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An Elementary Proof of the Baker-Campbell-Hausdorff-Dynkin Formula

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Six different proofs of this formula are given in [1–6]. We believe that our proof below is more elementary than any one of these six.

Let A be the algebra of formal power series in three independent variables t, x, y with rational coefficients, where we postulate that *t commutes with x and y but x and y do not commute.* We denote by D the operator of formal differentiation with respect to the variable t. With respect to the bracket operation $[a, b]=ab-ba$, A is a Lie algebra over Q. We say that $a\in A$ is a *Lie element* if it belongs to the Lie subalgebra of A generated by x and y. For $a\in A$ we denote by L_a and R_a the left and right multiplications by a in A, respectively. We put $\operatorname{Ad} a=L_a-R_a$. The power series $a\in A$ with zero constant term form an ideal A^+ of A. The maps $\exp: A^+\to 1+A^+$ and $\log: 1+A^+\to A^+$, defined by usual power series, are inverse to each other. We define $h\in A$ by $h=\log(e^{tx}e^{ty})$ and put $h=\sum_{n\geqq 1} h_n(x, y)t^n$.

Since $L_x R_x=R_x L_x$ we have

$$(\operatorname{Ad} x)^n(y)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{n-k}yx^k. \tag{1}$$

We compute that

$$\begin{aligned} e^x y e^{-x} &= \sum_{m,k\geqq 0}(-1)^k\frac{x^m y x^k}{m!k!} \\ &= \sum_{n\geqq 0}\frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{n-k}yx^k \\ &= \sum_{n\geqq 0}\frac{1}{n!}(\operatorname{Ad} x)^n(y)=e^{\operatorname{Ad} x}(y). \end{aligned} \tag{2}$$

We have $(De^h)e^{-h}=\big(D(e^{tx}e^{ty})\big)e^{-h}=(xe^h+e^h y)e^{-h}=x+e^h y e^{-h}$. Since $h\in A^+$ we can use (2) with x replaced by h to obtain

$$(De^h)e^{-h}=x+e^{\operatorname{Ad} h}(y). \tag{3}$$

By induction on m we find that

$$\sum_{r=0}^{m}(-1)^r\binom{n}{r}=(-1)^m\binom{n-1}{m} \quad \text{for } 0\leqq m\leqq n-1. \tag{4}$$

We also have

$$(De^h)e^{-h} = \left(\sum_{m\geqq 1} \frac{1}{m!} Dh^m\right)\left(\sum_{k\geqq 0} \frac{(-1)^k}{k!} h^k\right) = \sum_{n\geqq 1} \frac{1}{n!} \sum_{m=1}^{n} (-1)^{n-m} \binom{n}{m} (Dh^m) h^{n-m}. \tag{5}$$

The inner sum is equal to

$$\sum_{m=1}^{n} \sum_{k=0}^{m-1} (-1)^{n-m} \binom{n}{m} h^k (Dh) h^{n-k-1} = \sum_{k=0}^{n-1} \left(\sum_{m=k+1}^{n} (-1)^{n-m} \binom{n}{m} \right) h^k (Dh) h^{n-k-1}. \tag{6}$$

Using (4) we find that

$$\sum_{m=k+1}^{n} (-1)^{n-m} \binom{n}{m} = \sum_{r=0}^{n-k-1} (-1)^r \binom{n}{r} = (-1)^{n-k-1} \binom{n-1}{k}$$

and hence the expression (6) equals

$$\sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n-1}{k} h^k (Dh) h^{n-k-1}.$$

Using (1) we can rewrite (5) in the form

$$(De^h)e^{-h} = \sum_{n\geqq 1} \frac{1}{n!} (\mathrm{Ad}\, h)^{n-1} (Dh). \tag{7}$$

The power series $f = (e^x - 1)/x$ is invertible in A and we let $g = 1/f$. Then from (3) and (7) we obtain

$$Dh = g(\mathrm{Ad}\, h)(x + e^{\mathrm{Ad}\, h}(y)). \tag{8}$$

Theorem. *Each $h_n(x, y)$ is a Lie element.*

Proof. This is obvious from the recursion formulae for $h_n(x, y)$ that one obtains by using the differential Eq. (8). Indeed, the coefficient of t^n on the left-hand side is $(n+1)h_{n+1}(x, y)$ and on the right-hand side it is a linear combination of expressions of the form

$$\mathrm{Ad}\, h_{m_1}(x, y) \ldots \mathrm{Ad}\, h_{m_k}(x, y)(z)$$

where $z = x$ or y and $m_1 + \cdots + m_k = n$. Thus if $h_r(x, y)$ are Lie elements for $r \leqq n$ so is $h_{n+1}(x, y)$. Since $h \in A^+$ we have $h_0(x, y) = 0$ and so it is a Lie element.

To get explicit formulae for $h_n(x, y)$ one can now proceed as in Hochschild [4], Proposition 2.2, p. 110.

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