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Systems described by very rapid switching between two m-dissipative operators

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Motivation - finite-dimensional

The phenomenon of the dynamics of a system switching fast between two (or several) possible configurations is frequently encountered in control theory (especially sliding mode control) and in power electronics. In the finite-dimensional LTI version of a switched system, we consider two linear subsystems

$$\Sigma : \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases}$$

$$\widetilde{\Sigma} : \quad \begin{cases} \dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t), \\ y(t) = \tilde{C}x(t) + \tilde{D}u(t), \end{cases}$$

with input u, state x and output y. We consider $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$, where U, X and Y are finite dimensional inner product spaces.

Finite-dimensional LTI averaging

The switched system that we are interested in switches very fast (with higher frequency than the bandwidth of u, y) between Σ and $\tilde{\Sigma}$, and spends equal amounts of time in Σ and $\tilde{\Sigma}$.

The commonly used *average model* Σ_{ave} of the switched system is then given by a similar set of equations as Σ, but the four matrices A, B, C, D are replaced with

$$A_{ave} = \frac{1}{2}(A + \tilde{A}), \quad B_{ave} = \frac{1}{2}(B + \tilde{B}), \tag{1}$$

$$C_{ave} = \frac{1}{2}(C + \tilde{C}), \quad D_{ave} = \frac{1}{2}(D + \tilde{D}), \tag{2}$$

see Z. Artstein (1983), S. Ben-Yaakov (1994), J. Bentsman et al (1990), Daafouz et al (2014) and many others.

Justifying averaging by Lax-Phillips semigroups

We prefer to justify averaging as follows: any evolution of the system Σ can be regarded as being a trajectory of the associated *Lax-Phillips semigroup*, which acts on $\mathcal{X} = L^2((-\infty, 0]; Y) \times X \times L^2([0, \infty); U)$, see for instance O. Staffans and G. Weiss (2002). Here, the first component has the intuitive meaning of past output, the second component is the current state, and the third component is the future input. Then, the switched system jumps between these two Lax-Phillips semigroups. Thus, to average between the systems, we have in fact to average between two operator semigroups, as described below.

Averaging semigroups in finite dimensions

Averaging between two operator semigroups is a well researched topic, see for instance A. Batkai (2011), P.R. Chernoff (1968), T. Kato (1978), H.F. Trotter (1959). For the matrices $A, \tilde{A}$ and A_{ave} as above, the Lie product formula states that

$$e^{A_{ave}t} = \lim[e^{tA/2n}e^{tB/2n}]^n.$$

Note the interpretation of this formula as switching between A and B with switching period t/n. Similar results hold if we replace the matrix exponentials with operator semigroups, under certain conditions - as we discuss later.

Impedance passive and scattering passive systems

We recall the following results from Staffans (2002) and Tucsnak and Weiss (2014). (In the cited references, the infinite dimensional results are given, while here we are now only discussing the finite-dimensional case). The system Σ is *impedance passive*, meaning that $U = Y$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 \leq 2\mathrm{Re}\,\langle u(t), y(t)\rangle\,,$$

if and only if

$$T = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}$$

is *dissipative*, i.e., $T + T^* \leq 0$. The system Σ is *scattering passive*, meaning that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 \leq \|u(t)\|^2 - \|y(t)\|^2\,,$$

if and only if

$$\tilde{T} = \begin{bmatrix} A & B & 0 \\ 0 & -\frac{1}{2} & 0 \\ C & D & -\frac{1}{2} \end{bmatrix} \text{ is dissipative.}$$

Passive systems - continued

From the results just recalled it follows that if both Σ and $\tilde{\Sigma}$ are impedance (or scattering) passive, then also the average system Σ_{ave} is impedance (or scattering) passive. The purpose of this research is to provide tools needed to extend these results to the case when the systems Σ and $\tilde{\Sigma}$ are infinite-dimensional. We are mainly interested in the case when they are both impedance (or scattering) passive.

The infinite-dimensional version of the Lie formula is called *Trotter product formula* or *Trotter-Lie product formula*. A more general result, applying to general operator semigroups with an additional so-called *stability condition* is given in Corollary III.5.8 of the Engel-Nagel book. In 1968, Chernoff made an essential improvement to the Trotter product formula, stated below.

The Chernoff theorem

Theorem (Chernoff, 1968). Let $V(\cdot) : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous operator-valued family of contractions (thus, $\|V(t)\| \leq 1$ for all $t \geq 0$) such that $V(0) = I$. Define

$$Ax = \lim_{h \to 0,\, h > 0} \frac{V(h) - I}{h} x,$$

with domain $\mathcal{D}(A)$ consisting of those $x \in X$ for which the above limit exists. Assume that the exists a subspace $D \subset \mathcal{D}(A)$ such that D and $(\lambda_0 I - A)D$ are dense in X for some $\lambda_0 > 0$. Then A is closable and its closure $\overline{A}$ generates the operator semigroup given by

$$\mathbb{T}_t x = \lim_{n \to \infty} [V(t/n)]^n x \qquad \forall\, x \in X,\ t \geq 0.$$

The limit converges uniformly in t on any bounded interval.

