{E}^n$ to $\mathcal{E}^m$). Note that for our application with $\dim \mathcal{E} < \infty$ this result is actually more elementary than the general case as described above.

In any case, by the Banach-Alaoglŭ Theorem [75, page 68], norm-closed and bounded subsets of $\mathcal{L}(\mathcal{L}(\mathcal{S}^m, \mathcal{S}^n), \mathcal{L}(\mathcal{E}^m, \mathcal{E}^n))$ are compact in the weak-$*$ topology. Since we established above that $\{\Gamma_N(Z, W)\}$ is uniformly bounded in norm as $N \to \infty$ for each of the finitely many $Z, W \in \Omega$, it follows that we can find a subnet $\{\Gamma_\lambda\}$ of the sequence $\{\Gamma_N\}$ so that $\Gamma_\lambda(Z, W)$ converges weak-$*$ to an element $\Gamma(Z, W)$ in $\mathcal{L}(\mathcal{L}(\mathcal{S}^m, \mathcal{S}^n), \mathcal{L}(\mathcal{E}^m, \mathcal{E}^n))$. We need to check that Γ so defined is a cp nc kernel on Ω, i.e., we must check that the limiting Γ satisfies (2.5) and (2.13) given that each Γ_λ does, or, given that

$$Z \in \Omega_n,\ \widetilde{Z} \in \Omega_{\widetilde{n}},\ \alpha \in \mathbb{C}^{\widetilde{n} \times n} \text{ such that } \alpha Z = \widetilde{Z}\alpha,$$
$$W \in \Omega_m,\ \widetilde{W} \in \Omega_{\widetilde{m}},\ \beta \in \mathbb{C}^{\widetilde{m} \times m} \text{ such that } \beta W = \widetilde{W}\beta, \quad P \in \mathcal{L}(\mathcal{S})^{n \times m},$$

we must show that

$$\begin{aligned} &\alpha \Gamma_\lambda(Z, W)(P)\beta^* = \Gamma_\lambda(\widetilde{Z}, \widetilde{W})(\alpha P \beta^*) \text{ for all } \lambda \\ &\quad \Rightarrow \alpha \Gamma(Z, W)(P)\beta^* = \Gamma(\widetilde{Z}, \widetilde{W})(\alpha P \beta^*) \end{aligned} \tag{4.5}$$

as well as

$$\sum_{i,j=1}^n V_i^* \Gamma_\lambda(Z^{(i)}, Z^{(j)})(R_i^* R_j) V_j \succeq 0 \Rightarrow \sum_{i,j=1}^n V_i^* \Gamma(Z^{(i)}, Z^{(j)})(R_i^* R_j) V_j \succeq 0. \tag{4.6}$$

We now use the fact that weak-$*$ convergence on bounded sets is the same as pointwise weak-$*$ convergence as explained above. To verify (4.5), we fix a trace-class operator X from $\mathcal{E}^{\widetilde{n}}$ to $\mathcal{E}^{\widetilde{m}}$. From the assumption in (4.5) we then get

$$\operatorname{tr}(\alpha \Gamma_\lambda(Z, W)(P)\beta^* X) = \operatorname{tr}\left(\Gamma_\lambda(\widetilde{Z}, \widetilde{W})(\alpha P \beta^*) X\right) \text{ for all } \lambda.$$

Since $\Gamma_\lambda(Z, W) \to \Gamma(Z, W)$ in the pointwise weak-$*$ topology for each fixed $Z, W \in \Omega$, we may take the limit with respect to the net λ in this last expression to arrive at

$$\operatorname{tr}(\alpha \Gamma(Z, W)(P)\beta^* X) = \operatorname{tr}\left(\Gamma(\widetilde{Z}, \widetilde{W})(\alpha P \beta^*) X\right).$$

Since $X \in \mathcal{C}_1(\mathcal{E}^{\widetilde{n}}, \mathcal{E}^{\widetilde{m}})$ is arbitrary, we may peel X and the trace off to arrive at the desired conclusion in (4.5).

To verify (4.6), we let X be an arbitrary positive semidefinite trace-class operator in $\mathcal{L}(\mathcal{E})$. Then the hypothesis in (4.6) gives us

$$\operatorname{tr}\left(\left(\sum_{i,j=1}^{n} V_i^* \Gamma_\lambda(Z^{(i)}, Z^{(j)})(R_i^* R_j) V_j\right) X\right) \geq 0.$$

Again using the pointwise weak-$*$ convergence of $\Gamma_\lambda(Z,W)$ to $\Gamma(Z,W)$ for each $Z, W \in \Omega$, we may take the limit of this last expression to get

$$\operatorname{tr}\left(\left(\sum_{i,j=1}^{n} V_i^* \Gamma(Z^{(i)}, Z^{(j)})(R_i^* R_j) V_j\right) X\right) \geq 0.$$

As X is an arbitrary positive semidefinite trace-class operator on $\mathcal{E}$, we arrive at the conclusion of (4.6) as required.

It remains only to check that the kernel Γ so constructed provides an Agler decomposition (4.1) for the limit kernel K. Since $\{K_N(Z,W)\}$ is converging to $K(Z,W)$ in $\mathcal{L}(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n), \mathcal{L}(\mathcal{E}^m, \mathcal{E}^n))$-norm (where $Z \in \Omega_n$, $W \in \Omega_m$), it follows that the subnet $\{K_\lambda(Z,W)\}$ converges weak-$*$ (and hence pointwise weak-$*$ as well) to $K(Z,W)$. This together with the pointwise weak-$*$ convergence of Γ_λ and the fact that Γ_λ provides an Agler decomposition for K_λ for each λ leads to the conclusion that indeed Γ provides an Agler decomposition (4.1). This completes the proof of part (1) of Lemma 4.1. □

Proof of Lemma 4.1 part (2). Let $f \in \mathcal{T}(\Omega; \mathcal{E}_{\rm nc})$ and suppose $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathbb{C}^{n \times m}$. We must produce a cp nc kernel Γ lying in the class $\mathcal{T}^1(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc}, \mathcal{L}(\mathcal{S})_{\rm nc})$ so that

$$f(Z) P f(W)^* = \Gamma(Z,W)\left(P \otimes I_{\mathcal{S}} - Q(Z)(P \otimes I_{\mathcal{R}}) Q(W)^*\right). \tag{4.7}$$

Toward this goal, we need to introduce some auxiliary operators and spaces. Let $\mathcal{R}^{\otimes 2} := \mathcal{R} \otimes \mathcal{R}$ be the Hilbert-space tensor product of $\mathcal{R}$ with itself and inductively set $\mathcal{R}^{\otimes k} = \mathcal{R}^{\otimes(k-1)} \otimes \mathcal{R}$. The *Fock space* associated with $\mathcal{R}$ is then defined to be the Hilbert-space orthogonal direct sum

$$\mathbb{F}(\mathcal{R}) = \bigoplus_{k=0}^{\infty} \mathcal{R}^{\otimes k}$$

where we let $\mathcal{R}^{\otimes 0}$ be the space of scalars $\mathbb{C}$. Fix a linear functional ℓ on $\mathcal{S}$ of unit norm. For $Z \in \Omega_n$ define $Q_0(Z) \in \mathcal{L}(\mathcal{R}^n, \mathbb{C}^n)$ by

$$Q_0(Z) = (\ell \otimes I_n) Q(Z). \tag{4.8}$$

One can check that Q_0 is a nc function, i.e., $Q_0 \in \mathcal{T}^1(\Omega; \mathcal{L}(\mathcal{R}, \mathbb{C})_{\rm nc})$. For $W \in \Omega_m$, the operator $Q_0(W)^* \in \mathcal{L}(\mathbb{C}^m, \mathcal{R}^m)$. We identify $\mathcal{R}^m$ with the tensor product space $\mathbb{C}^m \otimes \mathcal{R}$ and then define an operator $L_{Q_0(W)^*} \in \mathcal{L}(\mathcal{R}^m, \mathcal{R}^m \otimes \mathcal{R})$ on an elementary tensor $c \otimes r$ ($c \in \mathbb{C}^m$ and $r \in \mathcal{R}$) by

$$L_{Q_0(W)^*} \colon c \otimes r \mapsto Q_0(W)^* c \otimes r. \tag{4.9}$$

We note the identifications

$$\mathcal{R}^m \otimes \mathcal{R} \cong (\mathbb{C}^m \otimes \mathcal{R}) \otimes \mathcal{R} \cong \mathbb{C}^m \otimes (\mathcal{R} \otimes \mathcal{R}) \cong (\mathcal{R} \otimes \mathcal{R})^m.$$

Then $L_{Q_0(W)^*}$, being of the form $Q_0(W)^* \otimes I_{\mathcal{R}}$, extends to a bounded operator from $\mathbb{C}^m \otimes \mathcal{R} \cong \mathcal{R}^m$ into $(\mathcal{R} \otimes \mathcal{R})^m$ with $\|L_{Q_0(W)^*}\| = \|Q_0(W)^*\|$.

We next define the generalized power $(L_{Q_0(W)^*})^{(k)} \colon \mathbb{C}^m \to (\mathcal{R}^{\otimes(k+1)})^m$ by

$$(L_{Q_0(W)^*})^{(k)} = (Q_0(W)^* \otimes I_{\mathcal{R}^{\otimes(k-1)}}) \cdots (Q_0(W)^* \otimes I_{\mathcal{R}}) Q_0(W)^*.$$

Note that $\|(L_{Q_0(W)^*})^{(k)}\| \le \|Q_0(W)^*\|^k$ where $\|Q_0(W)^*\| < 1$ and hence

$$\sum_{k=0}^{\infty} \|(L_{Q_0(W)^*})^{(k)}\| < \infty. \tag{4.10}$$

Then the adjoint $(L_{Q_0(W)^*})^{(k)*}$ maps $(\mathcal{R}^{\otimes k})^m$ into $\mathbb{C}^m$.

For $Z \in \Omega_n$, we define $H(Z) \colon \mathbb{F}(\mathcal{R})^n \to \mathcal{E}^n$ by

$$H(Z) = f(Z)\, \mathrm{row_{k \ge 0}[(L_{Q_0(Z)^*})^{(k)*}]}$$

Note that $H(Z)$ is bounded as an operator from $\mathbb{F}(\mathcal{R})^n$ to $\mathcal{E}^n$ due to the validity of the summability condition (4.10). From the fact that both f and Q_0 are nc functions, one can check that H so defined is a nc function ($H \in \mathcal{T}(\Omega; \mathcal{L}(\mathbb{F}(\mathcal{R}), \mathcal{E})_{\rm nc})$). For $W \in \Omega_m$, then $H(W)^*$ is given by

$$H(W)^* = \mathrm{col}_{k \ge 0}[(L_{Q_0(W)^*})^{(k)}] f(W)^* \colon \mathcal{E}^m \to \mathbb{F}(\mathcal{R})^m.$$

Define a representation from $\mathcal{L}(\mathcal{S})$ to $\mathcal{L}(\mathbb{F}(\mathcal{R}))$ by

$$\pi(X) = (\ell X \ell^*) I_{\mathbb{F}(\mathcal{R})}.$$

Define Γ from $\Omega \times \Omega$ to $\mathcal{L}(\mathcal{L}(\mathcal{S}), \mathcal{L}(\mathcal{Y}))_{\rm nc}$ by

$$\Gamma(Z,W)(X) = H(Z)(\mathrm{id}_{\mathbb{C}^{n \times m}} \otimes \pi(X)) H(W)^*. \tag{4.11}$$

Then Γ is a cp nc kernel ($\Gamma \in \widetilde{\mathcal{T}}^1(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc}, \mathcal{L}(\mathcal{S})_{\rm nc})$ since (4.11) exhibits a Kolmogorov decomposition (2.18) for Γ. It remains to check that Γ provides an Agler decomposition (4.7) for $D_{f,f}(Z,W)(P) = f(Z)Pf(W)^*$. To this end we compute

$$\begin{aligned}
&\Gamma(Z,W)\,(P\otimes I_{\mathcal{S}} - Q(Z)(P\otimes I_{\mathcal{R}})Q(W)^*)\\
&= f(Z)\left(\sum_{k=0}^{\infty}\left(L_{Q_0(Z)^*}\right)^{(k)*}(P\otimes I_{\mathcal{R}^{\otimes k}})\left(L_{Q_0(W)^*}\right)^{(k)}\right)f(W)^*\\
&\quad - f(Z)\left(\sum_{k=0}^{\infty}\left(L_{Q_0(Z)^*}\right)^{(k)*}Q(Z)\,(P\otimes I_{\mathcal{R}^{\otimes k}})\,Q(W)^*\left(L_{Q_0(W)^*}\right)^{(k)}\right)f(W)^*\\
&= f(Z)\left(\sum_{k=0}^{\infty}\left(L_{Q_0(Z)^*}\right)^{(k)*}(P\otimes I_{\mathcal{R}^{\otimes k}})\left(L_{Q_0(W)^*}\right)^{(k)}\right)f(W)^*\\
&- f(Z)\left(\sum_{k=0}^{\infty}\left(L_{Q_0(Z)^*}\right)^{(k)*}(Q(Z)\otimes I_{\mathcal{R}^{\otimes k}})\cdot\right.\\
&\qquad\qquad \left.\cdot\,(P\otimes I_{\mathcal{R}^{\otimes k+1}})\,(Q(W)^*\otimes I_{\mathcal{R}^{\otimes k}})\left(L_{Q_0(W)^*}\right)^{(k)}\right)f(W)^*\\
&= f(Z)\left(\sum_{k=0}^{\infty}\left(L_{Q_0(Z)^*}\right)^{(k)*}(P\otimes I_{\mathcal{R}^{\otimes k}})\left(L_{Q_0(W)^*}\right)^{(k)}\right)f(W)^*\\
&\quad - f(Z)\left(\sum_{k=1}^{\infty}\left(L_{Q_0(Z)^*}\right)^{(k)*}(P\otimes I_{\mathcal{R}^{\otimes k}})\left(L_{Q_0(W)^*}\right)^{(k)}\right)f(W)^*\\
&\quad = f(Z)Pf(W)^*
\end{aligned}$$

as wanted. □

The cone $\mathcal{C}_\rho$ for $\rho_0 < \rho < 1$. We shall actually need the following adjustment of the cone $\mathcal{C}$. Let ρ_0 be defined as in (4.3) and let ρ be any positive real number with $\rho_0 < \rho < 1$.

Define a subset $\mathcal{C}_\rho$ of $\mathfrak{X}$ to consist of all kernels $K \in \mathfrak{X}$ such that there exists cp nc kernels

$$\begin{aligned}
&\Gamma_0 \in \widetilde{\mathcal{T}}^1(\Omega;\mathcal{L}(\mathcal{E})_{\rm nc},\mathcal{L}(\mathcal{S})_{\rm nc}), \quad \Gamma_1 \text{ and } \Gamma_2 \in \widetilde{\mathcal{T}}^1(\Omega;\mathcal{L}(\mathcal{E})_{\rm nc},\mathcal{L}(\mathcal{E})_{\rm nc}),\\
&\Gamma_3 \in \widetilde{\mathcal{T}}^1(\Omega;\mathcal{L}(\mathcal{E})_{\rm nc},\mathbb{C}_{\rm nc})
\end{aligned}$$

which induce a ρ-refined Agler decomposition for K:

$$\begin{aligned}
K(Z,W)(P) =&\Gamma_0(Z,W)\left(P\otimes I_{\mathcal{S}} - \frac{1}{\rho^2}Q(Z)(P\otimes I_{\mathcal{R}})Q(W)^*\right)\\
&+\Gamma_1(Z,W)\left(P\otimes I_{\mathcal{E}} - (1-\rho)^2 a(Z)(P\otimes I_{\mathcal{Y}})a(W)^*\right)\\
&+\Gamma_2(Z,W)\left(P\otimes I_{\mathcal{E}} - (1-\rho)^2 b(Z)(P\otimes I_{\mathcal{U}})b(W)^*\right)\\
&+\Gamma_3(Z,W)\left(P - (1-\rho)^2\chi(Z)(P\otimes I_d)\chi(W)^*\right) \qquad (4.12)
\end{aligned}$$

Salient properties of the subset $\mathcal{C}_\rho$ are summarized in the next lemma.

Lemma 4.2. *Let $\rho_0 < \rho < 1$ with ρ_0 as in* (4.3)*. Then the subset $\mathcal{C}_\rho$ of $\mathfrak{X}$ has the following properties:*

(1) *$\mathcal{C}_\rho$ is a closed cone in $\mathfrak{X}$.*
(2) *Suppose that $K \in \mathfrak{X}$ has the property that K is in $\mathcal{C}_\rho$ for all ρ sufficiently close to 1 with $\rho_0 < \rho < 1$. Then $K \in \mathcal{C}$.*
(3) *The positive kernels $D_{f,f}$ (4.2) are in $\mathcal{C}_\rho$ for all ρ with $\rho_0 < \rho < 1$.*

Proof of Lemma 4.2 part 1: That $\mathcal{C}_\rho$ is invariant under positive rescalings and taking of sums is elementary; we conclude that indeed $\mathcal{C}_\rho$ is a cone.

That $\mathcal{C}_\rho$ is closed in $\mathfrak{X}$ once $\rho < 1$ is chosen sufficiently close to 1 can be proved in much the same way used to show that $\mathcal{C}$ is closed (part (1) of Lemma 4.1). By choosing $\rho < 1$ sufficiently close to 1, we can guarantee that

$$\begin{aligned} & I_{\mathcal{S}^n} - \frac{1}{\rho^2} Q(Z)Q(Z)^* \succeq \epsilon_0^2 I_{\mathcal{S}^n}, \\ & I_{\mathcal{E}^n} - (1-\rho)^2 a(Z)a(Z)^* \succeq \epsilon_0^2 I_{\mathcal{E}^n}, \\ & I_{\mathcal{E}^n} - (1-\rho)^2 b(Z)b(Z)^* \succeq \epsilon_0^2 I_{\mathcal{E}^n}, \\ & I_n - (1-\rho)^2 \chi(Z)\chi(Z)^* \succeq \epsilon_0^2 I_n \end{aligned} \tag{4.13}$$

for all of the finitely many points $Z \in \Omega$ (where $n = n_Z$ is chosen so that $Z \in \Omega_n$). We then see that

$$\begin{aligned} K_N(Z,Z)(I_n) = {} & \Gamma_{N,0}(Z,Z)(I_{\mathcal{S}^n} - (1/\rho^2)Q(Z)Q(Z)^*) \\ & + \Gamma_{N,1}(Z,Z)(I_{\mathcal{E}^n} - (1-\rho)^2 a(Z)a(Z)^*) \\ & + \Gamma_{N,2}(Z,Z)(I_{\mathcal{E}^n} - (1-\rho)^2 b(Z)b(Z)^*) \\ & + \Gamma_{N,3}(Z,Z)(I_n - (1-\rho)^2 \chi(Z)\chi(Z)^*) \succeq \end{aligned} \tag{4.14}$$

$$\epsilon_0^2 \left(\Gamma_{N,0}(Z,Z)(I_{\mathcal{S}^n}) + \Gamma_{N,1}(Z,Z)(I_{\mathcal{E}^n}) + \Gamma_{N,2}(Z,Z)(I_{\mathcal{E}^n}) + \Gamma_{N,3}(Z,Z)(I_n)\right) \tag{4.15}$$

and hence each of the quantities

$$\begin{aligned} & \|\Gamma_{N,0}(Z,Z)(I_{\mathcal{S}^n})\|_{\mathcal{L}(\mathcal{E}^n)}, \quad \|\Gamma_{N,1}(Z,Z)(I_{\mathcal{E}^n})\|_{\mathcal{L}(\mathcal{E}^n)}, \\ & \|\Gamma_{N,2}(Z,Z)(I_{\mathcal{E}^n})\|_{\mathcal{L}(\mathcal{E}^n)}, \quad \|\Gamma_{N,3}(Z,Z)(I_n)\|_{\mathcal{L}(\mathcal{E}^n)} \end{aligned} \tag{4.16}$$

is bounded above by $\frac{1}{\epsilon_0^2}\|K_N(Z,Z)(I_n)\|_{\mathcal{L}(\mathcal{E})}$. As this last quantity is uniformly bounded with respect to N, each of the cp nc kernels in the list is uniformly bounded in the appropriate operator norm and hence each sequence $\{\Gamma_{N,k}\}$ has a weak-$*$ convergent subnet $\{\Gamma_{\alpha,k}\}$ converging to some cp nc kernel Γ_k ($k = 0,1,2,3$). The fact that the foursome $\{\Gamma_{N,k} \colon k = 0,1,2,3\}$ provides a ρ-refined Agler decomposition (4.12) for K_N implies that the limiting foursome $\{\Gamma_k \colon k = 0,1,2,3\}$ provides a ρ-refined Agler decomposition for the limiting kernel K now proceeds as in the proof of Lemma 4.1 part (1), and hence the limiting kernel K still has a ρ-refined Agler decomposition as wanted. □

Proof of Lemma 4.2 part 2: Suppose that $K \in \mathfrak{X}$ is in $\mathcal{C}_\rho$ for all $\rho < 1$ subject to $\rho_{00} < \rho < 1$ for some ρ_{00} with $\rho_0 \leq \rho_{00} < 1$. Hence for each such

ρ there are cp nc kernels $\Gamma_{\rho,0}$, $\Gamma_{\rho,1}$, $\Gamma_{\rho,2}$, $\Gamma_{\rho,3}$ so that K has a decomposition as in (4.12) (with $\Gamma_{\rho,k}$ in place of Γ_k for $k = 0, 1, 2, 3$). The estimate (4.15) is uniform with respect to ρ for $\rho_0 < \rho < 1$ since the estimates (4.13) are uniform in ρ as ρ approaches 1 once ρ is sufficiently close to 1. By again following the proof of Lemma 4.1 part (1), we can find a subnet ρ_α of an increasing sequence $\{\rho_N\}_{N\in\mathbb{N}}$ of positive numbers with limit equal to 1 so that each $\Gamma_{\rho_\alpha,k}(Z, W)$ converges in the appropriate operator BW-topology to an operator $\Gamma_k(Z, W)$ for each $Z \in \Omega_n$, $W \in \Omega_m$ for each $k = 0, 1, 2, 3$. Since the convergence is with respect to the pointwise weak-$*$ topology, one can check as in the proof of Lemma 4.1 part (1) that the fact that the kernel K has a ρ-refined Agler decomposition provided by the foursome $\{\Gamma_{\rho,0}, \Gamma_{\rho,1}, \Gamma_{\rho,2}, \Gamma_{\rho,3}\}$ implies, upon taking a limit as $\rho_\alpha \to 1$, that K also has the limiting representation

$$\begin{aligned} K(Z, W)(P) = {} & \Gamma_0(Z, W)(P \otimes I_{\mathcal{S}} - Q(Z)(P \otimes I_{\mathcal{R}})Q(W)^*) \\ & + \Gamma_1(Z, W)(P \otimes I_{\mathcal{E}}) + \Gamma_2(Z, W)(P \otimes I_{\mathcal{E}}) + \Gamma_3(Z, W)(P). \end{aligned}$$

The last three terms on the right side of the equality are all cp nc kernels and hence have a standard Agler decomposition by Lemma 4.1 part (3) while the first term is in the form of an Agler decomposition. Thus each term is in $\mathcal{C}$ and, as $\mathcal{C}$ is a cone, the sum is again in $\mathcal{C}$, so K has a standard Agler decomposition as claimed. This concludes the proof of part (2) of Lemma 4.2 follows. □

Proof of Lemma 4.2 part 3: Apply part (2) of Lemma 4.1 with $\frac{1}{\rho}Q$ in place of Q to see that there is a cp nc kernel Γ_0 so that

$$D_{f,f}(Z, W)(P) = \Gamma_0(Z, Z)\left(I_{\mathcal{S}} \otimes P - \frac{1}{\rho^2}Q(Z)(I_{\mathcal{R}} \otimes P)Q(W)^*\right).$$

Then $D_{f,f}$ has a decomposition of the form (4.12) (with Γ_1, Γ_2 and Γ_3 all taken equal to zero) as required. □

The cone separation argument. We now have all the preliminaries needed to complete the proof of $(1') \Rightarrow (2)$ in Theorem 3.1 for Case 1. In this part we are assuming that a, b are nc functions on $\Omega_{\rm nc,full} \cap \mathbb{D}_Q$. To show that a, b has an Agler decomposition (3.5) on Ω, it suffices to show that the kernel

$$K_{a,b}(Z, W)(P) := a(Z)(P \otimes I_{\mathcal{Y}})a(W)^* - b(Z)(P \otimes I_{\mathcal{U}})b(W)^* \tag{4.17}$$

is in the cone $\mathcal{C}_\rho$ for all ρ subject to $\rho_0 < \rho < 1$ for some $\rho_{00} \le \rho_0$. Since the cone $\mathcal{C}_\rho$ is closed (by Lemma 4.2 part (1)), by the contrapositive formulation of the Hahn-Banach separation theorem adapted to the case of point/cone pair (see [75, Theorem 3.4 part (b)]), to show that $K_{a,b}$ is in the cone $\mathcal{C}_\rho$ it suffices to show: *if $\mathbb{L}$ is any continuous linear functional on the normed linear space of kernels $\mathfrak{X}$ such that*

$$\operatorname{Re} \mathbb{L}(K) \ge 0 \textit{ for all } K \in \mathcal{C}_\rho, \tag{4.18}$$

then also

$$\operatorname{Re}\mathbb{L}(K_{a,b}) \geq 0. \tag{4.19}$$

In general for $K \in \mathfrak{X}$ we define $\widehat{K} \in \mathfrak{X}$ by

$$\widehat{K}(Z,W)(P) = (K(W,Z)(P^*))^*.$$

Let us say that the kernel $K \in \mathfrak{X}$ is **Hermitian** if it is the case that $K = \widehat{K}$. Given a continuous linear functional $\mathbb{L}$ on $\mathfrak{X}$, define $\mathbb{L}_1 \colon \mathfrak{X} \to \mathbb{C}$ by

$$\mathbb{L}_1(K) = \frac{1}{2}(\mathbb{L}(K) + \overline{\mathbb{L}(\widehat{K})}).$$

Note that if K is a Hermitian kernel, then

$$\mathbb{L}_1(K) = \operatorname{Re}\mathbb{L}(K).$$

In particular, if K is cp, then K is Hermitian and $\mathbb{L}_1(K) = \operatorname{Re}\mathbb{L}(K)$.

