Proportional control based output tracking and disturbance rejection for a 1-D anti-stable wave equation

Hua-Cheng Zhou, George Weiss

School of Electrical Engineering, Tel Aviv University

In memory of U. Helmke and R. Kalman

Networks of Linear Systems and Operator Theory

Sde Boker

March 21, 2017
1. Problem formulation

2. Preliminary: no disturbance

3. Tracking controller design

4. A state observer based tracking controller design

5. Numerical simulation
1. Problem formulation

2. Preliminary: no disturbance

3. Tracking controller design

4. A state observer based tracking controller design

5. Numerical simulation
Output tracking and disturbance rejection for a 1-D anti-stable wave equation

Fig. 1: A anti-stable string system

- \(u\) is the control input signal;
- \(y_m\) is the output signal;
- \(d\) represents the unknown external disturbance,
- \(r\) is the reference signal.
More precisely, in this talk, we consider the following 1-D anti-stable wave equation with Neumann boundary control matched unknown disturbance:

\[
\begin{cases}
  w_{tt}(x, t) = w_{xx}(x, t), & 0 < x < 1, \ t > 0, \\
  w_x(0, t) = -qw_t(0, t), & t \geq 0, \text{ anti-stable boundary condition} \\
  w_x(1, t) = u(t) + d(t), & t \geq 0, \\
  w(x, 0) = w_0(x), \ w_t(x, 0) = w_1(x), & 0 \leq x \leq 1, \\
  y_o(t) = w(0, t), \ y_m(t) = w(1, t). 
\end{cases}
\] (1)

\((w, w_t)\) is the state, \(u\) is the control input signal, \(y_o\) is the output signal to be regulated. \(y_m\) is the measurement, parameter \(q \neq 1\) is a real positive constant. \(d\) represents the unknown external disturbance which is only supposed to satisfy \(d \in L^\infty(0, \infty)\). Let \(r\) be the known reference signal satisfying \(r \in W^{1, \infty}(0, \infty)\), where \(W^{1, \infty}(0, \infty) = \{\phi : \phi \in L^\infty(\phi, \infty), \ \phi' \in L^\infty(0, \infty)\}\).
Our aim is to design an output feedback regulator such that for all initial states of the systems (1), (i) all the internal signals should be bounded; (ii) the tracking error $e_y = y_o - r$ satisfies for some $M, \mu > 0$,

$$|e_y(t)| = |y_o(t) - r(t)| \leq M e^{-\mu t}, \text{ for all } t \geq 0. \quad (2)$$

Here, (ii) means that the tracking error is exponentially convergent to zero as time goes to infinity.

We consider system (1) in the state Hilbert space $\mathcal{H} = H^1(0, 1) \times L^2(0, 1)$ with the inner product given by, $\forall [\phi_i, \psi_i] \in \mathcal{H}, i = 1, 2$,

$$\langle [\phi_1, \psi_1], [\phi_2, \psi_2] \rangle_{\mathcal{H}} = \int_0^1 [\phi'_1(x)\phi'_2(x) + \psi_1(x)\psi_2(x)] dx + \phi_1(0)\phi_2(0).$$
In solving *Output regulation*(*output tracking*), traditional method is:

- Internal mode principle (IMP)

The number of frequencies and the frequencies are required to be known.

Our approach in this talk is:

- properties of time-delay system
- disturbance estimator and state observer

The disturbance is completely unknown, but the reference is known.
1. Problem formulation
2. Preliminary: no disturbance
3. Tracking controller design
4. A state observer based tracking controller design
5. Numerical simulation
Reformulate system (1) with \( d \equiv 0 \) by introducing the Riemann variables:

\[
\alpha(x, t) = \frac{1}{1 + q} \left[ w_t(x, t) - w_x(x, t) \right], \quad \beta(x, t) = w_t(x, t) + w_x(x, t),
\]

which leads to a new dynamics

\[
\begin{cases}
\alpha_t(x, t) = -\alpha_x(x, t), \\
\alpha(0, t) = \beta(0, t), & \text{right shift} \\
\beta_t(x, t) = \beta_x(x, t), \\
\beta(1, t) = u(t) + w_t(1, t), & \text{left shift} \\
w_t(0, t) = \frac{1}{1 - q} \beta(0, t).
\end{cases}
\]