The Chernoff theorem - continued

The formula in the last theorem is called *Chernoff product formula*, for further refinements see Theorem III.5.2 and its corillary in the book of Engel and Nagel (2000). For adding two m-dissipative operators A and B, many functions V are good candidates to use, for example $V(t) = \frac{1}{2}(e^{2tA} + e^{2tB})$, more refined: $V(t) = \frac{1}{2}\left[e^{tA/2}e^{tB}e^{tA/2} + e^{tB/2}e^{tA}e^{tB/2}\right]$. Notice that the Trotter-Lie formula corresponds to $V(t) = e^{tA}e^{tB}$.

T. Kato (1974,1978) developed a method for adding positive (unbounded) operators A, B based on the quadratic form sum C, defined by the formula $\langle Cu, u\rangle = \langle Au, u\rangle + \langle Bu, u\rangle$, interpreted in the form $\|C^{1/2}u\|^2 = \|A^{1/2}u\|^2 + \|B^{1/2}u\|^2$). Denote $D' = \mathcal{D}(A^{1/2}) \cap \mathcal{D}(B^{1/2})$ and let P' be the orthogonal projection of H onto H', the closure of D'. Then for every $x \in X$, $\lim\,(\exp(-tA/n)\exp(-tB/n))^n x = \exp(-tC)P'x$, uniformly in t on each compact subinterval of $(0, \infty)$.

A strange example

Example 1. Let $X = L^2(\mathbb{R})$ and define $A = \frac{\mathrm{d}}{\mathrm{d}\xi}$, with $\mathcal{D}(A) = \mathcal{H}^1(\mathbb{R})$, the generator of the left shift semigroup on X. For the continuous everywhere but nowhere differentiable *Weierstrass function* $p(x) = 2 + \sum_{n=1}^{\infty} 2^{-n}\cos(20^n \pi x)$, clearly, $1 \leq p(x) \leq 3$ for all $x \in \mathbb{R}$. Denote by P the corresponding pointwise multiplication operator, and set

$$B := P^{-1}AP, \quad \mathcal{D}(B) = \{\phi \in L^2(\mathbb{R}) \,|\, p\phi \in \mathcal{H}^1(\mathbb{R})\}.$$

Thus, B is similar to A via a bounded invertible operator and hence also generates an operator semigroup. However, it is not hard to check that $\mathcal{D}(A) \cap \mathcal{D}(B) = \{0\}$. We do not know if the Trotter-Lie formula converges for these two operators A, B.

The following example shows that the strong limit in Lie-Trotter formula does not always exist.

Another strange example

Example 2. Let $X = L^2(\mathbb{R})$. Define $A = \frac{\mathrm{d}}{\mathrm{d}\xi}$ with $\mathcal{D}(A) = \mathcal{H}^1(\mathbb{R})$, $B = \frac{\mathrm{d}}{\mathrm{d}\xi}$ with

$$\mathcal{D}(B) = \{f \in \mathcal{H}^1(-\infty, 0) \oplus \mathcal{H}^1(0, \infty) \,|\, f(0-) = -f(0+)\}.$$

Then it can be shown that the strong limit

$$\lim_{n\to\infty} [e^{tA/n} e^{tB/n}]^n$$

does not exist. DRAW PICTURE TO EXPLAIN THIS.

A new theorem

Theorem. Let A and B be two m-dissipative operators on a Hilbert space X. Assume that $\widetilde{D} = \mathcal{D}(A) \cap \mathcal{D}(B)$ is dense in X. Then there is an associated contraction semigroup $\mathbb{T}$ whose generator is an extension of $A + B$: for each $x_0 \in \widetilde{D}$, the function $t \to \mathbb{T}_t x_0$ is continuously differentiable and, for all $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{T}_t x_0 = \mathbb{T}_t (A + B) x_0 .$$

However, there is no uniqueness here for $\mathbb{T}$ -this can be seen by going back to Example 2.

Addition based on Chernoff's theorem

Definition. Let A and B be two semigroup generators on a Hilbert space X. Define

$$V(t) := \frac{e^{2tA} + e^{2tB}}{2}, \quad (A \oplus B)x = 2 \lim_{h \to 0,\, h>0} \frac{V(h) - I}{h} x,$$

$$\mathcal{D}\left(A \oplus B\right) := \left\{ x \in X \,\middle|\, \text{the above limit exists} \right\}.$$

It is clear that $\mathcal{D}(A) \cap \mathcal{D}(B) \subset \mathcal{D}(A \oplus B)$. We conjecture that $\mathcal{D}(A \oplus B)$ is always dense in X, but in general we cannot prove that $\mathcal{D}(A \oplus B)$ contains a vector other than 0.