Now assume that $\mathbb{L}$ is chosen so that (4.18) holds. Let $\mathcal{H}_{\mathbb{L}}^{\circ}$ be the vector space $\mathcal{T}(\Omega;\mathcal{E}_{\rm nc})$ of all $\mathcal{E}$-valued nc functions on Ω. Introduce a sesquilinear form on $\mathcal{H}_{\mathbb{L}}^{\circ}$ by

$$\langle f,\, g\rangle_{\mathcal{H}_{\mathbb{L}}^{\circ}} = \mathbb{L}_1(D_{f,g}) \tag{4.20}$$

where $D_{f,g} \in \mathfrak{X}$ is given by

$$D_{f,g}(Z,W)(P) = f(Z)\, P\, g(W)^*$$

for $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathbb{C}^{n\times m}$. We have observed in part (3) of Lemma 4.2 that cp kernels $D_{f,f}$ are in $\mathcal{C}_\rho$ and hence $\operatorname{Re}\mathbb{L}(D_{f,f}) \geq 0$ by the construction (4.18). But for any f, $D_{f,f} = \widehat{D}_{f,f}$, so by the above remarks we have $0 \leq \operatorname{Re}\mathbb{L}(D_{f,f}) = \mathbb{L}_1(D_{f,f})$. We conclude that the inner product is positive semidefinite. By modding out by any functions having zero self inner product and considering equivalence classes, we get a finite dimensional Hilbert space which we denote by $\mathcal{H}_{\mathbb{L}}$.

For $f \in \mathcal{H}_{\mathbb{L}}^{\circ}$, we let $[f]$ be the equivalence class of f in $\mathcal{H}_{\mathbb{L}}$. Since Ω is finite and $\mathcal{E}$ is finite-dimensional, we know by Lemma 2.10 that $\mathcal{H}_{\mathbb{L}}^{\circ} = \mathcal{T}(\Omega;\mathcal{E}_{\rm nc})$ is finite-dimensional. Denote by $\mathcal{B} := \{f_1,\dots,f_K\}$ any basis for $\mathcal{H}_{\mathbb{L}}^{\circ}$. Then certainly the collection of equivalence classes $\{[f_1],\cdots,[f_K]\}$ is a spanning set for $\mathcal{H}_{\mathbb{L}}$. By standard Linear Algebra, we can choose a subset, denoted after possible reindexing as $\mathcal{B}' := \{[f_1],\dots,[f_{K'}]\}$ for some $K' \leq K$, as a basis for $\mathcal{H}_{\mathbb{L}}$. Suppose that $f = \sum_{k=1}^{K'} c_k f_k$ has zero self-inner product in $\mathcal{H}_{\mathbb{L}}^{\circ}$, so

$$\left[\sum_{k=1}^{K'} c_k f_k\right] = \sum_{k=1}^{K'} c_k [f_k] = 0.$$

Since $\mathcal{B}'$ is linearly independent in $\mathcal{H}_{\mathbb{L}}$, it follows that $c_k = 0$ for $1 \leq k \leq K'$, from which it follows that $f = \sum_{k=1}^{K'} c_k f_k = 0$ in $\mathcal{T}(\Omega;\mathcal{E}_{\rm nc})$. We conclude that the $\mathbb{L}$-inner product (4.20) is in fact positive definite when restricted to $\mathcal{H}_{\mathbb{L}}^{\circ\prime} := \operatorname{span}\{f_1,\dots,f_{K'}\}$. We therefore may view the space $\mathcal{H}_{\mathbb{L}}$ as the space of bona fide functions $\mathcal{H}_{\mathbb{L}} \cong \mathcal{H}_{\mathbb{L}}^{\circ\prime}$ with inner product given by (4.20) and with orthonormal basis given by

$$\mathcal{B}' = \{f_1, \dots, f_{K'}\}. \tag{4.21}$$

With this convention in force, it follows that the point-evaluation maps

$$\mathbf{ev}_Z \colon f \mapsto f(Z) \tag{4.22}$$

are well-defined as operators from $\mathcal{H}_{\mathbb{L}}$ into $\mathcal{E}^n$ for each $Z \in \Omega_n$. As $\mathcal{H}_{\mathbb{L}}$ is also finite-dimensional, it follows that each such map $\mathbf{ev}_Z$ is bounded as an operator from $\mathcal{H}_{\mathbb{L}}$ into $\mathcal{E}^n$ as well.

For $\mathcal{X}$ an arbitrary separable coefficient Hilbert space, define a space $\mathcal{H}_{\mathbb{L},\mathcal{X}}$ to be the space of nc functions $\mathcal{T}(\Omega; \mathcal{L}(\mathcal{X}, \mathcal{E}))$ with inner product given by

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{H}_{\mathbb{L},\mathcal{X}}} = \mathbb{L}_1(D_{\mathbf{f},\mathbf{g}}) \tag{4.23}$$

where $D_{\mathbf{f},\mathbf{g}}$ is the kernel in $\widetilde{\mathcal{T}}^1(\Omega; \mathcal{L}(\mathcal{E})_{\rm nc}, \mathbb{C}_{\rm nc})$ given by

$$D_{\mathbf{f},\mathbf{g}}(Z, W)(P) = \mathbf{f}(Z)(P \otimes I_{\mathcal{X}})\mathbf{g}(W)^*. \tag{4.24}$$

For $x^* \in \mathcal{X}^* = \mathcal{L}(\mathcal{X}, \mathbb{C})$ and $Z \in \Omega$, define a function $\mathbf{x}^* \in \mathcal{T}(\Omega; \mathcal{L}(\mathcal{X}, \mathbb{C})_{\rm nc})$ as in Example 2.3 part (b) by

$$\mathbf{x}^*(Z) = \mathrm{id}_{\mathbb{C}^n} \otimes x^* \text{ if } Z \in \Omega_n.$$

For $f \in \mathcal{T}(\Omega; \mathcal{L}(\mathbb{C}, \mathcal{E})_{\rm nc})$ and $x^* \in \mathcal{X}^*$, it then follows that $Z \mapsto \mathbf{f}(Z) := f(Z) \cdot \mathbf{x}^*(Z)$, as the pointwise composition of nc functions, is itself a nc function in $\mathcal{T}(\Omega; \mathcal{L}(\mathcal{X}, \mathcal{E})_{\rm nc})$; this is explained in [23, Section 4.1] in the context of Schur multipliers or can be easily verified directly. Furthermore, if we have two elements f, g of $\mathcal{H}_{\mathbb{L}} = \mathcal{T}(\Omega; \mathcal{L}(\mathbb{C}, \mathcal{E})_{\rm nc})$ and two elements x^*, y^* of $\mathcal{X}^*$ and set $\mathbf{f} = f \circ \mathbf{x}^*$ and $\mathbf{g} = g \circ \mathbf{y}^*$ as above, then $\mathbf{f}$ and $\mathbf{g}$ are in $\mathcal{H}_{\mathbb{L},\mathcal{X}}$ with inner product given by

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{H}_{\mathbb{L},\mathcal{X}}} = \mathbb{L}_1\left(D_{\mathbf{f},\mathbf{g}}\right)$$

where, for $Z \in \Omega_n$ and $W \in \Omega_m$, $D_{\mathbf{f},\mathbf{g}}$ is given by

$$\begin{aligned} D_{\mathbf{f},\mathbf{g}}(Z, W)(P) &= f(Z)\left(\mathrm{id}_{\mathbb{C}^n} \otimes x^*\right)\left(P \otimes I_{\mathcal{X}}\right)\left(\mathrm{id}_{\mathbb{C}^m} \otimes y\right) g(W)^* \\ &= f(Z)Pg(W)^* \otimes x^*y \\ &= (f(Z)Pg(W)^*) \cdot x^*y \text{ (since } x^*y \in \mathbb{C}). \end{aligned}$$

We conclude that

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{H}_{\mathbb{L},\mathcal{X}}} = \langle f, g \rangle_{\mathcal{H}_{\mathbb{L}}} \cdot \langle x^*, y^* \rangle_{\mathcal{X}^*}.$$

It follows that the map

$$\iota \colon f \otimes x^* \mapsto f \cdot \mathbf{x}^*$$

extends to an isometry from the Hilbert-space tensor product $\mathcal{H}_{\mathbb{L}} \otimes \mathcal{X}^*$ into $\mathcal{H}_{\mathbb{L},\mathcal{X}}$.

To see that this isometry is onto, we proceed as follows. Let $\{x_\beta \colon \beta \in \mathfrak{B}\}$ be an orthonormal basis for $\mathcal{X}$ (so $\mathfrak{B} = \{1, \dots, \dim \mathcal{X}\}$ in case $\mathcal{X}$ is finite-dimensional, and $\mathfrak{B} = \mathbb{N}$ (the natural numbers) otherwise), and let $\mathbf{f}$ be an arbitrary element of $\mathcal{H}_{\mathbb{L},\mathcal{X}}$. We let $\mathbf{x}_\beta$ be the nc function in $\mathcal{T}(\Omega, \mathcal{L}(\mathbb{C}, \mathcal{X})_{\rm nc})$ given by

$$\mathbf{x}_\beta(Z) = \mathrm{id}_{\mathbb{C}^n} \otimes x_\beta \text{ if } Z \in \Omega_n$$

(again as in Example 2.3 part (b)). Then the pointwise composition $f_\beta := \mathbf{f} \cdot \mathbf{x}_\beta$ is a nc function in $\mathcal{H}_{\mathbb{L}}$. If we apply the construction of the previous paragraph to the pair f_β and x_β, we get the elementary-tensor nc function

$$f_\beta \cdot \mathbf{x}_\beta^* = (\mathbf{f} \cdot \mathbf{x}_\beta) \cdot \mathbf{x}_\beta^*$$

in $\iota(\mathcal{H}_{\mathbb{L}} \otimes \mathcal{X}^*)$. We claim that

$$\mathbf{f} = \sum_{\beta \in \mathfrak{B}} (\mathbf{f} \cdot \mathbf{x}_\beta) \cdot \mathbf{x}_\beta^* \tag{4.25}$$

with convergence of the series in $\mathcal{H}_{\mathbb{L},\mathcal{X}}$-norm in case $\dim \mathcal{X} = \infty$. It then follows that the span of the images of elementary tensors $\iota(f \otimes \mathbf{x}^*)$ is dense in $\mathcal{H}_{\mathbb{L},\mathcal{X}}$ and hence that the map ι extends to a unitary identification of $\mathcal{H}_{\mathbb{L}} \otimes \mathcal{X}^*$ with $\mathcal{H}_{\mathbb{L},\mathcal{X}}$.

To verify the claim (4.25), note first that since $\{\mathbf{x}_\beta^* \colon \beta \in \mathfrak{B}\}$ is an orthonormal set in $\mathcal{X}^*$, it follows that, for any $f, g \in \mathcal{H}_{\mathbb{L}}$,

$$\langle f \cdot \mathbf{x}_\beta^*,\, g \cdot \mathbf{x}_{\beta'}^* \rangle_{\mathcal{H}_{\mathbb{L},\mathcal{X}}} = \langle f, g \rangle_{\mathcal{H}_{\mathbb{L}}} \cdot \delta_{\beta,\beta'}$$

(where $\delta_{\beta,\beta'}$ is the Kronecker delta). Hence

$$\langle (\mathbf{f} \cdot \mathbf{x}_\beta) \cdot \mathbf{x}_\beta^*,\, (\mathbf{f} \cdot \mathbf{x}_{\beta'}) \cdot \mathbf{x}_{\beta'}^* \rangle_{\mathcal{H}_{\mathbb{L},\mathcal{X}}} = \delta_{\beta,\beta'} \mathbb{L}_1 \left(D_{\mathbf{f}\cdot\mathbf{x}_\beta, \mathbf{f}\cdot\mathbf{x}_\beta} \right)$$

where

$$\begin{aligned} D_{\mathbf{f}\cdot\mathbf{x}_\beta,\, \mathbf{f}\cdot\mathbf{x}_\beta}(Z,W)(P) &= \mathbf{f}(Z) \left(P \otimes x_\beta x_\beta^* \right) f(W)^* \\ &= D_{\mathbf{f},\, \mathbf{f}\cdot\mathbf{x}_\beta\cdot\mathbf{x}_\beta^*}(Z,W)(P) \end{aligned}$$

and hence

$$\langle (\mathbf{f} \cdot \mathbf{x}_\beta) \cdot \mathbf{x}_\beta^*,\, (\mathbf{f} \cdot \mathbf{x}_{\beta'}) \cdot \mathbf{x}_{\beta'}^* \rangle_{\mathcal{H}_{\mathbb{L},\mathcal{X}}} = \delta_{\beta,\beta'} \langle \mathbf{f},\, (\mathbf{f} \cdot \mathbf{x}_\beta) \cdot \mathbf{x}_\beta^* \rangle_{\mathcal{H}_{\mathbb{L},\mathcal{X}}}.$$

Let $K = \dim \mathcal{X}$ if $\dim \mathcal{X}$ is finite and $K \in \mathbb{N}$ arbitrary otherwise. Then it follows that

$$\left\| \mathbf{f} - \sum_{\beta=1}^{K} \mathbf{f} \cdot \mathbf{x}_\beta \cdot \mathbf{x}_\beta^* \right\|_{\mathcal{H}_{\mathbb{L},\mathcal{X}}}^2 = \|f\|_{\mathcal{H}_{\mathbb{L},\mathcal{X}}}^2 - \left\langle \mathbf{f},\, \sum_{\beta=1}^{K} \mathbf{f} \cdot \mathbf{x}_\beta \cdot \mathbf{x}_\beta^* \right\rangle_{\mathcal{H}_{\mathbb{L},\mathcal{X}}}.$$

Note next that

$$D_{\mathbf{f},\, \sum_{\beta=1}^{K} \mathbf{f}\cdot\mathbf{x}_\beta\cdot\mathbf{x}_\beta^*}(Z,W)(P) = \mathbf{f}(Z) \left(P \otimes \sum_{\beta=1}^{K} x_\beta x_\beta^* \right) \mathbf{f}(W)^*.$$

In case $K = \dim \mathcal{X} < \infty$, we have $P \otimes \sum_{\beta=1}^{K} x_\beta x_\beta^* = P \otimes I_{\mathcal{X}}$ and we conclude that $D_{\mathbf{f},\, \sum_{\beta=1}^{K} \mathbf{f}\cdot\mathbf{x}_\beta\cdot\mathbf{x}_\beta^*} = D_{\mathbf{f},\mathbf{f}}$ from which the claim (4.25) follows as wanted.

In case $\dim \mathcal{X} = \infty$, we use that $\sum_{\beta=1}^{\infty} x_\beta x_\beta^*$ converges strongly in $\mathcal{L}(\mathcal{X})$ to $I_\mathcal{X}$ from which it follows that

$$\lim_{K\to\infty} D_{\mathbf{f}, \sum_{\beta=1}^{K} \mathbf{f}\cdot \mathbf{x}_\beta \cdot \mathbf{x}_\beta^*} = D_{\mathbf{f},\mathbf{f}}$$

with convergence in the BW-topology on $\mathfrak{X}$. Due to the continuity of $\mathbb{L}_1$ with respect to the BW-topology on $\mathfrak{X}$, it then follows that

$$\lim_{K\to\infty} \|\mathbf{f} - \sum_{\beta=1}^{K} (\mathbf{f}\cdot \mathbf{x}_\beta)\cdot \mathbf{x}_\beta^* \|^2_{\mathcal{H}_{\mathbb{L},\mathcal{X}}} = 0$$

and the claim (4.25) follows in this case as well. In the sequel we shall freely use the resulting identification ι between $\mathcal{H}_\mathbb{L} \otimes \mathcal{X}^*$ and $\mathcal{H}_{\mathbb{L},\mathcal{X}}$ without explicit mention of the map ι.

For $Z \in \Omega$, we let n_Z denote the natural number n so that $Z \in \Omega_n$. We next define a Hilbert space $\boldsymbol{\mathcal{H}}$ by

$$\boldsymbol{\mathcal{H}} = \bigoplus_{Z\in\Omega} \mathcal{L}(\mathbb{C}^{n_z}, \mathcal{E}^{n_z})$$

Here $\mathcal{L}(\mathbb{C}^{n_z}, \mathcal{E}^{n_z})$ is given the Hilbert-Schmidt operator norm and the direct sum is a Hilbert-space direct sum. More generally, let $\boldsymbol{\mathcal{H}}_\mathcal{X}$ be the Hilbert space

$$\boldsymbol{\mathcal{H}}_\mathcal{X} = \bigoplus_{Z\in\Omega} \mathcal{L}(\mathcal{X}^{n_z}, \mathcal{E}^{n_z})$$

As $\mathcal{E}$ is finite-dimensional, any operator in $\mathcal{L}(\mathcal{X}^{n_z}, \mathcal{E}^{n_z})$ has finite Hilbert-Schmidt norm and we can again view $\boldsymbol{\mathcal{H}}_\mathcal{X}$ as a Hilbert space. The relation between $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{H}}_\mathcal{X}$ is analogous to that derived above between $\mathcal{H}_\mathbb{L}$ and $\mathcal{H}_{\mathbb{L},\mathcal{X}}$. Specifically, if $h = \bigoplus_{Z\in\Omega} h_Z \in \boldsymbol{\mathcal{H}}$ and $x^* \in \mathcal{X}^*$, then

$$h \cdot \mathbf{x}^* := \bigoplus_{Z\in\Omega} h_Z(\mathrm{id}_{\mathbb{C}^{n_z}} \otimes x^*) \in \boldsymbol{\mathcal{H}}_\mathcal{X}.$$

Moreover, given two pairs $h, k \in \boldsymbol{\mathcal{H}}$ and $x^*, y^* \in \mathcal{X}^*$,

$$\begin{aligned}
\langle h\cdot \mathbf{x}^*, k\cdot \mathbf{y}^*\rangle_{\boldsymbol{\mathcal{H}}_\mathcal{X}} &= \sum_{Z\in\Omega} \mathrm{tr}\left((\mathrm{id}_{\mathbb{C}^{n_z}} \otimes y) k_Z^* h_Z (\mathrm{id}_{\mathbb{C}^{n_z}} \otimes x^*)\right) \\
&= \sum_{Z\in\Omega} \mathrm{tr}\left(k_Z^* h_Z (\mathrm{id}_{\mathbb{C}^{n_z}} \otimes x^* y)\right) \\
&= \left(\sum_{Z\in\Omega} \mathrm{tr}(k_Z^* h_Z)\right) \cdot x^* y \text{ (since } x^* y \in \mathbb{C}) \\
&= \langle h,\, k\rangle_{\boldsymbol{\mathcal{H}}} \cdot \langle x^*, y^*\rangle_{\mathcal{X}^*}.
\end{aligned}$$

Thus the map $\boldsymbol{\iota}$ defined on elementary tensors by

$$\boldsymbol{\iota}\colon h \otimes x^* \mapsto h \cdot \mathbf{x}^*$$

extends to an isometry from the Hilbert-space tensor product $\boldsymbol{\mathcal{H}} \otimes \mathcal{X}^*$ into $\boldsymbol{\mathcal{H}}_\mathcal{X}$. To see that $\boldsymbol{\iota}$ is onto, we again work with an orthonormal basis

$\{x_\beta \colon \beta \in \mathfrak{B}\}$ for $\mathcal{X}$. Given any $\mathbf{h} = \bigoplus_{Z \in \Omega} \mathbf{h}_Z \in \boldsymbol{\mathcal{H}}_{\mathcal{X}}$, then $h_\beta = \mathbf{h} \cdot \mathbf{x}_\beta := \bigoplus_{Z \in \Omega} \mathbf{h}(Z) \, (\mathrm{id}_{\mathbb{C}^{n_z}} \otimes x_\beta)$ is in $\boldsymbol{\mathcal{H}}$. We claim that we can recover $\mathbf{h}$ as

$$\mathbf{h} = \sum_{\beta \in \mathfrak{B}} h_\beta \cdot \mathbf{x}_\beta^* = \sum_{\beta \in \mathfrak{B}} (\mathbf{h} \cdot \mathbf{x}_\beta) \cdot \mathbf{x}_\beta^* \tag{4.26}$$

with convergence of the series in the norm of $\boldsymbol{\mathcal{H}}_{\mathcal{X}}$ in the case that $\dim \mathcal{X} = \infty$. As a consequence it then follows that the span of images of elementary tensors $\boldsymbol{\iota}(h \otimes x^*)$ ($h \in \boldsymbol{\mathcal{H}}$ and $x^* \in \mathcal{X}^*$) is dense in $\boldsymbol{\mathcal{H}}_{\mathcal{X}}$ and hence $\boldsymbol{\iota}$ is onto. The claim (4.26) in turn can be seen as an immediate consequence of the identity

$$\sum_{\beta \in \mathfrak{B}} x_\beta x_\beta^* = I_{\mathcal{X}}$$

(with convergence n the strong topology of $\mathcal{L}(\mathcal{X})$ in case $\dim \mathcal{X} = \infty$). We hence identify $\boldsymbol{\mathcal{H}}_{\mathcal{X}}$ with the Hilbert-space tensor product $\boldsymbol{\mathcal{H}} \otimes \mathcal{X}^*$ without explicit mention of the identification map $\boldsymbol{\iota}$.

