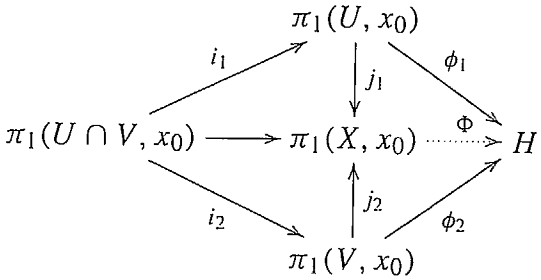
§70 The Seifert-van Kampen Theorem

We now return to the problem of determining the fundamental group of a space X that is written as the union of two open subsets U and V having path-connected intersection. We showed in §59 that, if $x_0 \in U \cap V$, the images of the two groups $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ in $\pi_1(X, x_0)$, under the homomorphisms induced by inclusion, generate the latter group. In this section, we show that $\pi_1(X, x_0)$ is, in fact, completely determined by these two groups, the group $\pi_1(U \cap V, x_0)$, and the various homomorphisms of these groups induced by inclusion. This is a basic result about fundamental groups. It will enable us to compute the fundamental groups of a number of spaces, including the compact 2-manifolds.

Theorem 70.1 (Seifert-van Kampen theorem). *Let $X = U \cup V$, where U and V are open in X; assume U, V, and $U \cap V$ are path connected; let $x_0 \in U \cap V$. Let H be a group, and let*

$$\phi_1 : \pi_1(U, x_0) \longrightarrow H \qquad \text{and} \qquad \phi_2 : \pi_1(V, x_0) \longrightarrow H$$

be homomorphisms. Let i_1, i_2, j_1, j_2 be the homomorphisms indicated in the following diagram, each induced by inclusion.



If $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then there is a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$.

This theorem says that if ϕ_1 and ϕ_2 are arbitrary homomorphisms that are "compatible on $U \cap V$," then they induce a homomorphism of $\pi_1(X, x_0)$ into H.

Proof. Uniqueness is easy. Theorem 59.1 tells us that $\pi_1(X, x_0)$ is generated by the images of j_1 and j_2. The value of Φ on the generator $j_1(g_1)$ must equal $\phi_1(g_1)$, and its value on $j_2(g_2)$ must equal $\phi_2(g_2)$. Hence Φ is completely determined by ϕ_1 and ϕ_2. To show Φ exists is another matter!

For convenience, we introduce the following notation: Given a path f in X, we shall use $[f]$ to denote its path-homotopy class in X. If f happens to lie in U, then $[f]_U$ is used to denote its path-homotopy class in U. The notations $[f]_V$ and $[f]_{U \cap V}$ are defined similarly.

Step 1. We begin by defining a set map ρ that assigns, to each *loop* f based at x_0 that lies in U or in V, an element of the group H. We define

$$\rho(f) = \phi_1([f]_U) \quad \text{if } f \text{ lies in } U,$$
$$\rho(f) = \phi_2([f]_V) \quad \text{if } f \text{ lies in } V.$$

Then ρ is well-defined, for if f lies in both U and V,

$$\phi_1([f]_U) = \phi_1 i_1([f]_{U\cap V}) \quad \text{and} \quad \phi_2([f]_V) = \phi_2 i_2([f]_{U\cap V}),$$

and these two elements of H are equal by hypothesis. The set map ρ satisfies the following conditions:

(1) If $[f]_U = [g]_U$, or if $[f]_V = [g]_V$, then $\rho(f) = \rho(g)$.

(2) If both f and g lie in U, or if both lie in V, then $\rho(f * g) = \rho(f) \cdot \rho(g)$.

The first holds by definition, and the second holds because ϕ_1 and ϕ_2 are homomorphisms.

Step 2. We now extend ρ to a set map σ that assigns, to each *path* f lying in U or V, an element of H, such that the map σ satisfies condition (1) of Step 1, and satisfies (2) when $f * g$ is defined.