Theorem. Let A and B be m-dissipative on a Hilbert space X. Let $A \oplus B$ be as in the above definition.

Then $A \oplus B$ is dissipative.

Addition of generators obtained by feedback

In the sequel, we show that if A and B are both closed-loop generators obtained from regular linear systems Σ and $\widetilde{\Sigma}$, with admissible output feedback operators, Σ and $\widetilde{\Sigma}$ have the same semigroup generator, and their feedthrough operators are zero, then the addition from the last definition is given by a straightforward formula. We will not use the notation A and B for our two semigroup generators, because in the sequel A and B will acquire a different meaning, in order to follow the terminology and notation of the paper Weiss (MCSS, 1994), which is needed here.

Let Σ and $\widetilde{\Sigma}$ be regular linear systems with generating operators $A, B, C, 0$ and $A, \widetilde{B}, \widetilde{C}, 0$, respectively (the 0 in the last position indicates that the feedthrough operators of these systems are 0). These systems have a common state space X and a common semigroup generator A, but their input spaces $U, \widetilde{U}$ and their output spaces $Y, \widetilde{Y}$ may be different.

Let

$$K \in \mathcal{L}(Y, U), \qquad \widetilde{K} \in \mathcal{L}(\widetilde{Y}, \widetilde{U})$$

be admissible feedback operators for Σ and $\widetilde{\Sigma}$, respectively. We denote by Σ^K and $\widetilde{\Sigma}^{\widetilde{K}}$ the corresponding closed-loop systems, which are known to be regular linear systems with feedthrough operators zero. We denote the generating operators of Σ^K and $\widetilde{\Sigma}^{\widetilde{K}}$ by $A^K, B^K, C^K, 0$ and by $A^{\widetilde{K}}, \widetilde{B}^{\widetilde{K}}, \widetilde{C}^{\widetilde{K}}, 0$, respectively. We know that

$$\begin{aligned} A^K &= A + BKC_\Lambda, \\ \mathcal{D}(A^K) &= \{x \in \mathcal{D}(C_\Lambda) \mid (A + BKC_\Lambda)x \in X\}, \end{aligned}$$

and of course a similar formula holds for $A^{\widetilde{K}}$. In the above formula, C_Λ is the Λ-extension of C, defined by

$$C_\Lambda x = \lim_{\lambda \to \infty, \lambda > 0} C\lambda(\lambda I - A)^{-1} x,$$

where $\mathcal{D}(C_\Lambda)$ consists of all $x \in X$ for which the limit exists.

Adding feedback generators

According to the well-known feedback theory for regular systems,

$$\begin{aligned}&(sI - A^K)^{-1} - (sI - A)^{-1}\\&= (sI - A^K)^{-1} B^K K C (sI - A)^{-1},\end{aligned} \tag{3}$$

for all complex s with $\mathrm{Re}\, s$ large enough. Of course, a similar formula holds for the resolvent of $A^{\widetilde{K}}$. Moreover, the Λ-extensions of the observation operators C and C^K are equal:

$$C^K_\Lambda x = \lim_{\lambda \to \infty, \lambda > 0} C^K \lambda(\lambda I - A^K)^{-1} x = C_\Lambda x$$

for all $x \in \mathcal{D}(C^K_\Lambda) = \mathcal{D}(C_\Lambda)$, and similarly for $\widetilde{\Sigma}^{\widetilde{K}}$.

Theorem. With the notation introduced in the last paragraph, for all $x \in \mathcal{D}(A^K \oplus A^{\widetilde{K}}) \cap \mathcal{D}(C_\Lambda) \cap \mathcal{D}(\widetilde{C_\Lambda})$,

$$\left[A^K \oplus A^{\widetilde{K}}\right] x = \left[2A + BKC_\Lambda + \widetilde{B}\widetilde{K}\widetilde{C_\Lambda}\right] x .$$

Theorem. Suppose that the operators A and B are m-dissipative on a Hilbert space X. Assume that there exists $s \in \rho(A \oplus B)$ and $\mathrm{Re}\, s > 0$. Then for any $x \in X$,

$$(sI - A \oplus B)^{-1} x = \lim_{\tau \to 0} \tau \left[2I - e^{-\frac{s\tau}{2}} (e^{A\tau} + e^{B\tau}) \right]^{-1} x.$$

We remark that since $A \oplus B$ is dissipative (see our earlier theorem), the existence s as in the theorem implies that $A \oplus B$ is m-dissipative.

Thank You