We shall also find it convenient to have a version of these spaces where the target space in the Z-slice is simply $\mathbb{C}$ rather than $\mathcal{E}^{n_z}$. Toward this end, note that we can choose an orthonormal basis $\{e_1, \dots, e_{\dim \mathcal{E}}\}$ to represent any operator $T \in \mathcal{L}(\mathbb{C}^{n_z}, \mathcal{E}^{n_z} \cong \mathbb{C}^{n_z \cdot (\dim \mathcal{E})})$ as an $k_Z \times n_Z$ matrix over $\mathbb{C}$, where we set

$$k_Z = (\dim \mathcal{E}) \cdot n_Z \tag{4.27}$$

(with $k_Z \cdot n_Z$ entries) while an operator $\widetilde{T} \in \mathcal{L}(\mathbb{C}^{k_z \cdot n_z}, \mathbb{C})$ is represented by a $1 \times k_Z \cdot n_Z$ matrix over $\mathcal{E}$ (again with $k_Z \cdot n_Z$ entries). In general, to map $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^k) \cong \mathbb{C}^{k \times n}$ bijectively to $\mathcal{L}(\mathbb{C}^{k \cdot n}, \mathbb{C}) \cong \mathbb{C}^{1 \times k \cdot n}$, we introduce the operator $\textbf{row-vec}_k$ (**row-vec** for *row-vectorization*) which reorganizes a $k \times n$ matrix into a $1 \times (k \cdot n)$ matrix as follows: if $T = [t_{ij}] = \begin{bmatrix} T_1 \\ \vdots \\ T_k \end{bmatrix}$ where $T_i = \begin{bmatrix} t_{i,1} & \cdots & t_{i,n} \end{bmatrix}$ is the i-th row of T for $1 \le i \le k$, we define

$$\textbf{row-vec}_k \colon \mathcal{L}(\mathbb{C}^n, \mathbb{C}^k) \cong \mathbb{C}^{k \times n} \to \mathcal{L}(\mathbb{C}^{k \cdot n}, \mathbb{C}) \cong \mathbb{C}^{1 \times k \cdot n}$$

by

$$\textbf{row-vec}_k \colon T = \begin{bmatrix} T_1 \\ \vdots \\ T_k \end{bmatrix} \mapsto \begin{bmatrix} T_1 & \cdots & T_k \end{bmatrix}.$$

We now introduce yet another Hilbert space $\widetilde{\boldsymbol{\mathcal{H}}}$ by

$$\widetilde{\boldsymbol{\mathcal{H}}} = \bigoplus_{Z \in \Omega} \mathcal{L}(\mathbb{C}^{k_z \cdot n_z}, \mathbb{C}) \cong \bigoplus_{Z \in \Omega} \mathbb{C}^{1 \times k_z \cdot n_z} \tag{4.28}$$

(k_Z as in (4.27)) and define an identification map $\mathbf{i} \colon \boldsymbol{\mathcal{H}} \to \widetilde{\boldsymbol{\mathcal{H}}}$ by

$$\mathbf{i} = [\mathrm{row}_{Z \in \Omega} \, \textbf{row-vec}_{k_z}] \colon \bigoplus_{Z \in \Omega} h_Z \mapsto \mathrm{row}_{Z \in \Omega} [\textbf{row-vec}_{k_z}(h_Z)].$$

For $\mathcal{X}$ a coefficient Hilbert space as above, we also introduce the companion Hilbert space

$$\widetilde{\boldsymbol{\mathcal{H}}}_{\mathcal{X}} = \bigoplus_{Z\in\Omega} \mathcal{L}(\mathcal{X}^{k_z \cdot n_z}, \mathbb{C}) \cong \widetilde{\mathcal{H}} \otimes \mathcal{X}^* \tag{4.29}$$

and extend the identification map $\mathbf{i}\colon \boldsymbol{\mathcal{H}} \to \widetilde{\boldsymbol{\mathcal{H}}}$ to an identification map between $\boldsymbol{\mathcal{H}}_{\mathcal{X}}$ and $\widetilde{\boldsymbol{\mathcal{H}}}_{\mathcal{X}}$ by

$$\mathbf{i}_{\mathcal{X}} = \mathbf{i} \otimes I_{\mathcal{X}^*} \colon \boldsymbol{\mathcal{H}}_{\mathcal{X}} \cong \boldsymbol{\mathcal{H}} \otimes \mathcal{X}^* \to \widetilde{\boldsymbol{\mathcal{H}}}_{\mathcal{X}} \cong \widetilde{\boldsymbol{\mathcal{H}}} \otimes \mathcal{X}^*.$$

Explicitly, the operator $\mathbf{i}_{\mathcal{X}}$ is constructed as follows. We use an orthonormal basis for $\mathcal{E}$ to view an element $T \in \mathcal{L}(\mathcal{X}^{n_z}, \mathcal{E}^{n_z})$ as an operator in $\mathcal{L}(\mathcal{X}^{n_z}, \mathbb{C}^{k_z}) \cong (\mathcal{L}(X, \mathbb{C})^{k_z \times n_z})$. We then apply the operator $\textbf{row-vec}_{k_z}$ to this $k_Z \times n_Z$ matrix over $\mathcal{L}(\mathcal{X}, \mathbb{C})$ to get a $1 \times (k_Z \cdot n_Z)$ matrix over $\mathcal{L}(\mathcal{X}, \mathbb{C})$. This last matrix in turn can be interpreted as an operator in $\mathcal{L}(\mathcal{X}^{k_z \times n_z}, \mathbb{C})$. Concretely we view the direct sum in (4.28) and in (4.29) as a row direct sum; then we view elements of $\widetilde{\boldsymbol{\mathcal{H}}}$ and of $\widetilde{\boldsymbol{\mathcal{H}}}_{\mathcal{X}}$ as long row vectors:

$$\widetilde{\boldsymbol{\mathcal{H}}} = \mathbb{C}^{1\times N}, \quad \widetilde{\boldsymbol{\mathcal{H}}}_{\mathcal{X}} = (\mathcal{X}^*)^{1\times N} \text{ where } N = \sum_{Z\in\Omega} k_Z \cdot n_Z. \tag{4.30}$$

We next introduce operators $\mathcal{I}\colon \mathcal{H}_{\mathbb{L}} \to \boldsymbol{\mathcal{H}}$ and more generally $\mathcal{I}_{\mathcal{X}}\colon \mathcal{H}_{\mathbb{L},\mathcal{X}} \to \boldsymbol{\mathcal{H}}_{\mathcal{X}}$ by

$$\mathcal{I}\colon f \mapsto \bigoplus_{Z\in\Omega} f(Z) \text{ for } f \in \mathcal{H}_{\mathbb{L}} \tag{4.31}$$

$$\mathcal{I}_{\mathcal{X}}\colon \mathbf{f} \mapsto \bigoplus_{Z\in\Omega} \mathbf{f}(Z) \text{ for } \mathbf{f} \in \mathcal{H}_{\mathbb{L},\mathcal{X}}. \tag{4.32}$$

We first note that both $\mathcal{I}$ *and* $\mathcal{I}_{\mathcal{X}}$ *are injective*; indeed if an element $f \in \mathcal{H}_{\mathbb{L}}$ has the property that $f(Z) = 0$ for all $Z \in \Omega$, then necessarily f is zero as an element of $\mathcal{H}_{\mathbb{L}}$, and a similar statement applying to an element $\mathbf{f}$ of $\mathcal{H}_{\mathbb{L},\mathcal{X}}$. Note next that when $\mathbf{f} \in \mathcal{H}_{\mathbb{L},\mathcal{X}}$ has the form of an elementary tensor $\mathbf{f} = f \cdot \mathbf{x}^* = \iota(f \otimes x^*)$ for an $f \in \mathcal{H}_{\mathbb{L}}$ and $x^* \in \mathcal{X}^*$, then

$$\mathcal{I}_{\mathcal{X}}\colon \iota(f \otimes x^*) \mapsto \boldsymbol{\iota}(\mathcal{I}f \otimes x^*).$$

We conclude that when ι and $\boldsymbol{\iota}$ are used to identify $\mathcal{H}_{\mathbb{L},\mathcal{X}}$ and $\boldsymbol{\mathcal{H}}_{\mathcal{X}}$ with the respective tensor-product spaces $\mathcal{H}_{\mathbb{L}} \otimes \mathcal{X}^*$ and $\boldsymbol{\mathcal{H}} \otimes \mathcal{X}^*$, then the map $\mathcal{I}_X$ assumes the operator elementary-tensor form

$$\mathcal{I}_{\mathcal{X}} = \mathcal{I} \otimes I_{\mathcal{X}^*}. \tag{4.33}$$

It is natural then to also define $\widetilde{\mathcal{I}}\colon \mathcal{H}_{\mathbb{L}} \to \widetilde{\boldsymbol{\mathcal{H}}}$ and $\widetilde{\mathcal{I}}_{\mathcal{X}}\colon \mathcal{H}_{\mathbb{L},\mathcal{X}} \to \widetilde{\boldsymbol{\mathcal{H}}}_{\mathcal{X}}$ by

$$\widetilde{\mathcal{I}} = \mathbf{i} \circ \mathcal{I}\colon f \mapsto \text{row}_{Z\in\Omega}[\textbf{row-vec}_{k_z}(f(Z))],$$

$$\widetilde{\mathcal{I}}_{\mathcal{X}} = \widetilde{\mathcal{I}} \otimes I_{\mathcal{X}^*}\colon \mathbf{f} \mapsto \text{row}_{Z\in\Omega}[\textbf{row-vec}_{k_z}(\mathbf{f}(Z))].$$

We shall be interested in all these constructions for the particular cases where $\mathcal{X} = \mathcal{R}$, $\mathcal{X} = \mathcal{S}$, $\mathcal{X} = \mathcal{U}$, and $\mathcal{X} = \mathcal{Y}$. Note the the case $\mathcal{X} = \mathbb{C}$ has already appeared explicitly: $\mathcal{H}_{\mathbb{L},\mathbb{C}} = \mathcal{H}_{\mathbb{L}}$, $\boldsymbol{\mathcal{H}}_{\mathbb{C}} = \boldsymbol{\mathcal{H}}$, $\widetilde{\boldsymbol{\mathcal{H}}}_{\mathbb{C}} = \widetilde{\boldsymbol{\mathcal{H}}}$, $\mathcal{I}_{\mathbb{C}} = \mathcal{I}$, $\widetilde{\mathcal{I}}_{\mathbb{C}} = \widetilde{\mathcal{I}}$, etc.

We define right multiplication operators

$$\begin{aligned} &M_Q^r \colon \mathcal{H}_{\mathbb{L},\mathcal{S}} \to \mathcal{H}_{\mathbb{L},\mathcal{R}}, \quad M_a^r \colon \mathcal{H}_{\mathbb{L},\mathcal{E}} \to \mathcal{H}_{\mathbb{L},\mathcal{Y}}, \\ &M_b^r \colon \mathcal{H}_{\mathbb{L},\mathcal{E}} \to \mathcal{H}_{\mathbb{L},\mathcal{U}}, \quad M_\chi^r \colon \mathcal{H}_{\mathbb{L}} \to \operatorname{row}_{1\le i\le d}[\mathcal{H}_{\mathbb{L}}] \end{aligned} \tag{4.34}$$

by

$$\begin{aligned} &M_Q^r \colon \mathbf{f}(Z) \mapsto \mathbf{f}(Z)Q(Z), \quad M_a^r \colon \mathbf{f}(Z) \mapsto \mathbf{f}(Z)a(Z), \\ &M_b^r \colon \mathbf{f}(Z) \mapsto \mathbf{f}(Z)b(Z), \quad M_\chi^r \colon f(Z) \mapsto f(Z)\chi(Z). \end{aligned}$$

From the defining form (4.12) for a kernel to be in $\mathcal{C}_\rho$ and the fact that $\operatorname{Re}\mathbb{L}$ is nonnegative on $\mathcal{C}_\rho$, we read off that

$$\begin{aligned} &\|M_Q^r\mathbf{f}\|^2_{\mathcal{H}_{\mathbb{L},\mathcal{R}}} \le \rho^2\|\mathbf{f}\|^2_{\mathcal{H}_{\mathbb{L},\mathcal{S}}}, \quad \|M_a^r\mathbf{f}\|^2_{\mathcal{H}_{\mathbb{L},\mathcal{E}}} \le \frac{1}{(1-\rho)^2}\|\mathbf{f}\|^2_{\mathcal{H}_{\mathbb{L},\mathcal{Y}}}, \\ &\|M_b^r\mathbf{f}\|^2_{\mathcal{H}_{\mathbb{L},\mathcal{E}}} \le \frac{1}{(1-\rho)^2}\|\mathbf{f}\|^2_{\mathcal{H}_{\mathbb{L},\mathcal{U}}}, \quad \|M_\chi^r f\|^2_{\operatorname{row}_{1\le i\le d}\mathcal{H}_{\mathbb{L}}} \le \frac{1}{(1-\rho)^2}\|f\|^2_{\mathcal{H}_{\mathbb{L}}} \end{aligned} \tag{4.35}$$

for all $\mathbf{f}$ or f in the space $\mathcal{H}_{\mathbb{L},\mathcal{X}}$ for the appropriate space $\mathcal{X}$, and hence each of the operators M_Q^r, M_a^r, M_b^r, M_χ^r is well-defined and bounded with $\|M_Q^r\| \le \rho < 1$ and $\|M_a^r\|$, $\|M_b^r\|$, $\|M_\chi^r\|$ all at most $\frac{1}{1-\rho} < \infty$ for $0 < \rho < 1$ with ρ sufficiently close to 1. Then we have the following intertwining relations

$$\begin{aligned} &\mathcal{I}_\mathcal{R} M_Q^r = \mathbf{Q}^r \mathcal{I}_\mathcal{S}, \\ &\mathcal{I}_\mathcal{Y} M_a^r = \mathbf{a}^r \mathcal{I}_\mathcal{E}, \\ &\mathcal{I}_\mathcal{U} M_b^r = \mathbf{b}^r \mathcal{I}_\mathcal{E}, \\ &\mathcal{I} M_{\chi_k}^r = \boldsymbol{\chi}_k^r \mathcal{I} \text{ for } k = 1, \dots, d \end{aligned} \tag{4.36}$$

where the operators $\mathbf{Q}^r$, $\mathbf{a}^r$, $\mathbf{b}^r$, $\boldsymbol{\chi}^r$ are given by

$$\begin{aligned} &\mathbf{Q}^r = \operatorname{diag}_{Z\in\Omega}[Q(Z)] \colon \bigoplus_{Z\in\Omega} \mathbf{h}_Z \mapsto \bigoplus_{Z\in\Omega} \mathbf{h}_Z Q(Z), \\ &\mathbf{a}^r = \operatorname{diag}_{Z\in\Omega}[a(Z)^r] \colon \bigoplus_{Z\in\Omega} \mathbf{h}_Z \mapsto \bigoplus_{Z\in\Omega} \mathbf{h}_Z a(Z), \\ &\mathbf{b}^r = \operatorname{diag}_{Z\in\Omega}[b(Z)^r] \colon \bigoplus_{Z\in\Omega} \mathbf{h}_Z \mapsto \bigoplus_{Z\in\Omega} \mathbf{h}_Z b(Z), \\ &\boldsymbol{\chi}_k^r = \operatorname{diag}_{Z\in\Omega}[\chi_k(Z)^r] \colon \bigoplus_{Z\in\Omega} h_Z \mapsto \bigoplus_{Z\in\Omega} h_Z Z_k \text{ for } k = 1, \dots, d. \end{aligned}$$

From the right-multiplier form of all these operators, one can deduce the next layer of intertwining relations:

$$\begin{aligned} \widetilde{\mathcal{I}}_{\mathcal{R}} M_Q^r &= (\operatorname{diag} \mathbf{Q})^r \widetilde{\mathcal{I}}_{\mathcal{S}}, \\ \widetilde{\mathcal{I}}_{\mathcal{Y}} M_a^r &= (\operatorname{diag} \mathbf{a})^r \widetilde{\mathcal{I}}_{\mathcal{E}}, \\ \widetilde{\mathcal{I}}_{\mathcal{U}} M_b^r &= (\operatorname{diag} \mathbf{b})^r \widetilde{\mathcal{I}}_{\mathcal{E}}, \\ \widetilde{\mathcal{I}}_{\mathcal{Y}} M_{\chi_k}^r &= (\operatorname{diag} \boldsymbol{\chi}_k)^r \widetilde{\mathcal{I}}. \end{aligned} \tag{4.37}$$

Here we view elements of $\widetilde{\boldsymbol{\mathcal{H}}}_{\mathcal{X}}$ as row matrix of length N over $\mathcal{X}^*$ as in (4.30) of the form $\widetilde{\mathbf{h}} = \operatorname{row}_{Z\in\Omega}[\mathbf{h}(Z)]$ where $\mathbf{h}(Z) \in (\mathcal{X}^*)^{1\times k_z}$ for $\mathcal{X}$ equal to any of $\mathcal{S}, \mathcal{R}, \mathcal{E}, \mathcal{U}, \mathcal{Y}, \mathbb{C}$ and then the operators $(\operatorname{diag} \mathbf{Q})^r$, $(\operatorname{diag} \mathbf{a})^r$, $(\operatorname{diag} \mathbf{b})^r$, $(\operatorname{diag} \boldsymbol{\chi}_k)^r$ are given by

$$\begin{aligned} &(\operatorname{diag} \mathbf{Q})^r \colon \operatorname{row}_{Z\in\Omega}[\widetilde{\mathbf{h}}(Z)] \mapsto \operatorname{row}_{Z\in\Omega}[\widetilde{\mathbf{h}}(Z) \cdot \operatorname{diag}_{1\le i\le k_z} Q(Z)], \\ &(\operatorname{diag} \mathbf{a})^r \colon \operatorname{row}_{Z\in\Omega}[\widetilde{\mathbf{h}}(Z)] \mapsto \operatorname{row}_{Z\in\Omega}[\widetilde{\mathbf{h}}(Z) \cdot \operatorname{diag}_{1\le i\le k_z} a(Z)], \\ &(\operatorname{diag} \mathbf{b})^r \colon \operatorname{row}_{Z\in\Omega}[\widetilde{\mathbf{h}}(Z)] \mapsto \operatorname{row}_{Z\in\Omega}[\widetilde{\mathbf{h}}(Z) \cdot \operatorname{diag}_{1\le i\le k_z} b(Z)], \\ &(\operatorname{diag} \boldsymbol{\chi}_k)^r \colon \operatorname{row}_{Z\in\Omega}[\widetilde{h}(Z)] \mapsto \operatorname{row}_{Z\in\Omega}[\widetilde{h}(Z) \cdot \operatorname{diag}_{1\le i\le k_z} \chi_k(Z)]. \end{aligned} \tag{4.38}$$

To get a matrix representation for $\widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{I}}_{\mathcal{X}}$, we wish to also represent elements of $\mathcal{H}_{\mathbb{L}}$ and $\mathcal{H}_{\mathbb{L},\mathcal{X}}$ as a space of row vectors. Toward this end, we use the orthonormal basis $\mathcal{B}' = \{f_1, \dots, f_{K'}\}$ (4.21) for $\mathcal{H}_{\mathbb{L}}$ to identify $\mathcal{H}_{\mathbb{L}}$ with the space of row vectors

$$\mathcal{H}_{\mathbb{L}} \cong \mathbb{C}^{1\times K'}.$$

Then the space $\mathcal{H}_{\mathbb{L},\mathcal{X}} \cong \mathcal{H}_{\mathbb{L}} \otimes \mathcal{X}^*$ can be identified with row vectors over $\mathcal{X}^*$:

$$\mathcal{H}_{\mathbb{L},\mathcal{X}} \cong (\mathcal{X}^*)^{1\times K'}.$$

Let us write $\vec{f}$ and $\vec{\mathbf{f}}$ for elements of $\mathcal{H}_{\mathbb{L}}$ and $\mathcal{H}_{\mathbb{L},\mathcal{X}}$ when viewed as a row matrix in $\mathbb{C}^{1\times K'}$ (respectively, $(\mathcal{X}^*)^{1\times K'}$). With these identifications, the operator $\mathcal{I} \in \mathcal{L}(\mathcal{H}_{\mathbb{L}}, \widetilde{\boldsymbol{\mathcal{H}}})$ is represented as right multiplication by a matrix $[\widetilde{\mathcal{I}}] \in \mathbb{C}^{K'\times N}$:

$$[\widetilde{\mathcal{I}}]^r \colon \vec{f} \mapsto \vec{f} \cdot [\widetilde{\mathcal{I}}].$$

Then $\widetilde{\mathcal{I}} \colon \mathcal{H}_{\mathbb{L}} \to \widetilde{\boldsymbol{\mathcal{H}}}$ has a matrix representation $[\widetilde{\mathcal{I}}] \in \mathbb{C}^{K'\times N}$ induced by these respective bases. Moreover the operator $M_{\chi_k}^r \in \mathcal{L}(\mathcal{H}_{\mathbb{L}})$ has a matrix representation $[M_{\chi_k}^r] \in \mathbb{C}^{K'\times K'}$ induced by the given basis for $\mathcal{H}_{\mathbb{L}}$ and similarly the operator $(\operatorname{diag} \boldsymbol{\chi}_k)^r$ has a matrix representation $[\operatorname{diag} \boldsymbol{\chi}_k]$ of the form $\operatorname{diag}_{Z\in\Omega}[\operatorname{diag}_{1\le i\le k_z}[\boldsymbol{\chi}_k]]$ induced by the given basis for $\widetilde{\boldsymbol{\mathcal{H}}}$. The last of the intertwining relations (4.37) implies the matricial intertwining relation

$$[\widetilde{\mathcal{I}}]^r [M_{\chi_k}^r]^r = [\operatorname{diag} \boldsymbol{\chi}_k]^r [\widetilde{\mathcal{I}}]^r, \tag{4.39}$$

or, directly in terms of the matrices

$$[M_{\chi_k}^r][\widetilde{\mathcal{I}}] = [\widetilde{\mathcal{I}}][\operatorname{diag} \boldsymbol{\chi}_k].$$

A consequence of the operator elementary-tensor form of $\widetilde{\mathcal{I}}_{\mathcal{X}}$ (4.33) is that the matrix representation for $\widetilde{\mathcal{I}}_{\mathcal{X}}$ is given by the same matrix $[\mathcal{I}_{\mathcal{X}}]$ (here we use that $\mathcal{X}^*$ as a vector space is a $\mathbb{C}$-module so multiplication of a matrix over $\mathcal{X}^*$ by a matrix over $\mathbb{C}$ makes sense as long as the sizes fit):

$$[\widetilde{\mathcal{I}}_{\mathcal{X}}]^r \colon \vec{\mathbf{f}} \mapsto \vec{\mathbf{f}} \cdot [\widetilde{\mathcal{I}}].$$