In this new framework, the wave equation is represented as the cascade of two transport PDEs, with one ODE being driven by the second of the two PDEs. The “ODE-part” of (4) with state \( w(0, t) \) has to be regulated to track the reference signal \( r \) asymptotically (exponentially) by feedback. It is seen that if the control input signal \( u(t) \) and the velocity signal \( w_t(1, t) \) are known, the solution \( \alpha, \beta \) of the transport equation can be solved explicitly. Particularly, \( \beta(0, t) \) takes the value \( u(t) + w_t(1, t) \) delayed by 1 unit of time.
Now, motivated by the “ODE-part” of (4) and the state \( w(0, t) \) needed to be regulated to \( r(t) \) asymptotically, we propose the following feedback controller

\[
u(t) = -w_t(1, t) + (1 - q)\dot{r}(t + 1) - ke_y(t),
\]

where \(-ke_y(t)\) is the error proportional feedback, the gain \( k \) is a tuning parameter, and \( e_y \) is the tracking error. Why do we propose such a control?

\[
\dot{e}_y(t) = -\frac{k}{1 - q}e_y(t - 1), \quad \forall t > 1,
\]

Set \( \sigma(t) = w(0, t) \). The closed-loop of system (4) becomes

\[
\begin{aligned}
\alpha_t(x, t) &= -\alpha_x(x, t), \\
\alpha(0, t) &= \beta(0, t), \\
\beta_t(x, t) &= \beta_x(x, t), \\
\beta(1, t) &= (1 - q)\dot{r}(t + 1) - k[\sigma(t) - r(t)], \\
\dot{\sigma}(t) &= \frac{1}{1 - q}\beta(0, t).
\end{aligned}
\]

We consider the system in \( \mathbb{H} = [L^2(0, 1)]^2 \times \mathbb{R} \) with the usual inner product.
Preliminary: Case study of $d(t) \equiv 0$

System (6) can be re-formulated in operator form:

$$\dot{Z}(:, t) = AZ(:, t) + Bf(t),$$

where $Z(:, t) = [\alpha(:, t), \beta(:, t), \sigma(t)], f(t) := (1 - q)\dot{r}(t + 1) + kr(t), B = [0, \delta_1, 0]$, and $A$ is a linear operator defined in $H^1$ by

$$A[\phi, \psi, h] = \left[ -\phi_x, \psi_x, \frac{1}{1 - q}\psi(0) \right], \quad \forall[\phi, \psi, h] \in D(A),$$

$$D(A) = \{[\phi, \psi, h] \in [H^1(0, 1)]^2 \times \mathbb{R} : \phi(0) = \psi(0), \psi(1) = -kh \}.$$
Preliminary: Case study of $d(t) \equiv 0$

**Lemma**

Let $k$ be a constant satisfying $\frac{k}{1-q} \in (0, \pi/2)$. $r \in W^{1,\infty}(0, \infty)$. The operator $A$ defined by (8) generates a $C_0$-semigroup $e^{At}$ on $\mathbb{H}$ and $B$ is admissible for $e^{At}$. Therefore, for any initial value $(\alpha(\cdot, 0), \beta(\cdot, 0), \sigma(0)) \in \mathbb{H}$, system (6) admits a unique solution $(\alpha(\cdot, t), \beta(\cdot, t), \sigma(t)) \in C(0, \infty; \mathbb{H})$ that is bounded, i.e., $\| (\alpha(\cdot, t), \beta(\cdot, t), \sigma(t)) \|_\mathbb{H} < +\infty$. Moreover, there exist two constants $M, \mu > 0$ such that $|\sigma(t) - r(t)| \leq Me^{-\mu t}$ for all $t \geq 0$.

The key step of the proof: $e_y(t) = \sigma(t) - r(t)$, it has

$$
\dot{e}_y(t) = -\frac{k}{1-q} e_y(t-1), \quad \forall t > 1,
$$

which is classical time delay system. It is well-known, by the frequency domain analysis, that system (9) is exponentially stable if and only if $0 < \frac{k}{1-q} < \frac{\pi}{2}$. Therefore, $|\sigma(t) - r(t)| \leq Me^{-\mu t}$ for all $t \geq 0$ holds with some $M, \mu > 0$. 

