Infinitesimal Calculus 3A, 2013-14: Topology in normed spaces

1. (a) Let $A = \left\{(x, y) \in \mathbb{R}^2 \mid 0 < x \wedge 0 < y < x^2\right\}$, and define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \mathbb{1}_A = \begin{cases} 1 & (x,y) \in A \\ 0 & \text{otherwise} \end{cases}$$

For all $h \in \mathbb{R}^2$ define $g_h : \mathbb{R} \to \mathbb{R}$ by $g_h(t) = f(th)$. Show that for all h, g_h is continuous at $t = 0$, but f is discontinuous at the origin. (Hint: show that every line through the origin contains an interval around it disjoint from A.)

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \frac{(x^2 + y^4)^3}{1 + x^6 y^4}.$$

Show that for all $k \in \mathbb{R}$, $\lim_{|x| \to \infty} f(x, kx) = \infty$. Does $\lim_{\|(x,y)\| \to \infty} f(x,y) = \infty$?

2. This exercise deals with continuity of functions defined on subsets of $\mathbb{R}^n$.
Let $A \subset \mathbb{R}^n$ be some subset and let $f : A \to \mathbb{R}^m$. Show that the following are equivalent:

(a) For all $x \in A$ and $\epsilon > 0$ there exists $\delta > 0$ such that for all $y \in A$ it holds that $\|y - x\| < \delta \implies \|f(y) - f(x)\| < \epsilon$.

(b) For every convergent sequence $(x_n) \subset A$ such that $x = \lim x_n \in A$ it holds that $f(x_n) \to f(x)$.

(c) For every open set $V \subset \mathbb{R}^m$ there exists some open $U \subset \mathbb{R}^n$ such that $f^{-1}[V] = A \cap U$.

A function satisfying these conditions is said to be continuous on A.

3. (a) Let X, Y be normed spaces, $A \subset X, B \subset Y$, and let $f : A \to B$ be a continuous bijection. Show that if A is compact then $g = f^{-1} : B \to A$ is also continuous. (A bijection whose inverse is continuous is called *open*. A continuous and open bijection is called *homeomorphism*.)

(b) In contrast, show that $f : [0, 2\pi) \to S^1 = \left\{x \in \mathbb{R}^2 \mid \|x\|_2 = 1\right\}$ defined $f(t) = (\cos t, \sin t)$ is a continuous bijection which is nonetheless not open.

4. Take $f : \mathbb{R}^{n+m} \to \mathbb{R}^k$ and assume the limit $\lim_{x \to x_0} f(x)$ exists. Denote $x_0 = (u_0, v_0)$ where $u_0 \in \mathbb{R}^n, v_0 \in \mathbb{R}^m$, and assume there exists $\delta > 0$ such that for all $\|u - u_0\| < \delta$ the limit $\lim_{v \to v_0} f(u, v)$ exists. Prove that

$$\exists \lim_{u \to u_0} \left(\lim_{v \to v_0} f(u,v) \right) = \lim_{x \to x_0} f(x).$$

5. Show that the following are real normed spaces (i.e., that they have a structure of a real vector space endowed with a norm):

(a) $C[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ with $\|f\|_1 = \int_0^1 |f(x)|\, dx$.

(b) $C[0,1]$ with $\|f\|_\infty = sup_{x \in [0,1]} |f(x)|$.

(c) $l^2(\mathbb{N}) = \left\{(x_n)_{n \geq 0} \subset \mathbb{R} \mid \sum_{n \geq 0} |x_n|^2 < \infty\right\}$ with $\|(x_n)_{n \geq 0}\|_2 = \sqrt{\sum_{n \geq 0} |x_n|^2}$.

(d) Given two real normed spaces V, W, the space

$$B(V,W) = \left\{ T \in \mathrm{Hom}(V,W) \mid \sup_{\|x\|_V=1} \|Tx\|_W < \infty \right\}$$

with $\|T\|_{op} = \sup_{\|x\|_V=1} \|Tx\|_W$.

6. (a) Continuing on the last part of the former question, show that for all $S, T \in B(V,V)$ it holds that $\|S \circ T\|_{op} \leq \|S\|_{op} \|T\|_{op}$.

   (b) Show that every $T \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ is continuous.

7. Let $U \subset \mathbb{R}^n$ be open, and $K \subset U$ be compact. Show that there exists a compact $K' \subset U$ such that $K \subset \mathrm{int}\, K'$.

8. Let $B = B_1(0,0) \subset \mathbb{R}^2$, and define $f : B \to \mathbb{R}^3$ by $f(x,y) = (x, y, \sqrt{1-x^2-y^2})$.

   (a) Prove that f is continuously differentiable.

   (b) Find $df_{(0,0)}$ and $df_{\left(\frac{1}{2},\frac{1}{3}\right)}$.

9. Take $p \in \mathbb{R}[t]$ and define $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $f(A) = p(A)$. Show that f is smooth (i.e., continuously differentiable infinitely many times).

10. Show that the determinant $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is differentiable, and find its differential at X. (Hint: show that for $a_1, \ldots, a_n, b \in \mathbb{R}^{1\times n}$ and $\alpha \in \mathbb{R}$ it holds that

$$\det \begin{bmatrix} a_1 \\ \vdots \\ a_{i-1} \\ a_i + \alpha b \\ a_{i+1} \\ \vdots \\ a_n \end{bmatrix} = \det \begin{bmatrix} a_1 \\ \vdots \\ a_{i-1} \\ a_i \\ a_{i+1} \\ \vdots \\ a_n \end{bmatrix} + \alpha \det \begin{bmatrix} a_1 \\ \vdots \\ a_{i-1} \\ b \\ a_{i+1} \\ \vdots \\ a_n \end{bmatrix},$$

and use that to express $\det(X+H) - \det X$.)

11. Define $\exp : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by

$$\exp(X) = I + \sum_{k=1}^{\infty} \frac{1}{k!} X^k$$

   (a) Show that exp is well-defined. That is, that for all $X \in M_n(\mathbb{R})$ the matrices $\exp_N(X) = I + \sum_{k=1}^{N} \frac{1}{k!} X^k$ form a convergent sequence in $M_n(\mathbb{R})$ (Hint: show that they form a Cauchy sequence.)

   (b) Show that exp is continuous.

   (c) Take $A \in M_n(\mathbb{R})$ and define $\gamma : \mathbb{R} \to M_n(\mathbb{R})$ by $\gamma(t) = \exp(tA)$. Show that A is the matrix representing $d\gamma_0$ in the standard basis.