When we identify $\mathcal{H}_{\mathbb{L},\mathcal{X}}$ with $(\mathcal{X}^*)^{1\times K'}$, the operators M_Q^r, M_a^r, M_b^r, $M_{\chi_k}^r$ in (4.34) then have representations as multiplication on the right by appropriate matrices

$$\begin{aligned} &[M_Q^r] \in \mathcal{L}(\mathcal{R},\mathcal{S})^{K'\times K'}, \quad [M_a^r] \in \mathcal{L}(\mathcal{Y},\mathcal{E})^{K'\times K'}, \\ &[M_b^r] \in \mathcal{L}(\mathcal{U},\mathcal{E})^{K'\times K'}, \quad [M_{\chi_k}^r] \in \mathbb{C}^{K'\times K'}. \end{aligned}$$

Note that the spaces $\widetilde{\boldsymbol{\mathcal{H}}}_{\mathcal{X}}$ are already represented as spaces of row vectors, namely $(\mathcal{X}^*)^{1\times N}$; the operators on these spaces are represented as right multiplication by matrices as in (4.38). Then the intertwining relations (4.37) assume the following matricial form:

$$\begin{aligned} &[\widetilde{\mathcal{I}}]^r[M_Q^r]^r = [\operatorname{diag} \mathbf{Q}]^r[\widetilde{\mathcal{I}}]^r, \\ &[\widetilde{\mathcal{I}}]^r[M_a^r]^r = [\operatorname{diag} \mathbf{a}]^r[\widetilde{\mathcal{I}}]^r, \\ &[\widetilde{\mathcal{I}}]^r[M_b^r]^r = [\operatorname{diag} \mathbf{b}]^r[\widetilde{\mathcal{I}}]^r, \\ &[\widetilde{\mathcal{I}}]^r[M_{\chi_k}^r]^r = [\operatorname{diag} \boldsymbol{\chi}_k]^r[\widetilde{\mathcal{I}}]^r, \end{aligned} \tag{4.40}$$

or, in terms of the matrices themselves,

$$\begin{aligned} &[M_Q^r][\widetilde{\mathcal{I}}] = [\widetilde{\mathcal{I}}][\operatorname{diag} \mathbf{Q}], \\ &[M_a^r][\widetilde{\mathcal{I}}] = [\widetilde{\mathcal{I}}][\operatorname{diag} \mathbf{a}], \\ &[M_b^r][\widetilde{\mathcal{I}}] = [\widetilde{\mathcal{I}}][\operatorname{diag} \mathbf{b}], \\ &[M_{\chi_k}^r][\widetilde{\mathcal{I}}] = [\widetilde{\mathcal{I}}][\operatorname{diag} \boldsymbol{\chi}_k]. \end{aligned}$$

Let us now define a point $\mathbf{Z}$ in the nc envelope $(\Omega)_{\rm nc}$ of Ω by

$$\mathbf{Z} = \operatorname{diag}_{Z\in\Omega}[\operatorname{diag}_{1,\dots,k_Z}[Z]].$$

Then the operator $\mathbf{Z}^r$ of right multiplication by $\mathbf{Z}$ on the row space $\mathbb{C}^{1\times N}$ is the same as the operator $[\operatorname{diag} \boldsymbol{\chi}]^r = \begin{bmatrix}\operatorname{diag} \boldsymbol{\chi}_1^r & \cdots & \operatorname{diag} \boldsymbol{\chi}_d^r\end{bmatrix}$, i.e., the matrix $\boldsymbol{\chi} = \begin{bmatrix}\operatorname{diag} \boldsymbol{\chi}_1 & \cdots & \operatorname{diag} \boldsymbol{\chi}_d\end{bmatrix}$ is in $(\Omega)_{\rm nc}$. As the right multiplication operator $[\widetilde{\mathcal{I}}]^r$ is injective and this is a matrix over $\mathbb{C}$, a consequence of the intertwining relation (4.39) is that the matrix

$$[M_\chi^r] = \begin{bmatrix}[M_{\chi_1}^r] & \cdots & [M_{\chi_d}^r]\end{bmatrix}$$

is in the full nc envelope $(\Omega)_{\rm nc,full}$. As Q is a nc function on $\Xi \supset \Omega_{\rm full}$, it follows that $Q(\boldsymbol{\chi})$ and $Q([M_\chi^r])$ are defined. Since nc functions respect intertwinings, a consequence of (4.39) is the intertwining relation

$$[\widetilde{\mathcal{I}}]^r Q([M_\chi^r])^r = Q([\operatorname{diag} \boldsymbol{\chi}])^r[\widetilde{\mathcal{I}}]^r. \tag{4.41}$$

since nc functions respect direct sums, it follows from the definitions that

$$Q([\operatorname{diag}\boldsymbol{\chi}]) = [\operatorname{diag}\mathbf{Q}].$$

Substitution of this relation back into (4.41) then gives

$$[\widetilde{\mathcal{I}}]^r\, Q([M^r_\chi])^r = [\operatorname{diag}\mathbf{Q}]\,[\widetilde{\mathcal{I}}]^r.$$

Comparison of this with the first of relations (4.40) then shows that

$$[\widetilde{\mathcal{I}}]^r\, Q([M^r_\chi])^r = [\widetilde{\mathcal{I}}]^r\,[M^r_Q]^r.$$

As $[\widetilde{\mathcal{I}}]^r$ is injective, we conclude that

$$Q([M^r_\chi]) = [M^r_Q].$$

We have already observed (see (4.35)) that $\|M^r_Q\| < 1$. We conclude that the point $[M^r_\chi]$ is not only in the full nc envelope $(\Omega)_{\rm nc,full}$ of Ω, but also is in the $\mathbb{D}_Q$-relative full nc envelope Ω' of Ω.

A quite similar analysis identifies the evaluation of the nc functions a, b at the point $[M^r_\chi]$:

$$a([M^r_\chi]) = [M^r_a], \quad b([M^r_\chi]) = [M^r_b]. \tag{4.42}$$

As we saw above that $[M^r_\chi]$ is in Ω', our standing hypothesis tells us that

$$a([M_{\chi^r}])a([M_{\chi^r}])^* - b([M_{\chi^r}])b([M_{\chi^r}])^* \succeq 0,$$

From the identities (4.42), this translates to

$$[M^r_a][M^r_a]^* - [M^r_b][M^r_b]^* \succeq 0.$$

In terms of matrix right-multiplication operators, this becomes

$$[M^r_a]^{r*}[M^r_a]^r - [M^r_b]^{r*}[M^r_b]^r \succeq 0.$$

In terms of the original coordinate-free presentation of $\mathcal{H}_{\mathbb{L},\mathcal{E}}$, this last expression translates to the statement

$$\|M^r_a\mathbf{f}\|^2_{\mathcal{H}_{\mathbb{L},\mathcal{Y}}} - \|M^r_b\mathbf{f}\|^2_{\mathcal{H}_{\mathbb{L},\mathcal{U}}} \geq 0$$

for all $\mathbf{f} \in \mathcal{H}_{\mathbb{L},\mathcal{E}}$, or, equivalently,

$$\mathbb{L}_1\left(D_{\mathbf{f}a,\mathbf{f}a} - D_{\mathbf{f}b,\mathbf{f}b}\right) \geq 0 \text{ for all } \mathbf{f} \in \mathcal{H}_{\mathbb{L},\mathcal{E}},$$

where

$$\left(D_{\mathbf{f}a,\mathbf{f}a} - D_{\mathbf{f}b,\mathbf{f}b}\right)(Z,W)(P) = \mathbf{f}(Z)K_{a,b}(Z,W)(P)\mathbf{f}(W)^*$$

(the kernel $K_{a,b}$ defined as in (4.17)). In particular we may choose

$$\mathbf{f}(Z) = \mathrm{id}_{\mathbb{C}^n} \otimes I_{\mathcal{E}} \text{ for } Z \in \Omega_n$$

to conclude that $\mathbb{L}_1(K_{a,b}) \geq 0$. We have thus arrived at condition (4.19) as required to complete the proof of Case 1.

Case 2: Reduction of the case of a general subset Ω and coefficient space $\mathcal{E}$ to the case of a finite subset Ω_α and finite-dimensional coefficient Hilbert space $\mathcal{E}_\alpha$. The general case can be reduced to the special situation of Case 1 as an application of a theorem of Kurosh that says that the limit of an inverse spectrum of nonempty compacta is a nonempty compactum (see [5, Theorem 2.56]). For the sake of completeness we go through the proof presented in [11, pages 73–75] adapted to the special case here.

We assume condition $(1')$ in the statement of the theorem, but now the point subset Ω of $\mathbb{D}_Q$ is not necessarily finite and the coefficient Hilbert space $\mathcal{E}$ is not necessarily finite-dimensional.

The directed set $\mathfrak{A}$. We let $\mathfrak{A}$ be the set of all pairs $(\Omega_0, \mathcal{E}_0)$ where Ω_0 is a finite subset of Ω and $\mathcal{E}_0$ is a finite-dimensional subspace of $\mathcal{E}$. It will be convenient to use lower case Greek letters, e.g. α, for a generic element of $\mathfrak{A}$. If $\alpha = (\Omega_0, \mathcal{E}_0) \in \mathfrak{A}$, let us write $\alpha_\Omega = \Omega_0$ and $\alpha_\mathcal{E} = \mathcal{E}_0$, so $\alpha = (\alpha_\Omega, \alpha_\mathcal{E})$. We define a partial order on $\mathfrak{A}$ as follows: given two elements α and β of $\mathfrak{A}$, we say that $\alpha \preceq \beta$ if both $\beta_\Omega \subset \alpha_\Omega$ (as finite subsets of Ω) and $\alpha_\mathcal{E} \subset \beta_\mathcal{E}$ (as finite-dimensional subspaces of $\mathcal{E}$). Then $\preceq$ satisfies reflexivity ($\alpha \preceq \alpha$) and transitivity ($\alpha \preceq \beta$, $\beta \preceq \gamma \Rightarrow \alpha \preceq \gamma$).

We claim that $\mathfrak{A}$ is a directed set, i.e.:

Claim: *Given any $\alpha, \beta \in \mathfrak{A}$, one can always find a $\gamma \in \mathfrak{A}$ so that both $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.*

To verify the Claim, let $\alpha = (\alpha_\Omega, \alpha_\mathcal{Y})$ and $\beta = (\beta_\Omega, \beta_\mathcal{Y})$ be in $\mathfrak{A}$. Then choose a finite set γ_Ω so that both $\alpha_\Omega \subset \gamma_\Omega$ and $\beta_\Omega \subset \gamma_\Omega$. Similarly, let $\gamma_\mathcal{Y}$ be any finite dimensional subspace containing both $\alpha_\mathcal{Y}$ and $\beta_\mathcal{Y}$. Then $\gamma = (\gamma_\Omega, \gamma_\mathcal{Y}) \in \mathfrak{A}$ has the property that both $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

The compact set $\mathbb{K}_\alpha$ for $\alpha \in \mathfrak{A}$. For each $\alpha \in \mathfrak{A}$, we let $\mathbb{K}_\alpha$ be the set of all cp nc kernels $\Gamma\colon \alpha_\Omega \times \alpha_\Omega \to \mathcal{L}(\mathcal{L}(\mathcal{S}), \mathcal{L}(\alpha_\mathcal{E}))_{\rm nc}$ so that

$$\begin{aligned}(I_n \otimes \Pi^{\mathcal{E}}_{\alpha_\mathcal{E}}\, (a(Z)(P \otimes I_\mathcal{Y})a(W)^* - b(Z)(P \otimes I_\mathcal{U})b(W)^*)\,|_{\alpha^m_\mathcal{E}} \\ = \Gamma(Z, W)\,((P \otimes I_\mathcal{S}) - Q(Z)(P \otimes I_\mathcal{R})Q(W)^*) \qquad (4.43)\end{aligned}$$

for all $Z \in \alpha_{\Omega,n}$, $W \in \alpha_{\Omega,m}$ and $P \in \mathbb{C}^{n\times m}$, where $\Pi^{\mathcal{E}}_{\alpha_\mathcal{E}} \colon \mathcal{E} \to \alpha_\mathcal{E}$ is the orthogonal projection of $\mathcal{E}$ onto $\alpha_\mathcal{E}$. The accomplishment of Case 1 of the proof is to confirm that $\mathbb{K}_\alpha$ is nonempty for all $\alpha \in \mathfrak{A}$. Furthermore, one can use the estimate (4.4) with $(I_n \otimes \Pi^{\mathcal{E}}_{\alpha_\mathcal{E}})\big(a(Z)(P \otimes I_\mathcal{Y})a(W)^* - b(Z)(P \otimes I_\mathcal{U})b(W)^*\big)|_{\alpha^m_\mathcal{E}}$ in place of $K_N(Z, Z)(I)$ to see that each $\mathbb{K}_\alpha$ is compact in the norm topology of $\mathfrak{X}$.

The restriction maps π^α_β. For α and β two elements of $\mathfrak{A}$ with $\beta \preceq \alpha$, we define a map $\pi^\alpha_\beta \colon \mathbb{K}_\alpha \to \mathbb{K}_\beta$ as follows: for $\Gamma \in \mathbb{K}_\alpha$, define $\pi^\alpha_\beta \Gamma \in \mathbb{K}_\beta$ by

$$\left(\pi^\alpha_\beta \Gamma\right)(Z, W)(R) = (I_n \otimes \Pi^\alpha_\beta)\Gamma(Z, W)(R)|_{\mathbb{C}^m \otimes \beta_\mathcal{Y}}$$

for $Z \in \beta_{\Omega,n}$, $W \in \beta_{\Omega,m}$, $R \in \mathcal{S}^{n\times m}$, where $\Pi^\alpha_\beta \colon \alpha_\mathcal{Y} \to \beta_\mathcal{Y}$ is the orthogonal projection. Then it is easily verified that the family of maps $\{\pi^\alpha_\beta \colon \beta \preceq \alpha\}$ satisfies the following properties:

$$\pi^\alpha_\alpha = \mathrm{id}_{\mathbb{K}_\alpha}, \qquad (4.44)$$

$$\gamma \preceq \beta,\ \beta \preceq \alpha \ \Rightarrow\ \pi^\alpha_\gamma = \pi^\beta_\gamma \circ \pi^\alpha_\beta. \qquad (4.45)$$

The sets $\mathbb{K}$, $\mathbb{K}^{\beta\alpha}$, and $\mathbb{K}^{\alpha}$. We let $\mathbb{K}$ be the Cartesian-product space

$$\mathbb{K} = \Pi_{\alpha\in\mathfrak{A}}\, \mathbb{K}_{\alpha}.$$

We denote an element of $\mathbb{K}$ by $\mathbf{\Gamma} = \{\Gamma_\alpha\}_{\alpha\in\mathfrak{A}}$ where each $\Gamma_\alpha \in \mathbb{K}_\alpha$. We endow $\mathbb{K}$ with the standard Cartesian-product topology (the weakest topology such that each projection $p_\alpha \colon \mathbf{\Gamma} \mapsto \Gamma_\alpha \in \mathbb{K}_\alpha$ is continuous). Since each fiber $\mathbb{K}_\alpha$ is nonempty and compact by the preceding discussion, it is a consequence of Tikhonov's Theorem (and the Axiom of Choice) that $\mathbb{K}$ is nonempty and compact. Given a pair of elements (β, α) from $\mathfrak{A}$ with $\beta \preceq \alpha$, we let $\mathbb{K}^{\beta\alpha}$ be the subset of $\mathbb{K}$ consisting of all elements $\mathbf{\Gamma} = \{\Gamma_\gamma\}_{\gamma\in\mathfrak{A}}$ such that $\pi^\alpha_\beta \Gamma_\alpha = \Gamma_\beta$. We can construct an element of $\mathbb{K}^{\beta\alpha}$ as follows: choose any kernel Γ_α in $\mathbb{K}_\alpha$ and define $\mathbf{\Gamma} = \{\Gamma_{\alpha'}\}_{\alpha'\in\mathfrak{A}} \in \mathbb{K}$ by:

$$\Gamma_{\alpha'} = \begin{cases} \Gamma_\alpha & \text{if } \alpha' = \alpha, \\ \pi^\alpha_\beta \Gamma_\alpha & \text{if } \alpha' = \beta, \\ \text{any element of } \mathbb{K}_{\alpha'} & \text{otherwise.} \end{cases}$$

In this way we see that $\mathbb{K}^{\beta\alpha}$ is nonempty for any pair of indices $\alpha, \beta \in \mathfrak{A}$ with $\beta \preceq \alpha$.

For α a single index in $\mathfrak{A}$, we define a subset $\mathbb{K}^\alpha$ by

$$\mathbb{K}^\alpha = \cap_{\beta\in\mathfrak{A}\colon \beta\preceq\alpha} \mathbb{K}^{\beta\alpha}.$$

A key point is that $\mathbb{K}^\alpha$ is always nonempty. This follows by a construction similar to that used to show that $\mathbb{K}^{\beta\alpha}$ is nonempty: choose any $\Gamma_\alpha \in \mathbb{K}_\alpha$ and define $\mathbf{\Gamma} = \{\Gamma_{\alpha'}\}_{\alpha'\in\mathfrak{A}}$ by

$$\Gamma_{\alpha'} = \begin{cases} \pi^\alpha_{\alpha'} \Gamma_\alpha & \text{if } \alpha' \preceq \alpha, \\ \text{any element of } \mathbb{K}_{\alpha'} & \text{otherwise.} \end{cases}$$

Then $\mathbf{\Gamma}$ so constructed is in $\mathbb{K}^\alpha$.

The next step is to argue that $\cap_{\alpha\in\mathfrak{A}} \mathbb{K}^\alpha$ is nonempty. To do this, we argue:

(1) Each $\mathbb{K}^\alpha$ is a closed subset of the compact space $\mathbb{K}$, and
(2) the collection of sets $\{\mathbb{K}^\alpha \colon \alpha \in \mathfrak{A}\}$ has the Finite Intersection Property: given any finitely many $\alpha^{(1)}, \dots, \alpha^{(N)} \in \mathfrak{A}$, then

$$\mathbb{K}^{\alpha^{(1)}} \cap \cdots \cap \mathbb{K}^{\alpha^{(N)}} \neq \emptyset.$$

It then follows from a standard compactness argument that $\bigcap_{\alpha\in\mathfrak{A}} \mathbb{K}^\alpha \neq \emptyset$.

To verify (1), we note that each $\mathbb{K}^{\beta\alpha}$ is closed by the definition of the Cartesian product topology and the fact that each map $\pi^\alpha_\beta \colon \mathbb{K}_\alpha \to \mathbb{K}_\beta$ is continuous.

To verify (2), suppose that we are given finitely many indices $\alpha^{(1)}, \dots, \alpha^{(N)}$ in $\mathfrak{A}$. We noted above that $\mathfrak{A}$ is a directed set; hence a simple induction argument using the transitivity property of $\mathfrak{A}$ enables us to find $\alpha^{(0)} \in \mathfrak{A}$ so that $\alpha^{(j)} \preceq \alpha^{(0)}$ for $j = 1, \dots, N$. Then we can use property (4.45) of the maps π^α_β to see that $\mathbb{K}^{\alpha^{(0)}} \subset \cap_{j=1}^N \mathbb{K}^{\alpha^{(j)}}$: if $\beta \preceq \alpha^{(j)}$, then since also $\alpha^{(j)} \preceq \alpha^{(0)}$ we have, for $\mathbf{\Gamma} = \{\Gamma_{\alpha'}\}_{\alpha' \in \mathfrak{A}}$ in $\mathbb{K}^{\alpha^{(0)}}$,

$$\pi_\beta^{\alpha^{(j)}} \Gamma_{\alpha^{(j)}} = \pi_\beta^{\alpha^{(j)}} \pi_{\alpha^{(j)}}^{\alpha^{(0)}} \Gamma_{\alpha^{(0)}} = \pi_\beta^{\alpha^{(0)}} \Gamma_{\alpha^{(0)}} = \Gamma_\beta$$

verifying that $\mathbf{\Gamma} \in \mathbb{K}^{\alpha^{(j)}}$ for each $j = 1, \dots, N$. As we verified above that each $\mathbb{K}^\alpha$ is nonempty, in particular $\mathbb{K}^{\alpha^{(0)}}$ is nonempty and hence $\bigcap_{j=1}^N \mathbb{K}^{\alpha^{(j)}} \neq \emptyset$.

Construction of Γ giving an Agler decomposition on all of Ω. As a result of the preceding paragraph, we can find an element $\mathbf{\Gamma} = \{\Gamma_{\alpha'}\}_{\alpha' \in \mathfrak{A}}$ in $\cap_{\alpha \in \mathfrak{A}} \mathbb{K}^\alpha$, i.e., an element $\mathbf{\Gamma}$ of $\mathbb{K}$ such that

$$\beta \preceq \alpha \text{ in } \mathfrak{A} \Rightarrow \pi^\alpha_\beta \Gamma_\alpha = \Gamma_\beta. \tag{4.46}$$

We use this $\mathbf{\Gamma}$ to construct a cp nc kernel Γ defined on all of $\Omega \times \Omega$ giving the desired Agler decomposition (3.5) for $K_{a,b}$ on all of Ω as follows. Given $Z \in \Omega_n$, $W \in \Omega_m$, $e = \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} \in \mathcal{E}^m$, $e' = \begin{bmatrix} e'_1 \\ \vdots \\ e'_n \end{bmatrix} \in \mathcal{E}^n$ and $R \in \mathcal{L}(\mathcal{S})^{n \times m}$, choose an element $\alpha \in \mathfrak{A}$ so that $Z, W \in \alpha_\Omega$ and $e_j, e'_i \in \alpha_\mathcal{E}$ for $1 \le j \le m$ and $1 \le i \le n$. We wish to define an operator $\Gamma(Z,W)(R) \in \mathcal{L}(\mathcal{E}^m, \mathcal{E}^n)$ so that

$$\langle \Gamma(Z,W)(R)e,\, e' \rangle_{\mathcal{E}^n} = \langle \Gamma_\alpha(Z,W)(R)e,\, e' \rangle_{(\alpha_\mathcal{E})^n}. \tag{4.47}$$

There are various issues to check.

First of all, we can always find an $\alpha \in \mathfrak{A}$ so that a given $Z \in \Omega_n$ and $W \in \Omega_m$ are in α_Ω and $e_j \in \alpha_\mathcal{E}$, $e'_i \in \alpha_\mathcal{E}$ for all $j = 1, \dots, m$ and $i = 1, \dots, n$. Set α_Ω equal to any finite subset containing Z and W, and choose $\alpha_\mathcal{E}$ to be any finite dimensional subspace of $\mathcal{E}$ containing the finite set of vectors $\{e_j,\, e'_i \colon 1 \le j \le m;\, 1 \le i \le n\}$.

Secondly the right-hand side of (4.47) is independent of the choice of index $\alpha \in \mathfrak{A}$. Indeed, if α' is another such index, choose $\alpha^{(0)} \in \mathfrak{A}$ with $\alpha \preceq \alpha^{(0)}$ and $\alpha' \preceq \alpha^{(0)}$. Then a simple application of the property (4.46) combined with the property (4.45) of the restriction maps π^α_β enables us to see that

$$\begin{aligned} \langle \Gamma_\alpha(Z,W)(R)e, e' \rangle_{(\alpha_\mathcal{E})^n} &= \langle \Gamma_{\alpha^{(0)}}(Z,W)(R)e, e' \rangle_{(\alpha_\mathcal{E}^{(0)})^n} \\ &= \langle \Gamma_{\alpha'}(Z,W)(R)e, e' \rangle_{(\alpha'_\mathcal{E})^n} \end{aligned}$$

as required.