Now, we go back to the closed-loop system of (1) without disturbance:

\[
\begin{aligned}
& w_{tt}(x, t) = w_{xx}(x, t), \quad 0 < x < 1, \quad t > 0, \\
& w_x(0, t) = -qw_t(0, t), \quad t \geq 0, \\
& w_x(1, t) = -w_t(1, t) + (1 - q)r(t + 1) - k[w(0, t) - r(t)], \quad t \geq 0.
\end{aligned}
\]  

(10)

**Main result I:**

**Theorem**

Let \( k \) be a constant satisfying \( \frac{k}{1 - q} \in (0, \pi/2) \). \( r \in W^{1,\infty}(0, \infty) \). For any initial value \((w(\cdot, 0), w_t(\cdot, 0)) \in \mathcal{H}\), system (10) admits a unique solution

\[
(w(\cdot, t), w_t(\cdot, t)) \in C(0, \infty; \mathcal{H})
\]

that is bounded, i.e.,

\[
\|(w(\cdot, t), w_t(\cdot, t))\|_\mathcal{H} < +\infty.
\]

Moreover, the output regulation is exponentially achieved, i.e., there exist two constants \( M, \mu > 0 \) such that

\[
|y_o(t) - r(t)| \leq Me^{-\mu t} \text{ for all } t \geq 0.
\]
A new stabilizing feedback control law for anti-stable wave equation, i.e., the control

\[ u(t) = -w_t(1, t) - kw(0, t) \text{ with } \frac{k}{1 - q} \in (0, \pi/2) \]

can stabilize exponentially system (1) with \( d \equiv 0 \), that is,

\[
\begin{aligned}
  w_{tt}(x, t) &= w_{xx}(x, t), \quad 0 < x < 1, \quad t > 0, \\
  w_x(0, t) &= -qw_t(0, t), \quad t \geq 0, \\
  w_x(1, t) &= -w_t(1, t) - kw(0, t), \quad t \geq 0,
\end{aligned}
\]

(11)
is exponentially stable if \( \frac{k}{1 - q} \in (0, \pi/2) \). This controller is very simpler than those in (A. Smyshlyaev and M. Krstic, SCL, 2009 and D. Bresch-Pietri and M. Krstic, Automatica, 2014), where the controller is much complicated, and backstepping approach and the state observer are used. It is worth to note that using our controller, the anti-damping constant \( q \) is allowed to be unknown but with known bounds \( q \) and \( \bar{q} \) such that \( q \in [q, \bar{q}] \) and either \( q > 1 \) or \( \bar{q} < 1 \). The proportional gain \( k \) is taken as the following ways: if \( q > 1 \), we take \( k \) such that \( \frac{k}{1 - q} \in (0, \pi/2) \); if \( \bar{q} < 1 \), we take \( k \) such that \( \frac{k}{1 - \bar{q}} \in (0, \pi/2) \).
The constant reference signal

If the reference signal is a constant signal $r$, then the control law (5) becomes
$$u(t) = -w_t(1, t) - k[w(0, t) - r]$$ with $k$ satisfying $k \in \left( \frac{1}{1-q}, \pi/2 \right)$, which is different from ([G. Weiss and V. Natarajan, 2016]) where the control is low gain integral control.

$$\begin{align*}
  w_{tt}(x, t) &= w_{xx}(x, t), \quad 0 < x < 1, \quad t > 0, \\
  w_x(0, t) &= -qw_t(0, t), \quad t \geq 0, \\
  w_x(1, t) &= -w_t(1, t) - k[w(0, t) - r], \quad t \geq 0.
\end{align*}$$

(12)

Noting that the anti-damping constant $q$ does not appear in the control law, it is seen from byproduct I that for the output $w(0, t)$ to be regulated to a constant signal $r$, the anti-damping constant $q$ can be unknown but with known bounds $\underline{q}$ and $\bar{q}$ such that $q \in [\underline{q}, \bar{q}]$ and either $\underline{q} > 1$ or $\bar{q} < 1$.