To begin, we choose, for each x in X, a path α_x from x_0 to x, as follows: If $x = x_0$, let α_x be the constant path at x_0. If $x \in U \cap V$, let α_x be a path in $U \cap V$. And if x is in U or V but not in $U \cap V$, let α_x be a path in U or V, respectively.

Then, for any path f in U or in V, we define a *loop* $L(f)$ in U or V, respectively, based at x_0, by the equation

$$L(f) = \alpha_x * (f * \bar{\alpha}_y),$$

where x is the initial point of f and y is the final point of f. See Figure 70.1. Finally, we define

$$\sigma(f) = \rho(L(f)).$$

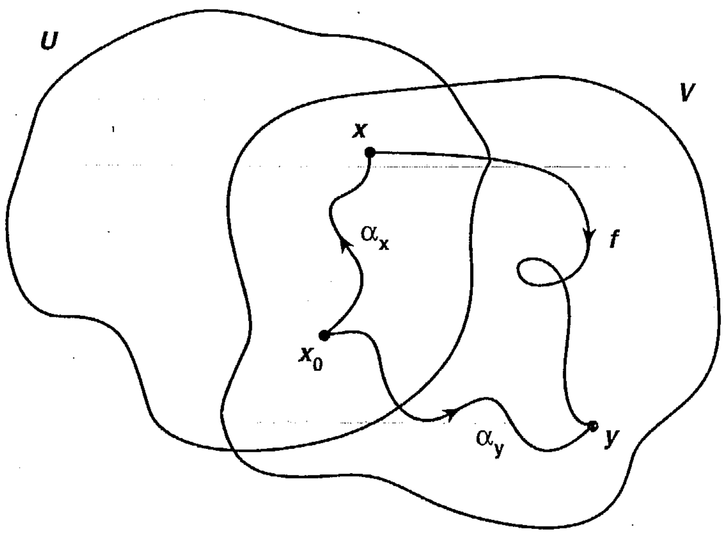


Figure 70.1

First, we show that σ is an extension of ρ. If f is a *loop* based at x_0 lying in either U or V, then

$$L(f) = e_{x_0} * (f * e_{x_0})$$

because α_{x_0} is the constant path at x_0. Then $L(f)$ is path homotopic to f in either U or V, so that $\rho(L(f)) = \rho(f)$ by condition (1) for ρ. Hence $\sigma(f) = \rho(f)$.

To check condition (1), let f and g be paths that are path homotopic in U or in V. Then the loops $L(f)$ and $L(g)$ are also path homotopic either in U or in V, so condition (1) for ρ applies. To check (2), let f and g be arbitrary paths in U or in V such that $f(1) = g(0)$. We have

$$L(f) * L(g) = (\alpha_x * (f * \bar{\alpha}_y)) * (\alpha_y * (g * \bar{\alpha}_z))$$

for appropriate points x, y, and z; this loop is path homotopic in U or V to $L(f * g)$. Then

$$\rho(L(f * g)) = \rho(L(f) * L(g)) = \rho(L(f)) \cdot \rho(L(g))$$

by conditions (1) and (2) for ρ. Hence $\sigma(f * g) = \sigma(f) \cdot \sigma(g)$.

Step 3. Finally, we extend σ to a set map τ that assigns, to an *arbitrary* path f of X, an element of H. It will satisfy the following conditions:

(1) If $[f] = [g]$, then $\tau(f) = \tau(g)$.

(2) $\tau(f * g) = \tau(f) \cdot \tau(g)$ if $f * g$ is defined.

Given f, choose a subdivision $s_0 < \cdots < s_n$ of $[0, 1]$ such that f maps each of the subintervals $[s_{i-1}, s_i]$ into U or V. Let f_i denote the positive linear map of $[0, 1]$ onto $[s_{i-1}, s_i]$, followed by f. Then f_i is a path in U or in V, and

$$[f] = [f_1] * \cdots * [f_n].$$

If τ is to be an extension of σ and satisfy (1) and (2), we must have

$$(*) \qquad \tau(f) = \sigma(f_1) \cdot \sigma(f_2) \cdots \sigma(f_n).$$

So we shall use this equation as our definition of τ.