One can now use standard techniques from Hilbert space theory (Principle of Uniform Boundedness and Riesz-Frechet Theorem—see [75]) to see that the well-defined sesquilinear form given by (4.47) arises from an operator $\Gamma(Z,W)(R) \in \mathcal{L}(\mathcal{E}^m, \mathcal{E}^n)$. Since the form is linear in R, the operator $\Gamma(Z,W)(R)$ is also linear in R. Since cp nc kernel properties involve only finitely many points Z, W at a time, it is easily verified that the fact that each Γ_α is a cp nc kernel on $\alpha_\Omega \times \alpha_\Omega$ implies that Γ is a cp nc kernel on $\Omega \times \Omega$. Finally, since each Γ_α gives rise to a local Agler decomposition for

$K_{a,b}$ on α_Ω compressed to $\alpha_{\mathcal{E}}$, it is easily argued that Γ gives rise to a global Agler decomposition (3.5) on all of Ω.

We have now finally completed the proof of $(1') \Rightarrow (2)$ in Theorem 3.1. □

Proof of (2) ⇒ (3) in Theorem 3.1. For this part we assume only that a, b are graded functions from Ω into $\mathcal{L}(\mathcal{Y}, \mathcal{E})_{\rm nc}$ and $\mathcal{L}(\mathcal{U}, \mathcal{E})_{\rm nc}$ respectively. We assume that we are given an Agler decomposition for $K_{a,b}$ as in (3.5). Let $\Gamma(Z, W)(T) = H(Z)\left((\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \pi)(T)\right) H(W)^*$ be the Kolmogorov decomposition (2.18) for Γ. Then (3.5) becomes

$$\begin{aligned} & a(Z)(P \otimes I_{\mathcal{Y}})a(W)^* - b(Z)(P \otimes I_{\mathcal{U}})b(W)^* \\ & \quad = H(Z)\left[(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \pi)(P \otimes I_{\mathcal{S}} - Q(Z)(P \otimes I_{\mathcal{R}})Q(W)^*)\right] H(W)^*. \quad (4.48) \end{aligned}$$

Since π is a unital $*$-representation, we have the simplification

$$(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \pi)(P \otimes I_{\mathcal{S}}) = P \otimes I_{\mathcal{X}}.$$

If we rebalance the identity (4.48) to eliminate all minus signs, we arrive at

$$\begin{aligned} & H(Z)[(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \pi)(Q(Z)(P \otimes I_{\mathcal{R}})Q(W)^*)]H(W)^* + a(Z)(P \otimes I_{\mathcal{Y}})a(W)^* \\ & \quad = H(Z)(P \otimes I_{\mathcal{X}})H(W)^* + b(Z)(P \otimes I_{\mathcal{U}})b(W)^*. \quad (4.49) \end{aligned}$$

We next write this as an inner-product identity

$$\begin{aligned} & \langle H(Z)[(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \pi)(Q(Z) \cdot P \cdot Q(W)^*)]H(W)^* e, e' \rangle + \langle a(Z) \cdot P \cdot a(W)^* e, e' \rangle \\ & \quad = \langle H(Z) \cdot P \cdot H(W)^* e, e' \rangle + \langle b(Z) \cdot P \cdot b(W)^* e, e' \rangle \quad (4.50) \end{aligned}$$

where e and e' are arbitrary vectors in $\mathcal{E}^m$ and $\mathcal{E}^n$ respectively, and the various inner products are in the space $\mathcal{E}^n$. We also simplify the notation by viewing the operator $P \otimes I_{\mathcal{R}}$ (as well as $P \otimes I_{\mathcal{Y}}$ and $P \otimes I_{\mathcal{X}}$) as multiplication by the scalar matrix P (which makes sense due the vector-space structure of $\mathcal{R}$, $\mathcal{Y}$, and $\mathcal{X}$).

Suppose next that P has a factorization

$$P = \beta^* \alpha \text{ where } \alpha \in \mathbb{C}^{k\times m},\ \beta \in \mathbb{C}^{k\times n}.$$

Then, by the definition (2.44) of the $(\mathcal{L}(\mathcal{S}, \mathcal{R}^k) \otimes_\pi \mathcal{X})_{\rm nc}$-inner product (see Theorem 2.19), we may rewrite (4.50) as an alternative inner-product identity:

$$\begin{aligned} & \left\langle \begin{bmatrix} \alpha \cdot Q(W)^* \otimes H(W)^* e \\ \alpha \cdot a(W)^* e \end{bmatrix}, \begin{bmatrix} \beta \cdot Q(Z)^* \otimes H(Z)^* e' \\ \beta \cdot a(Z)^* e' \end{bmatrix} \right\rangle \\ & \quad = \left\langle \begin{bmatrix} \alpha \cdot H(W)^* e \\ \alpha \cdot b(W)^* e \end{bmatrix}, \begin{bmatrix} \beta \cdot H(Z)^* e' \\ \beta \cdot b(Z)^* e' \end{bmatrix} \right\rangle, \quad (4.51) \end{aligned}$$

or equivalently (by recalling the definition (2.46) of L_T) as

$$\begin{aligned} & \left\langle \begin{bmatrix} L_{\alpha \cdot Q(W)^*} H(W)^* \\ \alpha \cdot a(W)^* \end{bmatrix} e,\ \begin{bmatrix} L_{\beta \cdot Q(Z)^*} H(Z)^* \\ \beta \cdot a(Z)^* \end{bmatrix} e' \right\rangle \\ & \quad = \left\langle \begin{bmatrix} \alpha \cdot H(W)^* \\ \alpha \cdot b(W)^* \end{bmatrix} e,\ \begin{bmatrix} \beta \cdot H(Z)^* \\ \beta \cdot b(Z)^* \end{bmatrix} e' \right\rangle, \quad (4.52) \end{aligned}$$

where the first inner product is in $(\mathcal{L}(\mathcal{S},\mathcal{R}^k)\otimes_\pi \mathcal{X})_{\rm nc}\oplus\mathcal{Y}^k$ and the second inner product is in $\mathcal{X}^k\oplus\mathcal{U}^k$. Furthermore, by part (3) of Theorem 2.19, we can identify the space $(\mathcal{L}(\mathcal{S},\mathcal{R}^k)\otimes_\pi \mathcal{X})_{\rm nc}$ with the k-fold direct sum space $((\mathcal{L}(\mathcal{S},\mathcal{R})\otimes_\pi \mathcal{X})_{\rm nc})^k$.

For $\alpha\in\mathbb{C}^{k\times m}$, $W\in\Omega_m$, $y\in\mathcal{Y}^m$, $m,k\in\mathbb{N}$ arbitrary, let us introduce the notation

$$\widehat{D}^{(k)}_{W,\alpha,e}=\begin{bmatrix}\alpha\cdot Q(W)^*\otimes H(W)^*e\\ \alpha\cdot a(W)^*e\end{bmatrix}\in\begin{bmatrix}(\mathcal{L}(\mathcal{S},\mathcal{R}^k)\otimes_\pi \mathcal{X})_{\rm nc}\\ \mathcal{Y}^k\end{bmatrix},$$
$$R^{(k)}_{W,\alpha,e}=\begin{bmatrix}\alpha\cdot H(W)^*e\\ \alpha\cdot b(W)^*e\end{bmatrix}\in\begin{bmatrix}\mathcal{X}^k\\ \mathcal{U}^k\end{bmatrix}.$$

We wish to use the identification map ι_k of the form (2.48) to identify the space $(\mathcal{L}(\mathcal{S},\mathcal{R}^k)\otimes_\pi \mathcal{X})_{\rm nc}$ with the space $((\mathcal{L}(\mathcal{S},\mathcal{R})\otimes_\pi \mathcal{X})_{\rm nc})^k$. Toward this end we note that

$$\iota_k\colon \alpha\cdot Q(W)^*\otimes H(W)^*e\mapsto\begin{bmatrix}\alpha_{1\cdot}\cdot Q(W)^*\otimes H(W)^*e\\ \vdots\\ \alpha_{k\cdot}\cdot Q(W)^*\otimes H(W)^*e\end{bmatrix}$$

where $\alpha_{i\cdot}=\begin{bmatrix}\alpha_{i1} & \cdots & \alpha_{im}\end{bmatrix}$ is the i-th row of α for $i=1,\dots,k$. In place of $\widehat{D}^{(k)}_{W,\alpha,e}$ we now work with

$$D^{(k)}_{W,\alpha,e}=\begin{bmatrix}\begin{bmatrix}\alpha_{1\cdot}\cdot Q(W)^*\otimes H(W)^*e\\ \vdots\\ \alpha_{k\cdot}\cdot Q(W)^*\otimes H(W)^*e\end{bmatrix}\\ \alpha\cdot a(W)^*e\end{bmatrix}\in\begin{bmatrix}((\mathcal{L}(\mathcal{S},\mathcal{R})\otimes \mathcal{X})_{\rm nc})^k\\ \mathcal{Y}^k\end{bmatrix},$$

or equivalently (by (2.42))

$$D^{(k)}_{W,\alpha,e}=\begin{bmatrix}\begin{bmatrix}L_{\alpha_{1\cdot}\cdot Q(W)^*}H(W)^*e\\ \vdots\\ L_{\alpha_{k\cdot}\cdot Q(W)^*}H(W)^*e\end{bmatrix}\\ \alpha\cdot a(W)^*e\end{bmatrix}\in\begin{bmatrix}((\mathcal{L}(\mathcal{S},\mathcal{R})\otimes \mathcal{X})_{\rm nc})^k\\ \mathcal{Y}^k\end{bmatrix},$$

and let $\mathcal{D}^{(k)}$ and $\mathcal{R}^{(k)}$ be the corresponding sets

$$\mathcal{D}^{(k)}=\{D^{(k)}_{W,\alpha,e}\colon W\in\Omega_{0,m},\alpha\in\mathbb{C}^{k\times m},e\in\mathcal{E}^m,\ m\in\mathbb{N}\},$$
$$\mathcal{R}^{(k)}=\{R^{(k)}_{W,\alpha,e}\colon W\in\Omega_{0,m},\alpha\in\mathbb{C}^{k\times m},e\in\mathcal{E}^m,\ m\in\mathbb{N}\}.$$

Then the content of (4.52) is that

$$\left\langle D^{(k)}_{W,\alpha,e},D^{(k)}_{Z,\beta,e'}\right\rangle_{((\mathcal{L}(\mathcal{S},\mathcal{R})\otimes_\pi\mathcal{X})_{\rm nc})^k\oplus\mathcal{Y}^k}=\left\langle R^{(k)}_{W,\alpha,e},\,R^{(k)}_{Z,\beta,e'}\right\rangle_{\mathcal{X}^k\oplus\mathcal{U}^k}\tag{4.53}$$

for any pairs of vectors $D^{(k)}_{W,\alpha,e},D^{(k)}_{Z,\beta,e'}\in\mathcal{D}^{(k)}$ and $R^{(k)}_{W,\alpha,e},R^{(k)}_{Z,\beta,e'}\in\mathcal{R}^{(k)}$. It follows that the mapping $\mathbf{V}^{(k)}\colon D^{(k)}_{W,V^*,e}\mapsto R^{(k)}_{W,V^*,e}$ extends by linearity and taking of limits to a well-defined isometry from $\boldsymbol{\mathcal{D}}^{(k)}:=\overline{\rm span}\,\mathcal{D}^{(k)}$ onto $\boldsymbol{\mathcal{R}}^{(k)}:=\overline{\rm span}\,\mathcal{R}^{(k)}$.

For the moment we are interested only in the special case $k = 1$: $\mathbf{V}^{(1)} \colon \boldsymbol{\mathcal{D}}^{(1)} \to \boldsymbol{\mathcal{R}}^{(1)}$, where

$$\boldsymbol{\mathcal{D}}^{(1)} \subset \begin{bmatrix} (\mathcal{L}(\mathcal{S},\mathcal{R}) \otimes_\pi \mathcal{X})_{\rm nc} \\ \mathcal{Y} \end{bmatrix}, \quad \boldsymbol{\mathcal{R}}^{(1)} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}.$$

Suppose for the moment that $\boldsymbol{\mathcal{D}}^{(1)\perp} = \left[\begin{smallmatrix} (\mathcal{L}(\mathcal{S},\mathcal{R}) \otimes_\pi \mathcal{X})_{\rm nc} \\ \mathcal{Y} \end{smallmatrix}\right] \ominus \boldsymbol{\mathcal{D}}^{(1)}$ and $\boldsymbol{\mathcal{R}}^{(1)\perp} = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \ominus \boldsymbol{\mathcal{R}}^{(1)}$ have the same dimension. Then we can extend $\mathbf{V}^{(1)}$ to a unitary operator

$$\mathbf{U}^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \colon \begin{bmatrix} (\mathcal{L}(\mathcal{S},\mathcal{R}) \otimes_\pi \mathcal{X})_{\rm nc} \\ \mathcal{Y} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}.$$

by letting $\mathbf{U}^*|_{\boldsymbol{\mathcal{D}}^{(1)}} = \mathbf{V}^{(1)} \colon \boldsymbol{\mathcal{D}}^{(1)} \to \boldsymbol{\mathcal{R}}^{(1)}$, letting $\mathbf{U}^*|_{\boldsymbol{\mathcal{D}}^{(1)\perp}}$ be any unitary transformation of $\boldsymbol{\mathcal{D}}^{(1)\perp}$ onto $\boldsymbol{\mathcal{R}}^{(1)\perp}$, and then using linearity to extend to a unitary map $\mathbf{U}^*$ from all of $(\mathcal{L}(\mathcal{S},\mathcal{R}) \otimes_\pi \mathcal{X})_{\rm nc} \oplus \mathcal{Y}$ onto $\mathcal{X} \oplus \mathcal{U}$. Even if the dimensions do not match, we can always artificially add on an infinite-dimensional direct-sum component $\mathcal{X}_0$ to $\mathcal{X}$ and to $\mathcal{L}(\mathcal{S},\mathcal{R}) \otimes_\pi \mathcal{X}$ to make the two codimensions have common dimension infinity, and then apply the construction just sketched above. Alternatively, one can get a contractive extension $\mathbf{U}^*$ by simply defining $\mathbf{U}^*|_{\boldsymbol{\mathcal{D}}^{(1)\perp}} = 0$ and extending by linearity; with this last construction one loses conservation of energy but avoids inflation of the state-space dimension.

In any case, we have now arrived at a contractive (possibly even unitary) colligation matrix $\mathbf{U}$ as in (3.6) so that

$$\mathbf{U}^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\Bigg|_{\boldsymbol{\mathcal{D}}^{(1)}} = \mathbf{V}^{(1)}.$$

For $k \in \mathbb{N}$ any positive integer, let $\mathbf{U}^{(k)} = I_k \otimes \mathbf{U}$ as in (3.8) (so $\mathbf{U}^{(1)} = \mathbf{U}$). Note that here we use the canonical identifications

$$\begin{bmatrix} ((\mathcal{L}(\mathcal{S},\mathcal{R}) \otimes_\pi \mathcal{X})_{\rm nc})^k \\ \mathcal{Y}^k \end{bmatrix} \cong \begin{bmatrix} (\mathcal{L}(\mathcal{S},\mathcal{R}) \otimes_\pi \mathcal{X})_{\rm nc} \\ \mathcal{Y} \end{bmatrix}^k, \quad \begin{bmatrix} \mathcal{X}^k \\ \mathcal{U}^k \end{bmatrix} \cong \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}^k \tag{4.54}$$

when convenient; the meaning should be clear from the context. Then it is clear that, for each $k \in \mathbb{N}$, $\mathbf{U}^{(k)*}$ is a contractive (or even unitary) linear transformation of $((\mathcal{L}(\mathcal{S},\mathcal{R}) \otimes_\pi \mathcal{X})_{\rm nc})^k \oplus \mathcal{Y}^k$ to $\mathcal{X}^k \oplus \mathcal{U}^k$.

The next goal is to verify that in addition $\mathbf{U}^{(k)*}$ extends $\mathbf{V}^{(k)}$:

$$\mathbf{U}^{(k)*}|_{\boldsymbol{\mathcal{D}}^{(k)}} = \mathbf{V}^{(k)}. \tag{4.55}$$

We are given that $\mathbf{U}^* \colon D^{(1)}_{W,v^*,e} \to R^{(1)}_{W,v^*,e}$ for each $W \in \Omega_m$, $v \in \mathcal{A}^{m \times 1}$ and $e \in \mathcal{E}^m$. We must show that

$$I_k \otimes \mathbf{U}^* \colon D^{(k)}_{W,\alpha,e} \to R^{(k)}_{W,\alpha,e} \tag{4.56}$$

for each $W \in \Omega_m$, $\alpha \in \mathbb{C}^{k \times m}$, $e \in \mathcal{E}^m$. Up to the identifications (4.54), note that we have the identities

$$D^{(k)}_{W,\alpha,e} = \begin{bmatrix} D^{(1)}_{W,\alpha_{1\cdot},e} \\ \vdots \\ D^{(1)}_{W,\alpha_{k\cdot},e} \end{bmatrix}, \quad R^{(k)}_{W,\alpha,e} = \begin{bmatrix} R^{(1)}_{W,\alpha_{1\cdot},e} \\ \vdots \\ R^{(1)}_{W,\alpha_{k\cdot},e} \end{bmatrix}.$$

Then by definition we have

$$\mathbf{U}^{(k)*} = I_k \otimes \mathbf{U}^* \colon D^{(k)}_{W,\alpha,e} = \begin{bmatrix} D^{(1)}_{W,\alpha_{1\cdot},e} \\ \vdots \\ D^{(1)}_{W,\alpha_{k\cdot},e} \end{bmatrix} \mapsto \begin{bmatrix} \mathbf{U}^* D^{(1)}_{W,\alpha_{1\cdot},e} \\ \vdots \\ \mathbf{U}^* D^{(1)}_{W,\alpha_{k\cdot},e} \end{bmatrix}$$
$$= \begin{bmatrix} R^{(1)}_{W,\alpha_{1\cdot},e} \\ \vdots \\ R^{(1)}_{W,\alpha_{k\cdot},e} \end{bmatrix} = R^{(k)}_{W,\alpha,e}$$

and hence (4.56) as well as (4.55) follow as wanted.

We now look at (4.55) for the special case where $k = m$ and $\alpha = I_m$. We therefore have

$$\begin{bmatrix} A^{(m)*} & C^{(m)*} \\ B^{(m)*} & D^{(m)*} \end{bmatrix} \begin{bmatrix} L_{Q(W)^*} H(W)^* e \\ a(W)^* e \end{bmatrix} = \begin{bmatrix} H(W)^* e \\ b(W)^* e \end{bmatrix} \tag{4.57}$$

where we use the notation in (3.8). From the first block row of (4.57) we get

$$A^{(m)*} L_{Q(W)^*} H(W)^* e + C^{(m)*} a(W)^* e = H(W)^* e.$$

We rearrange this to

$$(I - A^{(m)*} L_{Q(W)^*}) H(W)^* e = C^{(m)*} a(W)^* e.$$

As $\mathbf{U}^{(m)*}$ is contractive, it follows that $A^{(m)*}$ is contractive. As $W \in \Omega_m \subset \mathbb{D}_Q$, it follows that $Q(W)$ is strictly contractive; since $\|L_{Q(W)^*}\| = \|Q(W)^*\|$, we see that $L_{Q(W)^*}$ is strictly contractive as well. Hence $I - A^{(m)*} L_{Q(W)^*}$ is invertible. We may then solve for $H(W)^* e$ to get

$$H(W)^* e = (I - A^{(m)*} L_{Q(W)^*})^{-1} C^{(m)*} a(W)^* e.$$

Substituting this back into the second block row of (4.57) gives us

$$b(W)^* e = B^{(m)*} L_{Q(W)^*} (I - A^{(m)*} L_{Q(W)^*})^{-1} C^{(m)*} a(W)^* e + D^{(m)*} a(W)^* e,$$

Upon taking adjoints and replacing W with Z, we arrive at $b(Z) = a(Z) S(Z)$ with $S(Z)$ given by (3.7). □

Proof of (3) ⇒ (1) in Theorem 3.1: Let us now assume that we are given graded functions a, b from Ω to $\mathcal{L}(\mathcal{Y}, \mathcal{E})_{\rm nc}$ and $\mathcal{L}(\mathcal{U}, \mathcal{E})_{\rm nc}$ respectively along with the contractive (or even unitary) colligation matrix (3.6) so that S given by (3.7) satisfies the interpolation conditions (3.3). A key point is that the formula (3.7) actually makes sense for any $Z \in \mathbb{D}_{Q,n}$, not just

$Z \in \Omega_n$. From the tensored form of the formula (3.7), one can use the general results from [52] or check directly that S so defined on all of $\mathbb{D}_Q$ is indeed a nc function. Moreover, by general principles concerning feedback loading of a passive 2-port by a strictly dissipative 1-port (see e.g. [48]), it follows that the feedback connection is well-posed with resulting closed-loop input/output map $S(Z)$ having norm at most 1:

$$\|S(Z)\| \leq 1 \text{ for any } Z \in \mathbb{D}_Q$$

It follows that $S \in \mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$. This completes the proof of (3) $\Rightarrow$ (1) in Theorem 3.1. □

Remark 4.3. Let us consider the special case where $\Omega = \{Z^{(0)}\}$ consists of a single point in $\mathbb{D}_Q$ say of size $n \times n$ and where $\mathcal{E}$ is finite-dimensional. A close look at the proof of $(1') \Rightarrow (2)$ (in particular, look at (4.39)) in the proof of Theorem 3.1 shows that the hypothesis $(1')$ can be weakened to the following:

$(1'_0)$ *Let $\Omega'_0 \subset \Omega'$ (where $\Omega' = [\{Z^{(0)}\}]_{\text{full}} \cap \mathbb{D}_Q$) consist of those points $\widetilde{Z} \in \Omega'_m$ (where m is any natural number between 1 and $n \cdot \dim \mathcal{E}$) such that there is an injection $\mathcal{I} \in \mathbb{C}^{k \times m}$ so that*

$$\mathcal{I}\widetilde{Z} = \left(\bigoplus_{1}^{n \cdot \dim \mathcal{E}} Z^{(0)} \right) \mathcal{I}.$$

Assume that $a \in \mathcal{T}(\Omega'_0; \mathcal{L}(\mathcal{Y}, \mathcal{E}))$ and $b \in \mathcal{T}(\Omega'_0; \mathcal{L}(\mathcal{U}, \mathcal{E})$ are such that

$$a(Z)a(Z)^* - b(Z)b(Z)^* \succeq 0$$

for all $Z \in \Omega'_0$.

Then it is still the case that $(1'_0) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ and we can conclude with only $(1'_0)$ as the hypothesis that there is a Schur-Agler class solution $S \in \mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ of the single-point interpolation condition $a(Z^{(0)})S(Z^{(0)}) = b(Z^{(0)})$.

Remark 4.4. The proof of Theorem 3.1 verified the equivalence of the statements (1), $(1')$, (2), (3) by verifying the implications $(1) \Rightarrow (1') \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$. It is instructive to see to what extent some of the non-adjacent implications can be seen directly. We mention a couple of these possibilities.