Remark

If the objective is to design $u$ such that $w_t(0, t) \rightarrow \dot{r}(t)$ without requirement $w(0, t) \rightarrow r(t)$, we can simply use the way in ([P.O. Lamare and N. Bekiaris-Liberis, SCL, 2015]) to find a control law $u(t) = (1 - q)\dot{r}(t + 1)$. In general, the control $u(t) = (1 - q)\dot{r}(t + 1)$ cannot achieve $\lim_{t \rightarrow \infty} |w(0, t) - r(t)| = 0$. 

Outline

1. Problem formulation
2. Preliminary: no disturbance
3. Tracking controller design
4. A state observer based tracking controller design
5. Numerical simulation
Now, we consider the case where the disturbance is presented, i.e., $d(t) \neq 0$: In ([H.C. Zhou and G. Weiss, IFAC 2017]), the following disturbance estimator of system (1):

$$
\begin{align*}
v_t(x, t) &= v_{xx}(x, t), \quad 0 < x < 1, \quad t > 0, \\
v_x(0, t) &= -qv_t(0, t) + c_1[v(0, t) - w(0, t)], \quad t \geq 0, \\
v_x(1, t) &= u(t) - W_x(1, t), \quad t \geq 0, \\
v(x, 0) &= v_0(x), \quad v_t(x, 0) = v_1(x), \quad 0 \leq x \leq 1, \\
z_t(x, t) &= z_{xx}(x, t), \quad 0 < x < 1, \quad t > 0, \\
z_x(0, t) &= \frac{c_1}{1 - c_0} z(0, t) + \frac{c_0 - q}{1 - c_0} z_t(0, t), \quad t \geq 0, \\
z(1, t) &= v(1, t) + W(1, t) - w(1, t), \quad t \geq 0, \\
z(x, 0) &= z_0(x), \quad z_t(x, 0) = z_1(x), \quad 0 \leq x \leq 1, \\
W_t(x, t) &= -W_x(x, t), \quad 0 < x < 1, \quad W(0, t) = -c_0[v(0, t) - w(0, t)], \quad t > 0, \\
W(x, 0) &= W_0(x), \quad 0 \leq x \leq 1,
\end{align*}
$$

was used for output feedback exponential stabilization, where $c_0$ and $c_1$ are two design parameters so that $\frac{c_1}{1 - c_0} > 0$ and $\frac{c_0 - q}{1 - c_0} > 0$. From the above disturbance estimator, we get

$$
\tilde{z}_x(1, t) = z_x(1, t) + d(t) \in L^2(0, \infty), \text{ i.e., } -z_x(1, t) \approx d(t).
$$
Let $\sigma(t) = w(0, t)$. The closed-loop system becomes

\[
\begin{align*}
\alpha_t(x, t) &= -\alpha_x(x, t), \\
\alpha(0, t) &= \beta(0, t), \\
\beta_t(x, t) &= \beta_x(x, t), \\
\beta(1, t) &= z_x(1, t) + d(t) + (1 - q)\dot{r}(t + 1) - ke_y(t) \\
\dot{\sigma}(t) &= \frac{1}{1 - q} \beta(0, t).
\end{align*}
\]

\tag{14}

Lemma

Suppose that $d \in L^\infty(0, \infty)$, $r \in W^{1,\infty}(0, \infty)$. Let $k$ be a constant satisfying $\frac{k}{1 - q} \in (0, \pi/2)$. For any initial value $(\alpha(\cdot, 0), \beta(\cdot, 0), \sigma(0)) \in \mathbb{H}$, system (14) admits a unique solution $(\alpha(\cdot, t), \beta(\cdot, t), \sigma(t)) \in C(0, \infty; \mathbb{H})$ that is bounded, i.e., $\| (\alpha(\cdot, t), \beta(\cdot, t), \sigma(t)) \|_\mathbb{H} < +\infty$. Moreover, $\lim_{t \to \infty} |\sigma(t) - r(t)| = 0$. 
The key points of the proof: let \( e_y(t) = \sigma(t) - r(t) \), then \( e_y(t) \) satisfies

\[
\dot{e}_y(t) = -\frac{k}{1-q}e_y(t-1) + \frac{1}{1-q}\tilde{z}_x(1, t-1), \quad t > 1.
\]  \( \text{(15)} \)