We show that this definition is independent of the choice of subdivision. It suffices to show that the value of $\tau(f)$ remains unchanged if we adjoin a single additional point p to the subdivision. Let i be the index such that $s_{i-1} < p < s_i$. If we compute $\tau(f)$ using this new subdivision, the only change in formula $(*)$ is that the factor $\sigma(f_i)$ disappears and is replaced by the product $\sigma(f_i') \cdot \sigma(f_i'')$, where f_i' and f_i'' equal the positive linear maps of $[0, 1]$ to $[s_{i-1}, p]$ and to $[p, s_i]$, respectively, followed by f. But f_i is path homotopic to $f_i' * f_i''$ in U or V, so that $\sigma(f_i) = \sigma(f_i') \cdot \sigma(f_i'')$, by conditions (1) and (2) for σ. Thus τ is well-defined.

It follows that τ is an extension of σ. For if f already lies in U or V, we can use the trivial partition of $[0, 1]$ to define $\tau(f)$; then $\tau(f) = \sigma(f)$ by definition.

Step 4. We prove condition (1) for the set map τ. This part of the proof requires some care.

We first verify this condition in a special case. Let f and g be paths in X from x to y, say, and let F be a path homotopy between them. Let us assume the additional hypothesis that there exists a subdivision $s_0, \ldots, s_n$ of $[0, 1]$ such that F carries each rectangle $R_i = [s_{i-1}, s_i] \times I$ into either U or V. We show in this case that $\tau(f) = \tau(g)$.

Given i, consider the positive linear map of $[0, 1]$ onto $[s_{i-1}, s_i]$ followed by f or by g; and call these two paths f_i and g_i, respectively. The restriction of F to the rectangle R_i gives us a homotopy between f_i and g_i that takes place in either U or V, but it is not a path homotopy because the end points of the paths may move during the homotopy. Let us consider the paths traced out by these end points during the homotopy. We define β_i to be the path $\beta_i(t) = F(s_i, t)$. Then β_i is a path in X from $f(s_i)$ to $g(s_i)$. The paths β_0 and β_n are the constant paths at x and y, respectively. See Figure 70.2. We show that for each i,

$$f_i * \beta_i \simeq_p \beta_{i-1} * g_i,$$

with the path homotopy taking place in U or in V.

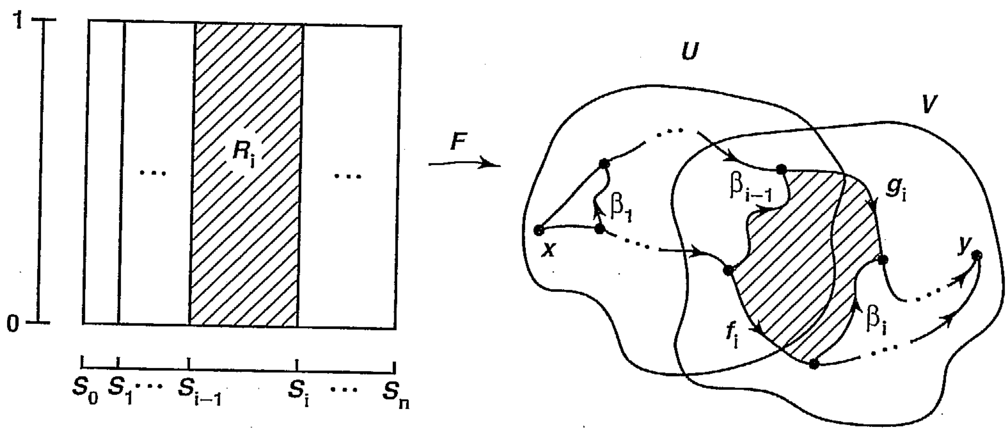


Figure 70.2

In the rectangle R_i, take the broken-line path that runs along the bottom and right edges of R_i, from $s_{i-1} \times 0$ to $s_i \times 0$ to $s_i \times 1$; if we follow this path by the map F, we obtain the path $f_i * \beta_i$. Similarly, if we take the broken-line path along the left and top edges of R_i and follow it by F, we obtain the path $\beta_{i-1} * g_i$. Because R_i is convex, there is a path homotopy in R_i between these two broken-line paths; if we follow by F, we obtain a path homotopy between $f_i * \beta_i$ and $\beta_{i-1} * g_i$ that takes place in either U or V, as desired.