(3) $\Rightarrow$ (2): We assume that a, b are nc functions on the $\mathbb{D}_Q$-relative full nc envelope $\Omega' := [\Omega]_{\text{full}} \cap \mathbb{D}_Q$ and that $S(Z)$ is given by the transfer-function realization formula (3.7) on all of $\mathbb{D}_Q$. Then we claim that the kernel $K_{a,b}$ (4.17) has the Agler decomposition (3.5) with cp nc kernel Γ given by

$$\Gamma(Z, W)(R) = H(Z)(R \otimes I_{\mathcal{X}})H(W)^*$$

with H given by

$$H(Z) = a(Z)C^{(n)}(I - L^*_{Q(Z)^*}A^{(n)})^{-1} \text{ for } Z \in \Omega_n.$$

To verify the claim, we first use the coisometry property of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to get

$$\begin{bmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{bmatrix} \begin{bmatrix} A^{(n)*} & C^{(n)*} \\ B^{(n)*} & D^{(n)*} \end{bmatrix} = \begin{bmatrix} I_{((\mathcal{L}(\mathcal{S},\mathcal{R})\otimes_\pi \mathcal{X})_{\rm nc})^n} & 0 \\ 0 & I_{\mathcal{Y}^n} \end{bmatrix}. \tag{4.58}$$

Now let $P \in \mathbb{C}^{n\times n}$. From the tensor-product structure of $\left[\begin{smallmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{smallmatrix}\right]$ we get the intertwining relation

$$\begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \cdot \begin{bmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{bmatrix} = \begin{bmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{bmatrix} \cdot \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}.$$

Multiplication of the identity (4.58) on the left by $\left[\begin{smallmatrix} P & 0 \\ 0 & P \end{smallmatrix}\right]$ then leads to

$$\begin{bmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{bmatrix} \cdot \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \cdot \begin{bmatrix} A^{(n)*} & C^{(n)*} \\ B^{(n)*} & D^{(n)*} \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \cdot \begin{bmatrix} I_{((\mathcal{L}(\mathcal{S},\mathcal{R})\otimes_\pi \mathcal{X})_{\rm nc})^n} & 0 \\ 0 & I_{\mathcal{Y}^n} \end{bmatrix}. \tag{4.59}$$

In the sequel to save space we suppress the $\cdot$ in the notation for the multiplication of a scalar matrix times an operator matrix of compatible block size. From the identity (4.59) we get the set of relations

$$\begin{aligned} & A^{(n)} P A^{(n)*} + B^{(n)} P B^{(n)*} = P \otimes I_{(\mathcal{L}(\mathcal{S},\mathcal{R})\otimes_\pi \mathcal{X})_{\rm nc}}, \\ & A^{(n)} P C^{(n)*} + B^{(n)} P D^{(n)*} = 0, \\ & C^{(n)} P A^{(n)*} + D^{(n)} P B^{(n)*} = 0, \\ & C^{(n)} P C^{(n)*} + D^{(n)} P \cdot D^{(n)*} = P \otimes I_{\mathcal{Y}}. \end{aligned}$$

We make use of these identities to then compute, for $Z, W \in \Omega_n$, $P \in \mathbb{C}^{n\times n}$,

$$\begin{aligned} & P \otimes I_{\mathcal{Y}} - S(Z) P S(W)^* = P \otimes I_{\mathcal{Y}} - (D^{(n)} + C^{(n)}(I - L^*_{Q(Z)^*} A^{(n)})^{-1} L^*_{Q(Z)^*} B^{(n)}) P \cdot \\ & \qquad \cdot (D^{(n)*} + B^{(n)*} L_{Q(W)^*}(I - A^{(n)*} L_{Q(W)^*})^{-1} C^{(n)*}) \\ & = P \otimes I_{\mathcal{Y}} - D^{(n)} P D^{(n)*} - C^{(n)}(I - L^*_{Q(Z)^*} A^{(n)})^{-1} L^*_{Q(Z)^*} B^{(n)} P D^{(n)*} \\ & \quad - D^{(n)} P B^{(n)*} L_{Q(W)^*}(I - A^{(n)*} P_{Q(W)^*})^{-1} C^{(n)*} \\ & \quad - C^{(n)}(I - L^*_{Q(Z)^*} A^{(n)})^{-1} L^*_{Q(Z)^*} B^{(n)} P B^{(n)*} L_{Q(W)^*}(I - A^{(n)*} L_{Q(W)^*})^{-1} C^{(n)*} \\ & = C^{(n)} P C^{(n)*} + C^{(n)}(I - L^*_{Q(Z)^*} A^{(n)})^{-1} L^*_{Q(Z)^*} A^{(n)} P C^{(n)*} \\ & + C^{(n)} P A^{(n)*} L_{Q(W)^*}(I - A^{(n)*} L_{Q(W)^*})^{-1} C^{(n)*} + \\ & C^{(n)}(I - L^*_{Q(Z)^*} A^{(n)})^{-1} L^*_{Q(Z)^*}(A^{(n)} P A^{(n)*} - P) L_{Q(W)^*}(I - A^{(n)*} L_{Q(W)^*})^{-1} C^{(n)*} \\ & = C^{(n)}(I - L^*_{Q(Z)^*} A^{(n)})^{-1} X (I - A^{(n)*} L_{Q(W)^*})^{-1} C^{(n)*} \end{aligned}$$

where

$$\begin{aligned} X &= (I - L^*_{Q(Z)^*} A^{(n)}) P (I - A^{(n)*} L_{Q(W)^*}) + L^*_{Q(Z)^*} A^{(n)} P (I - A^{(n)*} L_{Q(W)^*}) \\ & \qquad + (I - L^*_{Q(Z)^*} A^{(n)}) P A^{(n)*} L_{Q(W)^*} + L^*_{Q(Z)^*}(A^{(n)} P A^{(n)*} - P) L_{Q(W)^*} \\ & \quad = P \otimes I_{\mathcal{X}} - L^*_{Q(Z)^*} P L_{Q(W)^*}, \end{aligned}$$

and it follows that

$$P \otimes I_{\mathcal{Y}} - S(Z)PS(W)^* = H_0(Z)(P \otimes I_{\mathcal{X}} - L^*_{Q(Z)^*} P L_{Q(W)^*}) H_0(W)^* \quad (4.60)$$

with

$$H_0(Z) = C^{(n)}(I - L^*_{Q(Z)^*} A^{(n)})^{-1}.$$

As a consequence of the identity (2.47), we see that

$$L^*_{Q(Z)^*} P L_{Q(W)^*} = (\mathrm{id}_{\mathbb{C}^{n\times n}} \otimes \pi)(Q(Z)PQ(W)^*)$$

and hence (4.60) can be rewritten as

$$P \otimes I_{\mathcal{Y}} - S(Z)PS(W)^* = H_0(Z)(\mathrm{id}_{\mathbb{C}^{n\times n}} \otimes \pi)(P \otimes I_{\mathcal{X}} - Q(Z)PQ(W)^*)H_0(W)^*. \quad (4.61)$$

If the interpolation conditions (3.3) hold on Ω, then they continue to hold on Ω' by uniqueness of nc-function extensions to full nc envelopes (see Proposition 2.17). If we multiply (4.61) by $a(Z)$ on the left and by $a(W)^*$ on the right and use the interpolation condition (3.3) we arrive at the desired Agler decomposition (3.5) for the kernel $K_{a,b}$

$$K_{a,b}(Z,W) = H(Z)(\mathrm{id}_{\mathbb{C}^{n\times n}} \otimes \pi)(P \otimes I_{\mathcal{S}} - Q(Z)(P \otimes I_{\mathcal{R}})Q(W)^*)H(W)^*$$

with $H(Z) := a(Z)H_0(Z)$, at least for the square case where $Z \in \Omega'_n$, $W \in \Omega'_n$ and $P \in \mathbb{C}^{n\times n}$. To then arrive at the same identity with $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathbb{C}^{n\times m}$ with possibly $n \neq m$, one can apply the square case to the situation where $\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] \in \Omega'_{n+m}$ replaces both Z and W and $\left[\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right]$ replaces P, and then proceed as is done at the end of Section 3.6.

(2) $\Rightarrow$ (1′): If we assume that (2) holds in the strengthened form obtained in the previous paragraph, namely that there is a cp nc kernel $K\colon \Omega' \times \Omega' \to \mathcal{L}(\mathcal{L}(\mathcal{S}), \mathcal{L}(\mathcal{E}))_{\mathrm{nc}}$ so that (3.5) holds for $Z, W \in \Omega'$, then we can obtain condition (1′) easily by setting $Z = W \in \Omega'$ in (3.5) and using that $\|Q(Z)\| < 1$ along with the fact that Γ is a cp nc kernel on $\Omega' \times \Omega'$. It is not at all obvious how to get (1′) directly from (2) if one only assumes that the Agler decomposition (3.5) holds on Ω rather than on all of Ω'; this is part of the content of (2) $\Rightarrow$ (3) $\Rightarrow$ (1) $\Rightarrow$ (1′).

Remark 4.5. Consider the special case described in Example 2.5. We assume that the nc function Q is a matrix polynomial (a nc function with values in $\mathcal{L}(\mathcal{R}, \mathcal{S})_{\mathrm{nc}} = \mathcal{L}(\mathbb{C}^s, \mathbb{C}^r)_{\mathrm{nc}}$). We identify an associated Schur-Agler class function S with its power series representation:

$$S(Z) = \sum_{\mathfrak{a} \in \mathbb{F}^+_d} S_{\mathfrak{a}} \otimes Z^{\alpha}. \quad (4.62)$$

Here the coefficients S_α are operators from $\mathcal{U}$ to $\mathcal{Y}$ and the point $Z = (Z_1, \ldots, Z_d) \in \mathbb{D}_Q$, where $\mathbb{D}_Q$ consists of d-tuples of $n \times n$ matrices over $\mathbb{C}$ such that $\|Q(Z)\| < 1$.

In the original work of [21] the nc Schur-Agler class $\mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ (for the special case of linear $Q(z) = \sum_{k=1}^d L_k z_k$ with coefficient matrices L_k satisfying some additional conditions which we need not go into here) was defined to consist of noncommutative functions S such that $\|S(Z)\| \leq 1$ whenever $Z = (Z_1, \dots, Z_d)$ is a d-tuple of operators on a fixed separable infinite-dimensional Hilbert space $\mathcal{K}$ such that $\|Q(Z)\| < 1$ (where one can use the assumed nc power series representation for S to define the functional calculus $Z \mapsto S(Z)$). In general let us define the class $\mathcal{SA}_Q^\infty(\mathcal{U}, \mathcal{Y})$ to consist of those functions $S \in \mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ such that $\|S(Z)\| \leq 1$ whenever $Z \in \mathcal{L}(\mathcal{K})^d$ with $\|Q(Z)\| < 1$. Note that a priori from the definitions we have the containment $\mathcal{SA}_Q^\infty(\mathcal{U}, \mathcal{Y}) \subset \mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ (identify a matrix tuple $Z = (Z_1, \dots, Z_d) \in (\mathbb{C}^{n\times n})^d$ with an operator-tuple in $(\mathcal{L}(\mathcal{K}))^d$ by using some fixed orthonormal basis of $\mathcal{K}$ to embed $\mathbb{C}^n$ into $\mathcal{K}$). We note that in the proof of (4) $\Rightarrow$ (1) in Theorem 3.1, the function S given by the realization formula (3.7) is actually in the a priori smaller class $\mathcal{SA}_Q^\infty(\mathcal{U}, \mathcal{Y})$. We hence arrive at the following corollary of the proof of Theorem 3.1. A direct proof of this result for the special case where $Q(z) = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_d \end{bmatrix}$ is given in [8, Theorem 6.6].

Corollary 4.5.1 *The classes $\mathcal{SA}_Q^\infty(\mathcal{U}, \mathcal{Y})$ and $\mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ coincide.*

It appears that this result is new even for the commutative case. More precisely, let G be an open subset of $\mathbb{C}^d$, take the set Ξ^∞ to consist of commutative d-tuples $Z = (Z_1, \dots, Z_d)$ of operators on some fixed infinite-dimensional separable Hilbert space $\mathcal{K}$ such that the Taylor spectrum $\sigma(Z)$ of Z lies in G, let $\mathcal{R}$ to be a collection $\{F_k \colon k \in I\}$ of matrix valued functions $F_k(z) = [f_{k,ij}(z)]$ (where the row and column indices for each F_k have finite ranges, say $1 \leq i \leq m_k$ and $1 \leq j \leq n_k$) such that $\|F_k(z)\| < 1$ for $z \in G$, and set

$$\begin{aligned} & q(z) = \operatorname{diag}_{k\in I}[F_k(z)], \\ & \mathbb{D}_q^\infty = \{Z \in \Xi_\infty \colon \|q(Z)\| < 1\}, \\ & \mathcal{CSA}_q^\infty(\mathcal{U}, \mathcal{Y}) = \{S \colon \mathbb{D}_q \underset{\text{holo}}{\to} \mathcal{L}(\mathcal{U}, \mathcal{Y}) \colon \|S(Z)\| \leq 1 \text{ for } Z \in \mathbb{D}_q^\infty\}, \end{aligned}$$

where $Z \mapsto S(Z)$ is defined via the Taylor functional calculus. Then, as is explained in [60], restricting now to the scalar case $\mathcal{U} = \mathcal{Y} = \mathbb{C}$, $\mathcal{CSA}_q^\infty(\mathbb{C})$ is the unit ball of the dual operator algebra $\widetilde{\mathcal{A}}_\mathcal{R}$ associated with $\mathcal{R}$ as in [60]. Then results of [60] (using operator-algebra techniques quite different from ours) establish the commutative Agler-decomposition (3.10) (called *Agler factorization* in [60]) for the class $\mathcal{CSA}_q^\infty(\mathbb{C})$. The results of [60] also establish that the algebra $\widetilde{\mathcal{A}}_\mathcal{R}$ is **residually finite-dimensional**, i.e., the norm of an $n\times n$-matrix $[a_{ij}]$ over $\widetilde{A}$ is given by $\|[a_{ij}]\|_{\widetilde{\mathcal{A}}^{n\times n}} = \sup\{\|[\pi(a_{ij}]\|\}$ where the supremum is over all finite-dimensional representations π of $\mathcal{A}$. While this proves that

$$\mathcal{CSA}_q^\infty(\mathbb{C}) = \mathcal{CSA}_q(\mathbb{C}) \tag{4.63}$$

in many cases, there are examples of $\mathcal{R}$ for which not every finite-dimensional representation π is of the form $f \mapsto f(Z)$ for a commutative finite-matrix tuple $Z = (Z_1, \dots, Z_d) \in \mathbb{D}_q$ (see [60, Example 5.5]). A question asked in [60] is whether the identity (4.63) holds in general. As a result of Corollary 4.5.1, we conclude that the answer is indeed yes.

5. Multipliers between nc reproducing kernel Hilbert spaces

5.1. **Review of nc reproducing kernel Hilbert spaces and contractive multipliers.** Suppose that we are given two cp nc kernels K' and K, both defined on the Cartesian product $\mathbb{D} \times \mathbb{D}$ of a nc set $\mathbb{D} \subset \mathcal{V}_{\rm nc}$ where K' has values in $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{U}))_{\rm nc}$ while K has values in $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\rm nc}$ (here $\mathcal{A}$ is a C^*-algebra and $\mathcal{U}$ and $\mathcal{Y}$ are two auxiliary Hilbert spaces). Suppose next that S is a nc function on $\mathbb{D}$ with values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc}$. We say that S is a **multiplier** from $\mathcal{H}(K')$ to $\mathcal{H}(K)$, written as $S \in \mathcal{M}(K', K)$, if the operator M_S given by

$$(M_S f)(W) = S(W) f(W) \text{ for } W \in \Omega_m$$

maps $\mathcal{H}(K')$ boundedly into $\mathcal{H}(K)$. We can always multiply such an S by a positive scalar to arrange that $\|M_S\| \le 1$ in which case we say that S is a *contractive multiplier* and write $S \in \overline{\mathcal{B}}\mathcal{M}(K', K)$.

The following result appears in [23].

Theorem 5.1. *Given nc kernels K' and K from $\mathbb{D} \times \mathbb{D}$ to $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{U}))_{\rm nc}$ and $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\rm nc}$ respectively and given a nc function S from $\mathbb{D}$ to $\mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc}$, the following are equivalent:*

(1) *$S \in \overline{\mathcal{B}}\mathcal{M}(K', K)$.*
(2) *The de Branges-Rovnyak kernel $K_S^{\rm dBR}$ from $\mathbb{D}$ to $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\rm nc}$ associated with S given by*
$$K_S^{\rm dBR}(Z, W)(P) = K(Z, W)(P) - S(Z)\, K'(Z, W)(P)\, S(W)^* \tag{5.1}$$
is a cp nc kernel on $\mathbb{D}$.

5.2. **Interpolation by contractive multipliers.** Let us define the **nc Pick interpolation problem** as follows.

Noncommutative Pick Interpolation Problem: *Given cp nc kernels K' and K on $\mathbb{D} \times \mathbb{D}$ to respectively $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{U}))_{\rm nc}$ and $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\rm nc}$, and given a subset Ω of $\mathbb{D}$ together with a nc function S_0 from Ω into $\mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc}$, find a nc function S from all of $\mathbb{D}$ into $\mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc}$ so that*

(1) *$S|_\Omega = S_0$, and*
(2) *S is in the contractive multiplier class $\overline{\mathcal{B}}\mathcal{M}(K', K)$.*

An immediate consequence of Theorem 5.1 is the following result giving a necessary condition for solvability of the nc Pick interpolation problem.

Theorem 5.2. *Suppose that K, K', $\Omega \subset \mathbb{D}$ and S_0 form the data set for a solvable nc Pick interpolation problem. Then the associated de Branges-Rovnyak kernel $K_{S_0}^{\rm dBR}$ from $\Omega \times \Omega$ to $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\rm nc}$ is a cp nc kernel on Ω.*

Proof. By Theorem 5.1. If S is a solution of the nc Pick interpolation problem, then $K_S^{\rm dBR}$ is a cp nc kernel on $\mathbb{D}$, whence it follows that its restriction $K_{S_0}^{\rm dBR}$ to the nc subset $\Omega \times \Omega$ is a cp nc kernel on Ω. □

Let us now restrict our considerations to the case where $K' = I_{\mathcal{U}} \otimes k$ and $K = I_{\mathcal{Y}} \otimes k$ where k is a scalar cp nc kernel

$$k \colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathbb{C}))_{\rm nc}.$$

Inspired by the commutative case (see [5]) we make the following definition.

Definition 5.3. We say that the cp nc scalar kernel k on $\mathbb{D} \times \mathbb{D}$ is a **complete Pick nc kernel** if: given any coefficient Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ together with a subset Ω of $\mathbb{D}$ and a graded function S_0 from Ω to $\mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc}$, the necessary condition for existence of a solution of the associated nc Pick interpolation problem given by Theorem 5.2 is also sufficient, i.e., *the nc Pick interpolation problem has a solution if and only if the associated kernel*

$$K_{S_0}(Z,W)(P) = k(Z,W)(P) \otimes I_{\mathcal{Y}} - S_0(Z)\, (k(Z,W)(P) \otimes I_{\mathcal{U}})\, S_0(W)^* \quad (5.2)$$

is a cp kernel on Ω.

We now give a general example of a scalar kernel which turns out to be a complete Pick nc kernel. Specialize the general situation described in Subsection 2.2 to the case where the target space $\mathcal{S}$ for the nc function Q is taken to be $\mathcal{S} = \mathbb{C}$. Let us now use the notation Q_0 for the nc function on Ξ with values in $\mathcal{L}(\mathcal{R}, \mathbb{C}_{\rm nc})$ defining the nc domain $\mathbb{D}_{Q_0}$ as in (2.23). We mention that the Q_0 as in (4.8) appearing in the proof of Lemma 4.1 part (2) above is an example of such a Q_0. With notation as in the proof of Lemma 4.1 part (2), define a kernel k_{Q_0} on $\mathbb{D}_{Q_0}$ by

$$k_{Q_0}(Z,W)(P) = \sum_{k=0}^{\infty} (L_{Q_0(Z)^*})^{(k)*} (P \otimes I_{\mathcal{R}^{\otimes k}}) (L_{Q_0(W)^*})^{(k)}. \quad (5.3)$$

The following identities will be quite useful.

Proposition 5.4. *The scalar kernel* k_{Q_0} (5.3) *satisfies the following identities:*

$$k_{Q_0}(Z,W)(P) - Q_0(Z)\big(k_{Q_0}(Z,W)(P) \otimes I_{\mathcal{R}}\big) Q_0(W)^* = P, \quad (5.4)$$

$$k_{Q_0}(Z,W)\big(P - Q_0(Z)(P \otimes I_{\mathcal{R}}) Q_0(W)^*\big) = P. \quad (5.5)$$

Proof. To verify (5.4), note that

$$
\begin{aligned}
&Q_0(Z)\,(k_{Q_0}(Z,W)(P)\otimes I_{\mathcal{R}})\,Q_0(W)^* \\
&= \sum_{k=0}^{\infty} Q_0(Z)\left(\left[(L_{Q(Z)^*})^{(k)*}(P\otimes I_{\mathcal{R}^{\otimes k}})(L_{Q(W)^*})^{(k)}\right]\otimes I_{\mathcal{R}}\right) Q_0(W)^* \\
&= \sum_{k=0}^{\infty} Q_0(Z)\left((L_{Q_0(Z)^*})^{(k)*}\otimes I_{\mathcal{R}}\right)(P\otimes I_{\mathcal{R}^{\otimes k+1}})\left((L_{Q_0(W)^*})^{(k)}\otimes I_{\mathcal{R}}\right) Q_0(W)^* \\
&= \sum_{k=0}^{\infty}(L_{Q_0(Z)^*})^{(k+1)*}(P\otimes I_{\mathcal{R}^{\otimes k+1}})(L_{Q_0(W)^*})^{(k+1)} = k_{Q_0}(Z,W)(P) - P
\end{aligned}
$$

and (5.4) follows.

Similarly note that

$$
\begin{aligned}
&k_{Q_0}(Z,W)\big(Q_0(Z)(P\otimes I_{\mathcal{R}})Q_0(W)^*\big) \\
&= \sum_{k=0}^{\infty}(L_{Q_0(Z)^*})^{(k)*}\left(Q_0(Z)(P\otimes I_{\mathcal{R}})Q_0(W)^*\otimes I_{\mathcal{R}^{\otimes k}}\right)(L_{Q_0(W)^*})^{(k)} = \\
&\sum_{k=0}^{\infty}(L_{Q_0(Z)^*})^{(k)*}(Q_0(Z)\otimes I_{\mathcal{R}^{\otimes k}})(P\otimes I_{\mathcal{R}^{\otimes k+1}})(Q_0(W)^*\otimes I_{\mathcal{R}^{\otimes k}})(L_{Q_0(W)^*})^{(k)} \\
&= \sum_{k=0}^{\infty}\left(L_{Q_0(Z)^*}\right)^{(k+1)*}(P\otimes I_{\mathcal{R}^{\otimes k+1}})\left(L_{Q_0(W)^*}\right)^{(k+1)} = k_{Q_0}(Z,W)(P) - P
\end{aligned}
$$

verifying (5.5) □

Since the additional work is minimal, we shall consider the nc Pick interpolation problem for a pair of kernels $(k_{Q_0}\otimes I_{\mathcal{U}}, k_{Q_0}\otimes I_{\mathcal{Y}})$ in a more general left-tangential formulation:

Left-tangential nc Pick Interpolation Problem: *Given a nc function Q_0 from Ξ into $\mathcal{L}(\mathcal{R},\mathbb{C})_{\rm nc}$ defining the nc domain $\mathbb{D}_{Q_0}\subset\Xi$ as above along with a subset Ω of $\mathbb{D}_{Q_0}$ together with nc functions $a\in\mathcal{T}(\Omega';\mathcal{L}(\mathcal{Y},\mathcal{E})_{\rm nc})$ and $b\in\mathcal{T}(\Omega';\mathcal{L}(\mathcal{U},\mathcal{E})_{\rm nc})$ for auxiliary Hilbert spaces $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{E}$, where we set Ω' equal to the $\mathbb{D}_{Q_0}$-relative full nc envelope of Ω*

$$\Omega' = \Omega_{\rm nc,full}\cap\mathbb{D}_{Q_0}, \tag{5.6}$$

find a function S in the contractive multiplier class $\overline{\mathcal{B}}\mathcal{M}(k_{Q_0}\otimes I_{\mathcal{U}}, k_{Q_0}\otimes I_{\mathcal{Y}})$ so that

$$a(Z)S(Z) = b(Z) \text{ for all } Z\in\Omega. \tag{5.7}$$

Note that the special choice $a(Z) = I_{\mathcal{E}}$ and $b(Z) = S_0(Z)$, reduces the Left-Tangential nc Pick Interpolation Problem to the nc Pick Interpolation Problem.