where \( \tilde{z}_x(1, t) = z_x(1, t) + d(t) \in L^2(0, \infty) \). In the state space \( \mathcal{H}_0 = \mathbb{R} \times L^2(-1, 0) \), reformulate (15) as

\[
\dot{\chi}(t) = A_0\chi(t) + B_0\tilde{z}_x(1, t-1),
\]  \( \text{(16)} \)

where \( \chi(t) = (e_y(t), e_y(t+\cdot)) \), the operators \( A_0, B_0 \) are given by

\[
\left\{ \begin{array}{l}
A_0[h, \phi(\cdot)] = \left[ -\frac{k}{1-q}\phi(-1), \frac{d\phi}{d\theta} \right], \\
D(A_0) = \{ [h, \phi(\cdot)] \in \mathcal{H}_0 : \phi \in H^1(-1, 0), \phi(0) = h \}
\end{array} \right.
\]  \( \text{(17)} \)

and \( B_0 = [1, 0] \).

Since \( \frac{k}{1-q} \in (0, \pi/2) \), \( e^{A_0t} \) is exponentially stable. By the boundedness of \( B_0 \), \( B_0 \) is admissible for \( e^{A_0t} \). Since \( \tilde{z}_x(1, t) \in L^2(0, \infty) \), it follows that system (16) admits a unique solution \( (e_y(t), e_y(t+\cdot)) \in C(0, \infty; \mathcal{H}_0) \) satisfying

\[
\lim_{t \to \infty} \| (e_y(t), e_y(t+\cdot)) \|_{\mathcal{H}_0} = 0,
\]

which yields \( \lim_{t \to \infty} |e_y(t)| = 0 \).
The next lemma is from [Zhou and Weiss, 2017, IFAC].

**Lemma**

Let $A$ be the generator of exponential stable $C_0$-semigroup $e^{At}$ on the Hilbert space $X$. Assume that $B_i \in \mathcal{L}(U_i, X_{-1})$, $i = 1, 2, \ldots, n$ are admissible control operators for $e^{At}$. Then, the initial value problem

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n} B_i u_i(t), \quad x(0) = x_0, \quad u_i \in L^2_{loc}(0, \infty; U_i),$$

admits a unique solution $x \in C(0, \infty; X)$, which tends to zero as $t \to \infty$ if either $u_i \in L^2(0, \infty; U_i)$ or $\lim_{t \to \infty} \|u(t)\|_{U_i} = 0$, for $i = 1, 2, \ldots, n$, and is bounded if $u_i \in L^\infty(0, \infty; U_i)$, $i = 1, 2, \ldots, n$. Moreover, if there exist two constants $M_0, \mu_0 > 0$ such that $\|u\|_{U_i} \leq M_0 e^{-\mu_0 t}$, $i = 1, 2, \ldots, n$, then $\|x(t)\| \leq Me^{-\mu t}$ for some $M, \mu > 0$. 
Example

Let $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^n$. Consider system:

$$
\dot{x}(t) = Ax(t) + Bd(t)
$$

(18)

where $d \in \mathbb{R}$ is the disturbance. Suppose that $A$ is Hurwitz. The solution is given by

$$
x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}d(s)ds.
$$

which can be rewritten as

$$
x(t) = e^{At}x_0 + e^{A\frac{t}{2}} \int_0^{\frac{t}{2}} e^{A(\frac{t}{2}-s)}d(s)ds + \int_{\frac{t}{2}}^t e^{A(t-s)}d(s)ds.
$$

It is easy to verify that $x(t) \to 0$ as $t \to \infty$ if $d \in L^2(0, \infty)$. From this, if we design a stabilizing control law for $\dot{x}(t) = Ax(t) + B[u(t) + d(t)]$, it suffices to find a $u$ such that $u + d \in L^2(0, \infty)$. 