It follows from conditions (1) and (2) for σ that

$$\sigma(f_i) \cdot \sigma(\beta_i) = \sigma(\beta_{i-1}) \cdot \sigma(g_i),$$

so that

(**) $$\sigma(f_i) = \sigma(\beta_{i-1}) \cdot \sigma(g_i) \cdot \sigma(\beta_i)^{-1}.$$

It follows similarly that since β_0 and β_n are constant paths, $\sigma(\beta_0) = \sigma(\beta_n) = 1$. (For the fact that $\beta_0 * \beta_0 = \beta_0$ implies that $\sigma(\beta_0) \cdot \sigma(\beta_0) = \sigma(\beta_0)$.)

We now compute as follows:

$$\tau(f) = \sigma(f_1) \cdot \sigma(f_2) \cdots \sigma(f_n).$$

Substituting (**) in this equation and simplifying, we have the equation

$$\begin{aligned}\tau(f) &= \sigma(g_1) \cdot \sigma(g_2) \cdots \sigma(g_n)\\ &= \tau(g).\end{aligned}$$

Thus, we have proved condition (1) in our special case.

Now we prove condition (1) in the general case. Given f and g and a path homotopy F between them, let us choose subdivisions $s_0, \dots, s_n$ and $t_0, \dots, t_m$ of $[0, 1]$ such that F maps each subrectangle $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ into either U or V. Let f_j be the path $f_j(s) = F(s, t_j)$; then $f_0 = f$ and $f_m = g$. The pair of paths f_{j-1} and f_j satisfy the requirements of our special case, so that $\tau(f_{j-1}) = \tau(f_j)$ for each j. It follows that $\tau(f) = \tau(g)$, as desired.

Step 5. Now we prove condition (2) for the set map τ. Given a path $f * g$ in X, let us choose a subdivision $s_0 < \cdots < s_n$ of $[0, 1]$ containing the point $1/2$ as a subdivision point, such that $f * g$ carries each subinterval into either U or V. Let k be the index such that $s_k = 1/2$.

For $i = 1, \dots, k$, the positive linear map of $[0, 1]$ to $[s_{i-1}, s_i]$, followed by $f * g$, is the same as the positive linear map of $[0, 1]$ to $[2s_{i-1}, 2s_i]$ followed by f; call this map f_i. Similarly, for $i = k + 1, \dots, n$, the positive linear map of $[0, 1]$ to $[s_{i-1}, s_i]$, followed by $f * g$, is the same as the positive linear map of $[0, 1]$ to $[2s_{i-1} - 1, 2s_i - 1]$ followed by g; call this map g_{i-k}. Using the subdivision $s_0, \dots, s_n$ for the domain of the path $f * g$, we have

$$\tau(f * g) = \sigma(f_1) \cdots \sigma(f_k) \cdot \sigma(g_1) \cdots \sigma(g_{n-k}).$$

Using the subdivision $2s_0, \dots, 2s_k$ for the path f, we have

$$\tau(f) = \sigma(f_1) \cdots \sigma(f_k).$$

And using the subdivision $2s_k - 1, \dots, 2s_n - 1$ for the path g, we have

$$\tau(g) = \sigma(g_1) \cdots \sigma(g_{n-k}).$$

Thus (2) holds trivially.

Step 6. The theorem follows. For each loop f in X based at x_0, we define

$$\Phi([f]) = \tau(f).$$

Conditions (1) and (2) show that Φ is a well-defined homomorphism.

Let us show that $\Phi \circ j_1 = \phi_1$. If f is a loop in U, then

$$\begin{aligned}\Phi(j_1([f]_U)) &= \Phi([f])\\ &= \tau(f)\\ &= \rho(f) = \phi_1([f]_U),\end{aligned}$$

as desired. The proof that $\Phi \circ j_2 = \phi_2$ is similar. ∎