By combining Proposition 5.4 with Theorem 3.1, we arrive at the following solution of the Left-Tangential nc Pick Interpolation Problem.

Theorem 5.5. *Suppose that Q_0, Ω, a, b are data for a Left-Tangential nc Pick Interpolation Problem as above. Then the following are equivalent:*

(1) *The Left-Tangential nc Pick Interpolation Problem has a solution, i.e., there exists an $S\colon \mathbb{D}_{Q_0} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc}$ in the contractive multiplier class $\overline{\mathcal{B}}\mathcal{M}(k_0 \otimes I_{\mathcal{U}}, k_0 \otimes I_{\mathcal{Y}})$ satisfying the left tangential interpolation conditions* (5.7).

(1′) *The inequality*

$$a(Z)a(Z)^* - b(Z)b(Z)^* \succeq 0 \tag{5.8}$$

holds for all $Z \in \Omega'$.

(2) *The generalized de Branges-Rovnyak kernel $K_{a,b}^{\rm dBR}$ given by*

$$K_{a,b}^{\rm dBR}(Z,W)(P) = a(Z)\,(k_{Q_0}(Z,W)(P) \otimes I_{\mathcal{Y}})\,a(W)^* - b(Z)\,(k_{Q_0}(Z,W)(P) \otimes I_{\mathcal{U}})\,b(W)^* \tag{5.9}$$

is a cp nc kernel on Ω.

(3) *There exists an auxiliary Hilbert space $\mathcal{X}$ and a contractive (even unitary) colligation matrix*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{R} \otimes \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \tag{5.10}$$

so that the function S defined by

$$S(Z) = D^{(n)} + C^{(n)}(I - (Q_0(Z) \otimes I_{\mathcal{X}})A^{(n)})^{-1}(Q_0(Z) \otimes I_{\mathcal{X}})B^{(n)} \tag{5.11}$$

for $Z \in \Omega_n$, where

$$\begin{bmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{bmatrix} = \begin{bmatrix} I_n \otimes A & I_n \otimes B \\ I_n \otimes C & I_n \otimes D \end{bmatrix} : \begin{bmatrix} \mathcal{X}^n \\ \mathcal{U}^n \end{bmatrix} \to \begin{bmatrix} (\mathcal{R} \otimes \mathcal{X})^n \\ \mathcal{Y}^n \end{bmatrix}.$$

satisfies the Left-Tangential Interpolation condition (5.7) *on Ω.*

Moreover, the implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) hold under the weaker assumption that a and b are only graded (rather than nc) functions defined only from Ω to $\mathcal{L}(\mathcal{Y}, \mathcal{E})_{\rm nc}$ and $\mathcal{L}(\mathcal{U}, \mathcal{E})_{\rm nc}$ respectively.

Proof. When Theorem 3.1 is specialized to the case $Q = Q_0$, it is clear that condition (1′) in Theorem 3.1 coincides exactly with condition (1′) in Theorem 5.5. Furthermore, as a consequence of Remark 2.20 and Theorem 2.20, condition (3) in Theorem 3.1 simplifies exactly to condition (3) in Theorem 5.5 (as also observed in the proof of Theorem 3.3 above). We shall now show that (i) conditions (2) in Theorem 3.1 (for the case $Q = Q_0$) and in Theorem 5.5 coincide and (ii) conditions (1) in Theorem 3.1 (for the case $Q = Q_0$) and in Theorem 5.5 coincide. Once these correspondences are established, Theorem 5.5 follows as an immediate consequence of Theorem 3.1.

Equivalence of conditions (2). A consequence of the identity (5.5) is that

$$\begin{aligned} K^{\rm dBR}_{a,b}(Z,W)\,(P - Q_0(Z)(P\otimes I_{\mathcal R})Q_0(W)^*) \\ = a(Z)(P\otimes I_{\mathcal Y})a(W)^* - b(Z)(P\otimes I_{\mathcal U})b(W)^* \\ = K_{a,b}(Z,W)(P). \end{aligned}$$

Thus $K^{\rm dBR}_{a,b}$ being a cp nc kernel implies that $K_{a,b}$ has an Agler decomposition (3.5) (with $\Gamma = K^{\rm dBR}_{a,b}$ and with Q_0 in place of Q). Conversely, suppose that $K_{a,b}$ has an Agler decomposition (3.5) (with respect to Q_0 rather than Q). Then there is a cp nc kernel Γ so that

$$\begin{aligned} a(Z)(P\otimes I_{\mathcal Y})a(W)^* - b(Z)(P\otimes I_{\mathcal U})b(W)^* \\ = \Gamma(Z,W)\,(P - Q_0(Z)(P\otimes I_{\mathcal R})Q_0(W)^*)\,. \end{aligned} \tag{5.12}$$

Then

$$\begin{aligned} & K^{\rm dBR}_{a,b}(Z,W)(P) \\ & = a(Z)\,(k_{Q_0}(Z,W)(P)\otimes I_{\mathcal Y}))\,a(W)^* - b(Z)\,(k_{Q_0}(Z,W)(P)\otimes I_{\mathcal U})\,b(W)^* \\ & = \Gamma(Z,W)\,(k_{Q_0}(Z,W)(P) - Q_0(Z)(k_{Q_0}(Z,W)(P)\otimes I_{\mathcal R})Q_0(W)^*) \\ & \quad \text{(by (5.12) with } K_{Q_0}(Z,W)(P) \text{ in place of } P) \\ & = \Gamma(Z,W)(P) \text{ (by (5.4))} \end{aligned}$$

implying that $K^{\rm dBR}_{a,b} = \Gamma$ is a cp nc kernel. The equivalence of the respective conditions (2) now follows.

Equivalence of conditions (1): Note the interpolation condition (3.3) in Theorem 3.1 coincides with the interpolation condition (5.7) required in Theorem 5.5. To establish the equivalence of the respective conditions (1), it suffices to show that the Schur-Agler class $\mathcal{S}A_{Q_0}(\mathcal U,\mathcal Y)$ coincides with the contractive multiplier class $\overline{\mathcal B}\mathcal M(k_{Q_0}\otimes I_{\mathcal U}, k_{Q_0}\otimes I_{\mathcal Y})$.

By Theorem 5.1, the nc function S is in the contractive multiplier class $\overline{\mathcal B}\mathcal M(k_{Q_0}\otimes I_{\mathcal U}, k_{Q_0}\otimes I_{\mathcal Y})$ if and only if the associated de Branges-Rovnyak kernel

$$K^{\rm dBR}_S(Z,W)(P) := k_{Q_0}(Z,W)(P)\otimes I_{\mathcal Y} - S(Z)(k_{Q_0}(Z,W)(P)\otimes I_{\mathcal U})S(W)^*$$

is a cp nc kernel on $\mathbb D_{Q_0}$. On the other hand, by the equivalence of (1) and (2) in Theorem 3.1 for the special case $\Omega = \mathbb D_{Q_0}$, $\mathcal E = \mathcal Y$, $a(Z) = I_{\mathcal Y^n}$ for $Z \in \mathbb D_{Q_0,n}$ and $b(Z) = S(Z)$, we see that S is in the Schur-Agler class $\mathcal{S}A_{Q_0}(\mathcal U,\mathcal Y)$ if and only if the kernel K_S given by

$$K_S(Z,W)(P) := P\otimes I_{\mathcal Y} - S(Z)(P\otimes I_{\mathcal U})S(W)^*$$

has an Agler decomposition (3.5) (with Q_0 in place of Q). By the calculation in the proof of the equivalence of conditions (2) (now specialized to the setting where $\Omega = \mathbb D_{Q_0}$, $\mathcal E = \mathcal Y$, $a(Z) = I_{\mathcal Y^n}$ for $Z \in \mathbb D_{Q_0,n}$ and $b(Z) = S(Z)$) these latter two conditions are equivalent. We conclude that $\mathcal{S}A_{Q_0}(\mathcal U,\mathcal Y) = \overline{\mathcal B}\mathcal M(k_{Q_0}\otimes I_{\mathcal U}, k_{Q_0}\otimes I_{\mathcal Y})$ as required.

This completes the proof of Theorem 5.5. □

Specializing the equivalence (1) ⇔ (2) in Theorem 5.5 to the case $\mathcal{E} = \mathcal{Y}$, $a(Z) = I_{\mathcal{Y}^n}$ for $Z \in \Omega_n$ yields the following corollary.

Corollary 5.6. *The kernel k_{Q_0} given by* (5.3) *is a complete Pick nc kernel.*

Remark 5.7. It is well known (see [72, 4] as well as [5, Theorem 7.6]) that in the classical case the Szegő kernel has a certain universal property with respect to complete Pick kernels. An open question which we leave for future work is to find the analogue of this result for the noncommutative-function setting.

Another corollary is the transfer-function realization characterization of contractive multipliers in this nc Szegő-kernel setting.

Corollary 5.8. *A graded function S mapping $\mathbb{B}^d_{nc}$ to $\mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc}$ is a nc function which is in the contractive-multiplier class $\overline{\mathcal{B}}\mathcal{M}(I_{\mathcal{U}} \otimes k_{\rm dBR}, I_{\mathcal{Y}} \otimes k_{\rm dBR})$ if and only if S can be realized in the transfer-function realization form* (5.11) *for $Z \in \mathbb{D}_{Q_0,n}$.*

Proof. The assertion of this corollary amounts to the equivalence (1) ⇔ (3) in Theorem 5.5 for the special case where $\Omega = \mathbb{D}_{Q_0}$, $\mathcal{E} = \mathcal{Y}$, $a(Z) = I_{\mathcal{Y}^n}$ for $Z \in \mathbb{D}_{Q_0,n}$, $b(Z) = S(Z)$. □

6. Examples and applications of the Schur-Agler class interpolation and realization theorems

In this final section we collect some additional illustrative special cases and examples of the general theory.

6.1. **Single-point interpolation and Stein domination of interpolation node by interpolation value.** The following proposition amounts to a more explicit formulation of Corollary 3.8. To better formulate the result, it is convenient to introduce the following definition. For the second definition below, just as in Case 1 of the proof of Theorem 3.1, we assume that the ambient vector space $\mathcal{V}$ for the set of points Ξ is $\mathbb{C}^d$ (with d chosen sufficiently large); then $\mathcal{V}$ carries an operator-space structure via the identification $\mathbb{C}^d \cong \mathcal{L}(\mathbb{C}^d, \mathbb{C})$. Given a point $Z^{(0)} \in \mathbb{D}_{Q,n}$ and an operator $\Lambda_0 \in \mathcal{L}(\mathcal{U}, \mathcal{Y})^{n \times n} \cong \mathcal{L}(\mathcal{U}^n, \mathcal{Y}^n)$, let us say that

(1) Λ_0 **dominates** $Q(Z^{(0)})$ **in the sense of Stein**, written as $\Lambda_0 \succeq_{\mathcal{S}} Q(Z^{(0)})$, if

$$\begin{aligned} 0 \preceq P \in \mathbb{C}^{n \times n},\ & P \otimes I_{\mathcal{S}} - Q(Z^{(0)})(P \otimes I_{\mathcal{R}})Q(Z^{(0)})^* \succeq 0 \\ & \Rightarrow P \otimes I_{\mathcal{Y}} - \Lambda_0 (P \otimes I_{\mathcal{U}}) \Lambda_0^* \succeq 0, \end{aligned} \tag{6.1}$$

and

(2) Λ_0 **dominates** $Q(Z^{(0)})$ **in the strict-Stein sense**, written as $\Lambda_0 \succeq_{s\mathcal{S}} Q(Z^{(0)})$, if, whenever it is the case that there is a $\delta > 0$ so that for any $P \in \mathbb{C}^{n \times n}$ with $P \succeq 0$ such that the two inequalities

$$
\begin{aligned}
& P - (1-\delta^2) Z^{(0)} (P \otimes I_{\mathbb{C}^d}) Z^{(0)*} \succeq 0, \\
& (1-\delta^2) P \otimes I_{\mathcal{S}} - Q(Z^{(0)})(P \otimes I_{\mathcal{R}}) Q(Z^{(0)})^* \succeq 0 \qquad (6.2)
\end{aligned}
$$

hold, it follows that

$$P \otimes I_{\mathcal{Y}} - \Lambda_0 (P \otimes I_{\mathcal{U}}) \Lambda_0^* \succeq 0. \qquad (6.3)$$

Then we have the following refinement of Corollary 3.8.

Proposition 6.1. *Let Q and $\mathbb{D}_Q$ be as in Theorem 3.1 and suppose that $Z^{(0)} \in \mathbb{D}_{Q,n}$ and $\Lambda_0 \in \mathcal{L}(\mathcal{U}^n, \mathcal{Y}^n)$ for some $n \in \mathbb{N}$. Consider the following two statements:*

(1) *There exists a function S in the Schur-Agler class $\mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ satisfying the interpolation condition*

$$S(Z^{(0)}) = \Lambda_0. \qquad (6.4)$$

(2) *For $k = n \cdot \dim \mathcal{Y}$, $\bigoplus_1^k \Lambda_0$ dominates $\bigoplus_1^k Q(Z^{(0)})$ in the strict-Stein sense* (6.2)–(6.3).

Then (1) ⇒ (2). If we assume in addition that there exists a nc function S_0 (not necessarily contractive) on Ω_0' (defined as in Remark 4.3 with $\Omega = \{Z^{(0)}\}$) with $S_0(Z^{(0)}) = \Lambda_0$, then (2) ⇒ (1).

Proof. Suppose that the interpolation problem (6.4) has a Schur-Agler class solution $S \in \mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ and that $P \in \mathbb{C}^{k \cdot n \times k \cdot n}$ satisfies the hypotheses (6.2) for the strict-Stein dominance condition $\bigoplus_1^k \Lambda_0 \succeq_{s\mathcal{S}} \bigoplus_1^k Q(Z^{(0)})$, namely:

$$P - (1-\delta^2) \left(\bigoplus_1^k Z^{(0)} \right) (P \otimes I_{\mathbb{C}^d}) \left(\bigoplus_1^k Z^{(0)} \right)^* \succeq 0 \qquad (6.5)$$

and

$$(1-\delta^2) P \otimes I_{\mathcal{S}} - \left(\bigoplus_1^k Q(Z^{(0)}) \right) (P \otimes I_{\mathcal{R}}) \left(\bigoplus_1^k Q(Z^{(0)}) \right)^* \succeq 0. \qquad (6.6)$$

Let m be the rank of P and let $P = \mathcal{I}\mathcal{I}^*$ be a full-rank factorization of P with an injective $\mathcal{I}$ of size $k \cdot n \times m$. An application of the Douglas lemma (see [35]) applied to the condition (6.5) with $\mathcal{I}\mathcal{I}^*$ substituted for P gives the existence of a $\widetilde{Z} \in \mathcal{V}^{m \times m}$ with norm-squared at most $(1-\delta^2)^{-1}$ solving the factorization problem

$$\mathcal{I} \widetilde{Z} = \left(\bigoplus_1^k Z^{(0)} \right) (\mathcal{I} \otimes I_{\mathbb{C}^d}). \qquad (6.7)$$

By the "respects intertwinings" property of the nc function Q, we then have

$$(\mathcal{I} \otimes I_{\mathcal{S}})Q(\widetilde{Z}) = \left(\bigoplus_1^k Q(Z^{(0)}) \right) (\mathcal{I} \otimes I_{\mathcal{R}}). \tag{6.8}$$

A similar application of the Douglas lemma applied to the inequality (6.6) assures us that there is an operator $Y \in \mathcal{L}(\mathcal{R}^m, \mathcal{S}^m)$ with $\|Y\|^2 \le 1 - \delta^2 < 1$ such that

$$(\mathcal{I} \otimes I_{\mathcal{S}})Y = \left(\bigoplus_1^k Q(Z^{(0)}) \right) (\mathcal{I} \otimes I_{\mathcal{R}}). \tag{6.9}$$

As $\mathcal{I}$ is injective, we see from (6.8) and (6.9) that $Y = Q(\widetilde{Z})$. As $\|Q(\widetilde{Z})\| = \|Y\| < 1$, we conclude from (6.7) that $\widetilde{Z} \in \Omega_0' \subset \mathbb{D}_Q$. As S by assumption is in the nc Schur-Agler class $\mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$, we conclude that $\|S(\widetilde{Z})\| \le 1$. From the intertwining relation (6.7) and the fact that the nc function S respects direct sums and intertwinings, we see that we must have

$$(\mathcal{I} \otimes I_{\mathcal{Y}})S(\widetilde{Z}) = \left(\bigoplus_1^k S(Z^{(0)}) \right) (\mathcal{I} \otimes I_{\mathcal{U}}).$$

From the fact that $\|S(\widetilde{Z})\| \le 1$, i.e., $I_{\mathcal{Y}^m} - S(\widetilde{Z})S(\widetilde{Z})^* \succeq 0$, we get

$$\begin{aligned} P \otimes I_{\mathcal{Y}} &- \left(\bigoplus_1^k \Lambda_0 \right) (P \otimes I_{\mathcal{U}}) \left(\bigoplus_1^k \Lambda_0 \right)^* \\ &= \mathcal{I}\mathcal{I}^* \otimes I_{\mathcal{Y}} - \left(\bigoplus_1^k S(Z^{(0)}) \right) (\mathcal{I}\mathcal{I}^* \otimes I_{\mathcal{U}}) \left(\bigoplus_1^k S(Z^{(0)}) \right)^* \\ &= (\mathcal{I} \otimes I_{\mathcal{Y}}) \left(I_{\mathcal{Y}^m} - S(\widetilde{Z})S(\widetilde{Z})^* \right) (\mathcal{I}^* \otimes I_{\mathcal{Y}}) \succeq 0 \end{aligned}$$

and hence

$$P \otimes I_{\mathcal{Y}} - \left(\bigoplus_1^k \Lambda_0 \right) (P \otimes I_{\mathcal{U}}) \left(\bigoplus_1^k \Lambda_0 \right)^* \succeq 0, \tag{6.10}$$

i.e., the inequality (6.3) holds with $\bigoplus_1^k \Lambda_0$ in place of Λ_0. This completes the proof that the strict-Stein dominance condition $\bigoplus_1^k \Lambda_0 \succeq_{s\mathcal{S}} \bigoplus_1^k Q(Z^{(0)})$ holds.

Conversely, suppose that there is a nc function S_0 on Ω_0' with $S(Z^{(0)}) = \Lambda_0$ and that $\bigoplus_1^k \Lambda_0 \succeq_{s\mathcal{S}} \bigoplus_1^k Q(Z^{(0)})$. By Remark 4.3 applied to the case $a(Z^{(0)}) = I_{\mathcal{Y}^n}$, $b(Z^{(0)}) = \Lambda_0$, $\mathcal{E} = \mathcal{Y}$, to verify that there exists an $S \in \mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ with $S(Z^{(0)}) = \Lambda_0$, it suffices to show that the function S_0 must in fact be contractive. Let $\widetilde{Z}$ be a point of Ω_0'. Then there is an injective $\mathcal{I} \in \mathbb{C}^{k \cdot n \times m}$ (for some m with $1 \le m \le k \cdot n$) so that

$$\mathcal{I}\widetilde{Z} = \left(\bigoplus_1^k Z^{(0)} \right) (\mathcal{I} \otimes I_{\mathbb{C}^d}). \tag{6.11}$$

If $\widetilde{Z}$ has $\|\widetilde{Z}\|^2 \leq \frac{1}{1-\delta^2}$ for some $\delta < 1$, it follows that $I_{\mathbb{C}^m} - (1-\delta^2)\widetilde{Z}\widetilde{Z}^* \succeq 0$. It then follows that

$$\begin{aligned} &\mathcal{I}\mathcal{I}^* - (1-\delta^2)\left(\bigoplus_1^k Z^{(0)}\right)(\mathcal{I}\mathcal{I}^* \otimes I_{\mathbb{C}^d})\left(\bigoplus_1^k Z^{(0)}\right)^* \\ &\quad = \mathcal{I}\left(I_{\mathbb{C}^m} - (1-\delta^2)\widetilde{Z}\widetilde{Z}^*\right)\mathcal{I}^* \succeq 0. \end{aligned}$$

If we set $P = \mathcal{I}\mathcal{I}^*$, we then see that $P \succeq 0$ in $\mathbb{C}^{k\cdot n \times k \cdot n}$ satisfies (6.5). For this $\widetilde{Z}$ to be in Ω_0', we also require that $\|Q(\widetilde{Z})\| < 1$. Choosing $\delta < 1$ sufficiently close to 1, we may assume that $\|Q(\widetilde{Z})\|^2 < 1-\delta^2$, or $(1-\delta^2)I_{\mathcal{S}^m} - Q(\widetilde{Z})Q(\widetilde{Z})^* \succeq 0$. Combining this with the intertwining (6.11) then leads to the conclusion

$$\begin{aligned} &(1-\delta^2)\mathcal{I}\mathcal{I}^* \otimes I_{\mathcal{S}} - \left(\bigoplus_1^k Q(Z^{(0)})\right)(\mathcal{I}\mathcal{I}^* \otimes I_{\mathcal{R}})\left(\bigoplus_1^k Q(Z^{(0)})\right)^* \\ &\quad = (\mathcal{I} \otimes I_{\mathcal{S}})\left((1-\delta^2)I_{\mathcal{S}^m} - Q(\widetilde{Z})Q(\widetilde{Z})^*\right)(\mathcal{I}^* \otimes I_{\mathcal{S}}) \succeq 0 \end{aligned}$$

and consequently P also satisfies (6.6). As the inequalities (6.5)–(6.6) amount to the assumptions (6.2) in the strict-Stein dominance condition $\bigoplus_1^k \Lambda_0 \succeq_{s\mathcal{S}} \bigoplus_1^k Q(Z^{(0)})$, our standing hypothesis that this strict-Stein dominance condition holds implies that that (6.3) holds (with $\bigoplus_1^k Q(Z^{(0)})$ in place of $Z^{(0)}$ and $\bigoplus_1^k \Lambda_0$ in place of Λ_0), i.e., that

$$\mathcal{I}\mathcal{I}^* \otimes I_{\mathcal{Y}} - \left(\bigoplus_1^k \Lambda_0\right)(\mathcal{I}\mathcal{I}^* \oplus I_{\mathcal{U}})\left(\bigoplus_1^k \Lambda_0\right)^* \succeq 0.$$

An application of the Douglas lemma [35] then gives us the existence of an operator $Y \in \mathcal{L}(\mathcal{U}^k, \mathcal{Y}^k)$ with $\|Y\| \leq 1$ solving the factorization problem

$$(\mathcal{I} \otimes I_{\mathcal{Y}})Y = \left(\bigoplus_1^k \Lambda_0\right)(\mathcal{I} \otimes I_{\mathcal{U}}). \tag{6.12}$$

As S_0 is a nc function on Ω_0' with $S_0(Z^{(0)}) = \Lambda_0$, from the relation (6.11) we deduce that

$$(\mathcal{I} \otimes I_{\mathcal{Y}})S_0(\widetilde{Z}) = \left(\bigoplus_1^k \Lambda_0\right)(\mathcal{I} \otimes I_{\mathcal{U}}). \tag{6.13}$$

As $\mathcal{I}$ is injective, from (6.12) and (6.13) we deduce that $S_0(\widetilde{Z}) = Y$ and hence $\|S_0(\widetilde{Z})\| = \|Y\| \leq 1$. As $\widetilde{Z}$ was an arbitrary point in Ω_0', we see that in fact S_0 is contractive on Ω_0'. It now follows from the content of Remark 4.3 that there exists a nc Schur-Agler class function $S \in \mathcal{SA}_Q(\mathcal{U}, \mathcal{Y})$ satisfying the interpolation condition $S(Z^{(0)}) = \Lambda_0$. □

In the setting of Section 5 where $\mathcal{S} = \mathbb{C}$ and $Q(z) = Q_0(z)$ in the notation there, it is possible to use the complete positivity condition in statement (2) of Theorem 5.5 to get a more definitive version of the result in Proposition 6.1 for this case.