With the feedback, the closed-loop system of (1) becomes:

\[
\begin{align*}
    w_{tt}(x, t) &= w_{xx}(x, t), \\
    w_x(0, t) &= -qw_t(0, t), \\
    w_x(1, t) &= z_x(1, t) - w_t(1, t) + (1 - q)\dot{r}(t + 1) - k[w(0, t) - r(t)] + d(t), \\
    v_{tt}(x, t) &= v_{xx}(x, t), \\
    v_x(0, t) &= -qv_t(0, t) + c_1[v(0, t) - w(0, t)], \\
    v_x(1, t) &= z_x(1, t) - w_t(1, t) + (1 - q)\dot{r}(t + 1) - k[w(0, t) - r(t)] - W_x(1, t), \\
    z_{tt}(x, t) &= z_{xx}(x, t), \\
    z_x(0, t) &= \frac{c_1}{1 - c_0}z(0, t) + \frac{c_0 - q}{1 - c_0}z_t(0, t), \quad z(1, t) = v(1, t) + W(1, t) - w(1, t), \\
    W_t(x, t) &= -W_x(x, t), \quad W(0, t) = -c_0[v(0, t) - w(0, t)].
\end{align*}
\]
Main result II:

**Theorem**

Suppose that \( \frac{c_0 - q}{1 - c_0} > 0 \), \( \frac{c_1}{1 - c_0} > 0 \), \( d \in L^\infty(0, \infty) \), \( r \in W^{1,\infty}(0, \infty) \), and the gain \( k \) is a constant satisfying \( \frac{k}{1 - q} \in (0, \pi/2) \). For any initial value \( ((w, w_t, v, v_t, z, z_t, W)(\cdot, 0)) \in \mathcal{H}^3 \times H^1(0, 1) \) with the compatibility conditions

\[
z(1, 0) - v(1, 0) - W(1, 0) + w(1, 0) = 0, \quad W(0, 0) + c_0[v(0, 0) - w(0, 0)] = 0,
\]

system (19) admits a unique solution

\[
(w(\cdot, t), w_t(\cdot, t), v(\cdot, t), v_t(\cdot, t), z(\cdot, t), z_t(\cdot, t), W(\cdot, t)) \in C(0, \infty; \mathcal{H}^3 \times H^1(0, 1))
\]

that is bounded, i.e., there exists a constant \( M > 0 \) such that

\[
\|(w(\cdot, t), w_t(\cdot, t), v(\cdot, t), v_t(\cdot, t), z(\cdot, t), z_t(\cdot, t), W(\cdot, t))\|_{\mathcal{H}^3 \times H^1(0, 1)} \leq M
\]

for all \( t \geq 0 \). Moreover, the output regulation is achieved, i.e.,

\[
\lim_{t \to \infty} |y_o(t) - r(t)| = 0.
\]
Outline

1. Problem formulation
2. Preliminary: no disturbance
3. Tracking controller design
4. A state observer based tracking controller design
5. Numerical simulation
In this section, we employ the following state observer ([H.C. Zhou and G. Weiss, IFAC 2017]) for system (1)

\[
\begin{align*}
\hat{w}_i(x, t) &= \hat{w}_{xx}(x, t), \quad 0 < x < 1, \quad t > 0, \\
\hat{w}_x(0, t) &= -q\hat{w}_i(0, t) + c_1[\hat{w}(0, t) - w(0, t)], \quad t \geq 0, \\
\hat{w}_x(1, t) &= u(t) - z_x(1, t) - Y_x(1, t), \quad t \geq 0, \\
\hat{w}(x, 0) &= \hat{w}_0(x), \quad \hat{w}_i(x, 0) = \hat{w}_1(x), \quad 0 \leq x \leq 1, \\
Y_t(x, t) &= -Y_x(x, t), \quad 0 < x < 1, \quad t > 0, \\
Y(0, t) &= -c_0[\hat{w}(0, t) - w(0, t)], \quad t \geq 0, \\
Y(x, 0) &= Y_0(x), \quad 0 \leq x \leq 1,
\end{align*}
\]

(20)

where \(c_1\) and \(c_2\) are two design parameters. \(-z_x(1, t)\) plays the role of disturbance since \(-z_x(1, t) \approx d(t)\). Using the above state observer, we get that

\[
(\hat{w}(x, t), \hat{w}_i(x, t)) \approx (w(x, t), w_i(x, t)).
\]
A state observer based tracking controller design

Now, we propose the following feedback controller

\[ u(t) = z_x(1, t) + Y_x(1, t) + \hat{w}_t(1, t) + (1 - q)\dot{r}(t + 1) - k[\hat{w}(0, t) - r(t)]. \tag{21} \]

This control \( u(t) \) is implementable.

