Preliminaries

(*) We've seen in the practical session the following property of $\varphi_{i_1} \wedge \ldots \wedge \varphi_{i_k}$:

Let $V_1, \ldots, V_k \in \mathbb{R}^n$, it holds that $\varphi_{i_1} \wedge \ldots \wedge \varphi_{i_k}(V_1, \ldots, V_k) = \det(M_{i_1,..,i_k})$.

With $M_{i_1,..,i_k}$ being the K columns of the matrix whose rows are $V_1, \ldots, V_k$.

(**) Let $\omega \epsilon \Lambda^k(\mathbb{R}^n)$, and $f: \mathbb{R}^n \to \mathbb{R}^m$ continuously differentiable. Then $f^*\omega \epsilon \Lambda^k(\mathbb{R}^m)$ such that $f^*\omega(V_1, \ldots, V_k) = \omega(dfV_1, \ldots, dfV_k)$. Lets note it as $\omega\ o\ df$.

(***) Let $\omega \epsilon \Lambda^n(\mathbb{R}^n)$, then $\omega = f\ dx_1 \wedge \ldots \wedge dx_n$ and $\int'_{[0,1]^k} \omega = \int'_{[0,1]^k} f$.

(#) Let $C: [0,1]^k \to \mathbb{R}^n$ be a singular k-cube (?) then $\int'_c \omega = \int'_{[0,1]^k} C^*\omega$

Main part

Now, Let $\omega \epsilon \Lambda^{k-1}(\mathbb{R}^k)$ and $C = I^k$ the identity cube. In order to calculate $\int'_{dC} \omega$ it suffices to calculate for $\omega = f\ dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_k$ from linearity.

For a typical summand in dC we have:

'$I^k_{[i,\alpha]}: [0,1]^{k-1} \to \mathbb{R}^k$. $I^k_{[i,\alpha]}(x_1, .., x_{k-1}) = (x_1, ., x_i, \alpha, x_{j+1}., x_{k-1})$

And the integral is as follows:

$$\int'_{I^k_{[i,\alpha]}} \omega \underset{from\ (\#)}{=} \int'_{I^{k-1}} (I^k_{[i,\alpha]}{}^* \omega) = \int'_{I^{k-1}} (I^k_{[i,\alpha]}{}^* f\ dx_1 \wedge .. \widehat{dx_j} .. \wedge dx_k) =$$

$$= \underset{from\ (**)}{=} \int'_{I^{k-1}} (f o I^k_{[i,\alpha]})(dx_1 \wedge .. \widehat{dx_j} .. \wedge dx_k\ o\ dI^k_{[i,\alpha]}).$$

We'll note that $dI^k_{[i,\alpha]} = \begin{vmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 0 & & \\ & & & \ddots & \\ & 0 & & 1 & \\ & & & & 1 \end{vmatrix}$ $a\ (kXk-1) matrix$ with the i'th row being 0.

From (***) we know how to integrate a K-1 form over a K-1 dimensional space. All is left to note is that for $i \neq j$, we get that:

$$dx_1 \wedge .. \widehat{dx_j} .. \wedge dx_n\ o\ dI^k_{[i,\alpha]}(V_1, \ldots, V_{k-1}) = dx_1 \wedge .. \widehat{dx_j} .. \wedge dx_k (dI^k_{[i,\alpha]} V_1, \ldots, dI^k_{[i,\alpha]} V_{k-1}) =$$

$$= \underset{from\ (*)}{=} \det(M_{1,..,i,,\hat{j},..k-1})$$

But for every $dI^k_{[i,\alpha]} V_m$, that i'th coordinate is 0. So the i'th column of $M_{1,..,i,,\hat{j},..k-1}$ is 0, and we get $\det(M_{1,..,i,,\hat{j},..k-1})$=0. Hence $I^k_{[i,\alpha]}{}^* f\ dx_1 \wedge .. \widehat{dx_j} .. \wedge dx_k$ is the zero k-1 form, and $\int'_{I^k_{[i,\alpha]}} \omega = 0$.

On the other hand if $i = j$ then for the standard basis $e_1, \dots, e_{k-1}$ it holds that $dx_1 \wedge .. \widehat{dx_j} .. \wedge dx_k\left(dI_{[i,\alpha]}^k e_1, \dots, dI_{[i,\alpha]}^k e_{k-1}\right) = dx_1 \wedge .. \widehat{dx_j} .. \wedge dx_k(e_1', \dots, e_{k-1}') =$ $\det\left(M_{1,.\hat{j}.,k-1}\right) = det(I_{k-1})$=1 . with e'=e concatenated with 0 in j'th entry.

So $dx_1 \wedge .. \widehat{dx_j} .. \wedge dx_k\left(dI_{[i,\alpha]}^k V_1, \dots, dI_{[i,\alpha]}^k V_{k-1}\right)$ is **<u>THE</u>** base element for $\Lambda^{k-1}\left(\mathbb{R}^{k-1}\right)$.

And from (***) we get $\int_{[0,1]^{k-1}}' \left(foI_{[i,\alpha]}^k\right)(dx_1 \wedge .. \widehat{dx_j} .. \wedge dx_k \; o \; dI_{[i,\alpha]}^k)$=$\int_{[0,1]^{k-1}}' \left(foI_{[i,\alpha]}^k\right) =$

'$\int_{[0,1]^k}' f(x_1, ., x_i, \alpha, x_{j+1} ., x_{k-1})$ $\boxdot$