Proposition 6.2. *Suppose that Q_0 is as in Section 5, $Z^{(0)} \in \mathbb{D}_{Q_0,n}$ and $\Lambda_0 \in \mathcal{L}(\mathcal{U},\mathcal{Y})^{n\times n}$. Then the following conditions are equivalent:*

(1) *There exists S in the Schur-Agler class $\mathcal{SA}_{Q_0}(\mathcal{U},\mathcal{Y})$, or equivalently in the contractive multiplier class $\overline{\mathcal{B}}\mathcal{M}(k_{Q_0}\otimes I_{\mathcal{U}}, k_{Q_0}\otimes I_{\mathcal{Y}})$, satisfying the interpolation condition*

$$S(Z^{(0)}) = \Lambda_0. \tag{6.14}$$

(2) *$\bigoplus_1^n \Lambda_0$ dominates $\bigoplus_1^n Q_0(Z^{(0)})$ in the sense of Stein* (6.1).

Remark 6.3. As explained in the proof of Theorem 5.5 ("equivalence of conditions (1)"), the contractive multiplier class $\overline{\mathcal{B}}\mathcal{M}(k_{Q_0}\otimes I_{\mathcal{U}}, k_{Q_0}\otimes I_{\mathcal{Y}})$ coincides with the Schur-Agler class $\mathcal{SA}_{Q_0}(\mathcal{U},\mathcal{Y})$. Thus Proposition 6.1 applied to this case tells us that the interpolation problem (6.14) has a contractive multiplier solution $S \in \overline{\mathcal{B}}\mathcal{M}(k_{Q_0}\otimes I_{\mathcal{U}}, k_{Q_0}\otimes I_{\mathcal{Y}})$ if and only if the strict-Stein dominance $\bigoplus_1^k \Lambda_0 \succeq_{s\mathcal{S}} \bigoplus_1^k Q(Z^{(0)})$ with $k = n\cdot \dim \mathcal{Y}$ (together with an extra hypothesis for the converse direction) holds. The content of Proposition 6.2 is that, for the special case where $Q = Q_0$ has target space $\mathcal{S} = \mathbb{C}$, the result of Proposition 6.1 holds with strict-Stein dominance (6.2)–(6.3) replaced by simple Stein dominance (6.1), $n \le k = n\cdot \dim \mathcal{E}$ replacing k, and with removal of the extra hypothesis for the converse direction.

Proof of Proposition 6.2. By the equivalence of (1) and (2) in Theorem 5.5, we see that there is a contractive multiplier solution $S \in \overline{\mathcal{B}}\mathcal{M}(k_{Q_0}\otimes I_{\mathcal{U}}, k_{Q_0}\otimes I_{\mathcal{Y}})$ of the interpolation condition $S(Z^{(0)}) = \Lambda_0$ if and only if the map

$$P \mapsto k_{Q_0}(Z^{(0)}, Z^{(0)})(P)\otimes I_{\mathcal{Y}} - \Lambda_0\left(k_{Q_0}(Z^{(0)}, Z^{(0)})(P)\otimes I_{\mathcal{U}}\right)\Lambda_0^*$$

is a completely positive map from $\mathbb{C}^{n\times n}$ into $\mathcal{L}(\mathcal{Y})^{n\times n}$. By a result of M.-D. Choi (see [68, Theorem 3.14], it suffices to check that this map in n-positive, i.e., that the map

$$\begin{aligned} P \mapsto\ & k_{Q_0}\left(\bigoplus_1^n Z^{(0)}, \bigoplus_1^n Z^{(0)}\right)(P)\otimes I_{\mathcal{Y}} \\ & - \left(\bigoplus_1^n \Lambda_0\right)\left(k_{Q_0}\left(\bigoplus_1^n Z^{(0)}, \bigoplus_1^n Z^{(0)}\right)(P)\otimes I_{\mathcal{U}}\right)\left(\bigoplus_1^n \Lambda_0\right)^* \end{aligned} \tag{6.15}$$

is a positive map from $\mathbb{C}^{n^2\times n^2}$ into $\mathcal{L}(\mathcal{Y})^{n^2\times n^2}$. If we set

$$R = k_{Q_0}\left(\bigoplus_1^n Z^{(0)}, \bigoplus_1^n Z^{(0)}\right)(P),$$

then according to the identity (5.4) we recover P from R via

$$P = R - \left(\bigoplus_1^n Q_0(Z^{(0)})\right) R \left(\bigoplus_1^n Q_0(Z^{(0)})\right)^* .$$

Then the condition that the map (6.15) be positive can be reformulated as:

$$R \succeq 0 \text{ such that } R - \left(\bigoplus_1^n Q(Z^{(0)})\right) R \left(\bigoplus_1^n Q_0(Z^{(0)})\right)^* \succeq 0$$

$$\Rightarrow \quad R - \left(\bigoplus_1^n \Lambda_0\right) R \left(\bigoplus_1^n \Lambda_0\right)^* \succeq 0$$

This in turn amounts to the Stein dominance condition $\bigoplus_1^n \Lambda_0 \succeq_{\mathcal{S}} \bigoplus_1^n Q(Z^{(0)})$ (see (6.1)). ☐

Remark 6.4. Propositions 6.1 and 6.2 were inspired by the work of Cohen-Lewkowicz [30, 31] on the so-called Lyapunov order on real symmetric matrices and the connection of this with the Pick-matrix criterion for interpolation by positive real odd functions (roughly, the real right-half-plane analogue of the classical case of our topic here).

6.2. **Finite-Point Left-Tangential Pick Interpolation Problem.** Let us consider the special case of Theorem 5.5 where $\mathcal{V} = \Xi = \mathbb{C}^d$. We then write points Z in $\mathcal{V}_{\mathrm{nc},n} = (\mathbb{C}^d)^{n\times n}$ as d tuples $Z = (Z_1, \dots, Z_d) \in (\mathbb{C}^{n\times n})^d \cong (\mathbb{C}^d)^{n\times n}$. As a nc function Q on $\mathbb{C}^d_{\mathrm{nc}}$ we choose $Q = Q_{\mathrm{row}}$ given by

$$Q_{\mathrm{row}}(Z) = Q(Z_1, \dots, Z_d) = \begin{bmatrix} Z_1 & \cdots & Z_d \end{bmatrix}.$$

The resulting nc domain $\mathbb{D}_{Q_{\mathrm{row}}}$ then amounts to the nc operator ball

$$\mathbb{B}^d_{\mathrm{nc}} = \amalg_{n=1}^\infty \{Z = (Z_1, \dots, Z_d) \in (\mathbb{C}^{n\times n})^d \colon Z_1 Z_1^* + \cdots + Z_d Z_d^* \prec I_n\}.$$

For this case the formula for the generalized Szegő kernel can be written out more concretely in coordinate form as

$$k_{Q_{\mathrm{row}}}(Z, W)(P) = \sum_{\mathfrak{a} \in \mathbb{F}_d^+} Z^{\mathfrak{a}} P W^{*\mathfrak{a}^\top}$$

where we use nc functional calculus conventions as in Theorem 1.4 in the Introduction: for $\mathfrak{a} = (i_1, \dots, i_N)$ in the unital free semigroup $\mathbb{F}_d^+$ and for $Z = (Z_1, \dots, Z_d)$ and $W = (W_1, \dots, W_d)$ in $\mathbb{B}^d_{\mathrm{nc}}$,

$$Z^{\mathfrak{a}} = Z_{i_N} \cdots Z_{i_1}, \quad W^* = (W_1^*, \dots, W_d^*), \quad W^{*\mathfrak{a}^\top} = W_{i_1}^* \cdots W_{i_N}^*.$$

Then nc functions $S \in \mathcal{T}(\mathbb{B}^d_{\mathrm{nc}}; \mathcal{L}(\mathcal{U}, \mathcal{Y})_{nc})$ are given by a power-series representation as in (1.7)

$$S(Z) = \sum_{\mathfrak{a} \in \mathbb{F}_d^+} S_{\mathfrak{a}} \otimes Z^{\mathfrak{a}} \text{ for } Z = (Z_1, \dots, Z_d) \in \mathbb{B}^d_{\mathrm{nc}}. \tag{6.16}$$

We consider the **Finite-Point Left-Tangential Pick Interpolation Problem** for the contractive multiplier class associated with the kernel $k_{Q_{\mathrm{row}}}$:

Given $Z^{(1)}, \dots, Z^{(N)} \in \mathbb{B}^d_{\rm nc}$ *along with vectors* $A_i \in \mathcal{L}(\mathcal{U}^{n_i}, \mathcal{E}_i)$, $B_i \in \mathcal{L}(\mathcal{Y}^{n_i}, \mathcal{E}_i)$ *(*n_i *chosen so that* $Z^{(i)} \in \mathbb{B}^d_{{\rm nc},n_i}$*), find* $S \in \overline{\mathcal{B}}\mathcal{M}(k_{Q_{\rm row}} \otimes I_{\mathcal{U}}, k_{Q_{\rm row}} \otimes I_{\mathcal{Y}})$ *with*

$$A_i S(Z^{(i)}) = B_i \text{ for } i = 1, \dots, N.$$

Using the nc function structure, one can reduce any finite-point problem to a single-point problem; specifically take the single-point data set $(Z^{(0)}, A_0, B_0)$ to be

$$Z^{(0)} = \begin{bmatrix} Z^{(1)} & & \\ & \ddots & \\ & & Z^{(N)} \end{bmatrix}, \quad A_0 = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{bmatrix}, \quad B_0 = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_N \end{bmatrix}.$$

Thus we simplify the discussion here by considering only the single-point version of the Left-Tangential Pick Interpolation Problem with data set $\{Z^{(0)}, A_0, B_0\}$. The equivalence of (1) $\Leftrightarrow$ (2) in Theorem 5.5 leads to the following result.

Theorem 6.5. *Suppose that we are given the data set* $(Z^{(0)}, A_0, B_0)$ *for a Single-Point Left-Tangential Pick Interpolation Problem. Then the following are equivalent:*

(1) *The interpolation problem has a solution, i.e., there exists* S *in* $\overline{\mathcal{B}}\mathcal{M}(k_{Q_{\rm row}} \otimes I_{\mathcal{U}}, k_{Q_{\rm row}} \otimes I_{\mathcal{U}})$ *with*

$$A_0 S(Z^{(0)}) = B_0. \tag{6.17}$$

(2) *The map*

$$P \mapsto \sum_{\mathfrak{a} \in \mathbb{F}^+_d} \Big(A_0 (Z^{(0)\mathfrak{a}} P Z^{(0)*\mathfrak{a}^\top} \otimes I_{\mathcal{Y}}) A_0^* - B_0 (Z^{(0)\mathfrak{a}} P Z^{(0)*\mathfrak{a}^\top} \otimes I_{\mathcal{U}}) B_0^* \Big) \tag{6.18}$$

is completely positive.

This particular nc Pick interpolation problem has already been considered by some other authors. Our result Theorem 6.5 agrees with the result of Muhly-Solel in [62, Theorem 6.3] and is a slight variation of a result of Popescu [71, Corollary 2.3]. We mention that Muhly-Solel actually considered a much more general setting where the multiplier algebra $\mathcal{M}(k_{Q_{\rm row}} \otimes I_{\mathcal{U}}, k_{Q_{\rm row}} \otimes I_{\mathcal{Y}})$ is replaced by the generalized Hardy algebra H^∞_E associated with a correspondence E over a von Neumann algebra M (see also [61, 63]). An interesting topic for future research is to get a better understanding about how this general von Neumann-algebra correspondence setting fits with the free nc-function setting used here; preliminary steps in such a program have already been made in [16, 65, 67].

6.3. **NC-function versus Left-Tangential Operator-Argument point evaluation: the nc ball setting.** A popular formalism for handling intricate univariate matrix-valued interpolation problems over the years has been to make use of a **Left-Tangential Operator Argument** point-evaluation (see e.g. [20, 44]). A nc version of the Left Tangential Operator Argument point evaluation (as well as right and two-sided versions which we need not go into here) was introduced in [19], along with a study of associated interpolation problems. We now describe one such result for the nc ball setting in this Subsection.

Suppose that $Z^{(0)} = (Z_1^{(0)}, \dots, Z_d^{(0)})$ is a point in $\mathbb{B}^d_{\text{nc},n}$, X is in operator in $\mathcal{L}(\mathcal{Y}, \mathbb{C}^n)$, and S is a nc operator-valued function in $\mathcal{T}(\mathbb{B}^d_{\text{nc}}; \mathcal{L}(\mathcal{U}, \mathcal{Y})_{\text{nc}})$ with associated formal power series $S(z) = \sum_{\mathfrak{a} \in \mathbb{F}_d^+} S_{\mathfrak{a}} z^{\mathfrak{a}}$. We define $(XS)^{\wedge L}(Z^{(0)})$ (the Left-Tangential Operator Argument evaluation of S at $Z^{(0)}$ in direction X) by

$$(XS)^{\wedge L}(Z^{(0)}) = \sum_{\mathfrak{a} \in \mathbb{F}_d^+} Z^{(0)\mathfrak{a}^\top} X S_{\mathfrak{a}} \in \mathcal{L}(\mathcal{U}, \mathbb{C}^n). \tag{6.19}$$

Note that in contrast to the nc-function point evaluation (6.16), the power on $Z^{(0)}$ involves $\mathfrak{a}^\top$ rather than $\mathfrak{a}$ and all multiplications in (6.19) are operator compositions (no tensor products). Given an interpolation data set $(Z^{(0)}, X, Y)$, where $Z^{(0)}$, X are as above along with an operator Y in $\mathcal{L}(\mathcal{Y}, \mathbb{C}^n)$, the **Left-Tangential Operator Argument**(**LTOA**) interpolation problem is: *find $S \in \overline{\mathcal{B}}\mathcal{M}(k_{Q_{\text{row}}} \otimes \mathcal{U}, k_{Q_{\text{row}}} \otimes \mathcal{Y})$ (or equivalently in $\mathcal{SA}_{Q_{\text{row}}}(\mathcal{U}, \mathcal{Y})$ so that the Left-Tangential Operator Argument interpolation condition*

$$(XS)^{\wedge L}(Z^{(0)}) = Y \tag{6.20}$$

holds. The solution is as follows (see [19, Theorem 7.4] as well as [32, Theorem 3.4] for a different but equivalent formulation).

Theorem 6.6. *Suppose that we are given the data set $(Z^{(0)}, X, Y)$ for a LTOA interpolation problem for $S \in \overline{\mathcal{B}}\mathcal{M}(k_{Q_{\text{row}}} \otimes \mathcal{U}, k_{Q_{\text{row}}} \otimes \mathcal{Y})$ as above. Then the LTOA interpolation problem has a solution if and only if*

$$\sum_{\mathfrak{a} \in \mathbb{F}_d^+} Z^{(0)\mathfrak{a}} (XX^* - YY^*) Z^{(0)*\mathfrak{a}^\top} \succeq 0.$$

There is a curious connection between the nc-function Left-Tangential Pick Interpolation Problem versus the LTOA Interpolation Problem which we now discuss. Let us again restrict to the scalar nc Schur class $\mathcal{SA}_{Q_{\text{row}}}(\mathbb{C}) = \overline{\mathcal{B}}\mathcal{M}(k_{Q_{\text{row}}})$. Choose a point $Z^{(0)} \in \mathbb{B}^d_{\text{nc},n}$ along with an a matrix $\Lambda_0 \in \mathbb{C}^n$ and consider the single-point interpolation problem: *find $s \in \mathcal{SA}_{Q_{\text{row}}}(\mathbb{C})$ so that the interpolation condition*

$$s(Z^{(0)}) = \Lambda_0. \tag{6.21}$$

holds. According to Theorem 6.5, this interpolation problem has a solution if and only if the map from $\mathbb{C}^{n \times n}$ into $\mathcal{L}(\mathcal{E})$ given by

$$P \mapsto \sum_{\mathfrak{a} \in \mathbb{F}_d^+} \left(Z^{(0)\mathfrak{a}} P Z^{(0)*\mathfrak{a}^\top} - \Lambda_0 Z^{(0)\mathfrak{a}} P Z^{(0)*\mathfrak{a}^\top} \Lambda_0^* \right) \tag{6.22}$$

is completely positive. As the domain for this map is $\mathbb{C}^{n \times n}$, the Choi criterion (see [68, Theorem 3.14]) gives a test for complete positivity in terms of positive definiteness of a single operator: *the map* (6.22) *is completely positive if and only if the block matrix*

$$\left[\sum_{\mathfrak{a} \in \mathbb{F}_d^+} \left(Z^{(0)\mathfrak{a}} e_\kappa e_{\kappa'}^* Z^{(0)*\mathfrak{a}^\top} - \Lambda_0 Z^{(0)\mathfrak{a}} e_\kappa e_{\kappa'}^* Z^{(0)*\mathfrak{a}^\top} \Lambda_0^* \right) \right]_{\kappa, \kappa' \in \mathfrak{B}} \tag{6.23}$$

is positive semidefinite, where $\{e_\kappa \colon \kappa \in \mathfrak{B}\}$ is the standard basis for $\mathbb{C}^n$.

On the other hand, we can view the interpolation condition as a twisted version of a LTOA interpolation condition as follows. Note that the interpolation condition (6.21) can also be expressed as

$$s(Z^{(0)}) e_\kappa = \Lambda_0 e_\kappa \text{ for each } \kappa \in \mathfrak{B}.$$

Writing this condition out in terms of a series and using that the coefficients $s_\mathfrak{a}$ are scalar, we get

$$\sum_{\mathfrak{a} \in \mathbb{F}_d^+} s_\mathfrak{a} Z^{(0)\mathfrak{a}} e_\kappa = \sum_{\mathfrak{a} \in \mathbb{F}_d^+} Z^{(0)\mathfrak{a}} e_\kappa s_\mathfrak{a} = \Lambda_0 e_\kappa \text{ for each } \kappa \in \mathfrak{B}. \tag{6.24}$$

Let us introduce the twisted LTOA point evaluation

$$(XS)^{\wedge \tau \circ L}(Z) = \sum_{\mathfrak{a} \in \mathbb{F}_d^+} Z^\mathfrak{a} X S_\mathfrak{a}, \tag{6.25}$$

i.e., the formula (6.19) but with the power of $Z^{(0)}$ equal to $\mathfrak{a}$ instead of $\mathfrak{a}^\top$. Then the nc-function interpolation condition (6.21) can be reexpressed as the set of nc twisted LTOA interpolation conditions

$$(e_\kappa s)^{\tau \circ L}(Z^{(0)}) = \Lambda_0 e_\kappa \text{ for } \kappa \in \mathfrak{B}.$$

If we introduce the column vector $E = \operatorname{col}_{\kappa \in \mathfrak{B}}[e_\kappa]$, we can convert the problem to a single twisted LTOA interpolation condition

$$(Es)^{\wedge \tau \circ L}(\oplus_{\kappa \in \mathfrak{B}} Z^{(0)}) = \left(\bigoplus_{\kappa \in \mathfrak{B}} \Lambda_0 \right) E. \tag{6.26}$$

By our previous analysis we know that the positive semidefiniteness of the matrix (6.23) is necessary and sufficient for there to be a solution $s \in \mathcal{S}A_{Q_{\rm row}}(\mathbb{C})$ of the interpolation condition (6.26).

The same data set $(\oplus_{\kappa \in \mathfrak{B}} Z^{(0)}, E, (\bigoplus_1^n \Lambda_0) E)$ is the data set for a (untwisted) LTOA interpolation problem: *find* $s \in \mathcal{S}A_{Q_{\rm row}}(\mathbb{C})$ *such that*

$$(Es)^{\wedge L} \left(\oplus_{\kappa \in \mathfrak{B}} Z^{(0)} \right) = \left(\bigoplus_{\kappa \in \mathfrak{B}} \Lambda_0 \right) E.$$

or equivalently, *such that*

$$\sum_{\mathfrak{a} \in \mathbb{F}_d^+} s_{\mathfrak{a}^\top} Z^\mathfrak{a} = \Lambda_0. \tag{6.27}$$

The solution criterion for this problem is positive semidefiniteness of the block matrix

$$\left[\sum_{\mathfrak{a} \in \mathbb{F}_d^+} Z^{(0)\mathfrak{a}} (e_\kappa e_{\kappa'}^* - \Lambda_0 e_\kappa e_{\kappa'}^* \Lambda_0^*) Z^{(0)* \mathfrak{a}^\top} \right]_{\kappa, \kappa' \in \mathfrak{B}} . \tag{6.28}$$

or equivalently (by the Choi test), complete positivity of the map

$$P \mapsto \sum_{\mathfrak{a} \in \mathbb{F}_d^+} Z^{(0)\mathfrak{a}} (P - \Lambda_0 P \Lambda_0^*) Z^{(0)* \mathfrak{a}^\top}. \tag{6.29}$$

Note that the problems (6.26) and (6.27) are the same in case the components $Z_1^{(0)}, \dots, Z_d^{(0)}$ commute with each other (so $Z^{\mathfrak{a}} = Z^{\mathfrak{a}^\top}$). A consequence of the "respects intertwinings" condition for nc functions is that Λ_0 must be in the double commutant of the collection $Z_1^{(0)}, \dots, Z_d^{(0)}$ if $\Lambda_0 = S(Z^{0)})$ for a nc function S. Thus, for the case of commutative d-tuple $Z^{(0)}$, positivity of the matrix (6.23) in fact implies that $\Lambda^{(0)}$ commutes with each $Z_k^{(0)}$ and the matrices (6.23) and (6.28) are the same, consistent with positivity of either being the solution criterion for existence of a solution to the same problem (6.21) or (6.27). In case $Z^{(0)}$ is not a commutative tuple, we are led to the conclusion that the interpolation conditions (6.21) and (6.27) are different problems with each having its own independent solution criterion, positive semidefiniteness of (6.23) and of (6.28) respectively. For the case of commuting variables, nc-function point-evaluation (or Riesz-Dunford) interpolation conditions can be reduced to the older theory of LTOA interpolation conditions and one can recover the solution criterion of one from the solution criterion for the other; this point is explored in more detail in [24]. The recent paper of Norton [67] explores similar connections between the interpolation theory of Constantinescu-Johnson [32] and that of Muhly-Solel [62].

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