- \( \hat{w}_t(1, t), Y_x(1, t) \) is from the state observer system
- \( z_x(1, t) \) is from the disturbance estimator system
- \( r(t) \) is the known reference signal
A state observer based tracking controller design

We go back to the closed-loop system (1) under the feedback (21)

\[
\begin{align*}
  w_{\prime\prime}(x, t) &= w_{xx}(x, t), \\
  w_x(0, t) &= -qw_t(0, t), \\
  w_x(1, t) &= z_x(1, t) + Y_x(1, t) - \hat{w}_t(1, t) + (1 - q)\dot{r}(t + 1) - k[\hat{w}(0, t) - r(t)] + d(t), \\
  v_{\prime\prime}(x, t) &= v_{xx}(x, t), \\
  v_x(0, t) &= -qv_t(0, t) + c_1[v(0, t) - w(0, t)], \\
  v_x(1, t) &= z_x(1, t) + Y_x(1, t) - \hat{w}_t(1, t) + (1 - q)\dot{r}(t + 1) - k[\hat{w}(0, t) - r(t)] - W_x(1, t), \\
  z_{\prime\prime}(x, t) &= z_{xx}(x, t), \\
  z_x(0, t) &= \frac{c_1}{1 - c_0}z(0, t) + \frac{c_0 - q}{1 - c_0}z_t(0, t), \\
  z(1, t) &= v(1, t) + W(1, t) - w(1, t), \\
  W_t(x, t) &= -W_x(x, t), \quad W(0, t) = -c_0[v(0, t) - w(0, t)], \\
  \hat{w}_{\prime\prime}(x, t) &= \hat{w}_{xx}(x, t), \\
  \hat{w}_x(0, t) &= -q\hat{w}_t(0, t) + c_1[\hat{w}(0, t) - w(0, t)], \\
  \hat{w}_x(1, t) &= -\hat{w}_t(1, t) + (1 - q)\dot{r}(t + 1) - k[\hat{w}(0, t) - r(t)], \\
  Y_t(x, t) &= -Y_x(x, t), \quad Y(0, t) = -c_0[\hat{w}(0, t) - w(0, t)].
\end{align*}
\]
Main result III:

Theorem

Suppose that \( \frac{c_1}{1-c_0} > 0, \frac{c_0-q}{1-c_0} > 0, \) and \( d \in L^\infty(0, \infty), r \in W^{1,\infty}(0, \infty). \)

Suppose that \( k \) is a constant satisfying \( \frac{k}{1-q} \in (0, \pi/2). \) For any initial value \((w, w_t, \hat{w}, \hat{w}_t, v, v_t, z, z_t, W, Y)(\cdot, 0) \in \mathcal{H}^4 \times [H^1(0, 1)]^2\) with the compatibility conditions

\[
\begin{align*}
  z(1, 0) - v(1, 0) - W(1, 0) + w(1, 0) &= 0, \\
  W(0, 0) + c_0[v(0, 0) - w(0, 0)] &= 0, \\
  Y(0, 0) + c_0[\hat{w}(0, 0) - w(0, 0)] &= 0,
\end{align*}
\]

system (22) admits a unique solution

\[
(w, w_t, \hat{w}, \hat{w}_t, v, v_t, z, z_t, W, Y)(\cdot, t) \in C(0, \infty; \mathcal{H}^4 \times [H^1(0, 1)]^2)
\]

that is bounded, i.e.,

\[
\sup_{t \geq 0} \| (w, w_t, \hat{w}, \hat{w}_t, v, v_t, z, z_t, W, Y)(\cdot, t) \|_{\mathcal{H}^4 \times [H^1(0, 1)]^2} \leq M_0, \text{ with some } M_0 > 0.
\]

Moreover, the output regulation is achieved, i.e., there exist two constants \( M, \mu > 0 \) such that

\[
|y_o(t) - r(t)| \leq M e^{-\mu t}, \text{ for all } t \geq 0.
\]
Byproduct II: An output feedback exponentially stabilizing control law

Reference signal \( r(t) \equiv 0 \), the control input becomes

\[
 u(t) = z_x(1, t) + Y_x(1, t) - \hat{w}_t(1, t) - k\hat{w}(0, t). 
\]

The closed-loop is

\[
\begin{align*}
 w_{tt}(x, t) &= w_{xx}(x, t), \\
 w_x(0, t) &= -qw_t(0, t), \\
 w_x(1, t) &= z_x(1, t) + Y_x(1, t) - \hat{w}_t(1, t) - k\hat{w}(0, t) + d(t), \\
 v_{tt}(x, t) &= v_{xx}(x, t), \\
 v_x(0, t) &= -qv_t(0, t) + c_1[v(0, t) - w(0, t)], \\
 v_x(1, t) &= z_x(1, t) + Y_x(1, t) - \hat{w}_t(1, t) - k\hat{w}(0, t) - W_x(1, t), \\
 z_{tt}(x, t) &= z_{xx}(x, t), \\
 z_x(0, t) &= \frac{c_1}{1 - c_0}z(0, t) + \frac{c_0 - q}{1 - c_0}z_t(0, t), \\
 z(1, t) &= v(1, t) + W(1, t) - w(1, t), \\
 W_t(x, t) &= -W_x(x, t), \quad W(0, t) = -c_0[v(0, t) - w(0, t)], \\
 \hat{w}_{tt}(x, t) &= \hat{w}_{xx}(x, t), \\
 \hat{w}_x(0, t) &= -q\hat{w}_t(0, t) + c_1[\hat{w}(0, t) - w(0, t)], \quad \hat{w}_x(1, t) = -\hat{w}_t(1, t) - k\hat{w}(0, t), \\
 Y_t(x, t) &= -Y_x(x, t), \quad Y(0, t) = -c_0[\hat{w}(0, t) - w(0, t)].
\end{align*}
\]
Byproduct II: An output feedback exponentially stabilizing control law

**Theorem**

Suppose that the reference signal \( r \equiv 0 \). Suppose that \( \frac{c_1}{1-c_0} > 0 \), \( \frac{c_0-q}{1-c_0} > 0 \) and \( d \in L^\infty(0, \infty) \). Suppose that \( k \) is a constant satisfying \( \frac{k}{1-q} \in (0, \pi/2) \). For any initial value \( ((w, w_t, \hat{w}, \hat{w}_t, v, v_t, z, z_t, W, Y)(\cdot, 0)) \in H^4 \times [H^1(0, 1)]^2 \), the solution of system (23) satisfies

\[
\| (w(\cdot, t), w_t(\cdot, t), \hat{w}(\cdot, t), \hat{w}_t(\cdot, t), Y(\cdot, t)) \|_{H^2 \times H^1(0,1)} \leq Me^{-\mu t}
\]

with some \( M, \mu > 0 \) for all \( t \geq 0 \), and

\[
\| (v(\cdot, t), v_t(\cdot, t), z(\cdot, t), z_t(\cdot, t), W(\cdot, t)) \|_{H^2 \times H^1(0,1)} < +\infty.
\]

**Remark**

This byproduct improves significantly the results in [Feng and Guo, 2016 IEEE TAC] and [Zhou and Weiss, 2017, IFAC].
Outline

1. Problem formulation
2. Preliminary: no disturbance
3. Tracking controller design
4. A state observer based tracking controller design
5. Numerical simulation
Fig. 3: The displacement $w(x, t)$ and the velocity $w_t(x, t)$ of $w$-system.
Simulation for anti-stable wave equations

*Fig. 4:* The displacement $v(x, t)$ and the velocity $v_t(x, t)$ of $v$-system.
Fig. 5: The displacement $z(x, t)$ and the velocity $z_t(x, t)$ of $w$-system.
Fig. 6: The displacement $Y(x, t)$ and $W(x, t)$. 
Simulation for anti-stable wave equations

Fig. 7: The displacement $\hat{w}(x, t)$ and the velocity $\hat{w}_t(x, t)$ of $w$-system.
Fig. 8: The reference signal $r(t)$, the output $w(0, t)$ and the control law $u(t)$. 

Simulation for anti-stable wave equations


Thanks for your attention!