Harmonic analysis on local fields - introduction

We saw that $\mathbb{Q}$ can be completed with respect to the norms $|\ |_p$, and denoted this completion by $\mathbb{Q}_p$. Explicitly one can think of elements of $\mathbb{Q}_p$ as Laurant series 'in p':

$$\sum_{i=-N}^{\infty} a_i p^i \quad , \quad 0 \leq a_i \leq p-1$$

As $|\ |_p$ satisfies the ultra-metric inequality, it follows that

$$\mathbb{Z}_p := \{ x \in \mathbb{Q}_p \mid |x|_p \leq 1 \}$$

is a ring.

Exercise: (i) $\mathbb{Z}_p$ is compact and is local with $p\mathbb{Z}_p$ unique maximal ideal).

(ii) $\{p^n \mathbb{Z}_p\}_{n \in \mathbb{N}}$ is a basis of neighbourhoods of 0, these are all compact & open & closed.

Thus $\mathbb{Q}_p$ is a locally compact field: the operations $+, \cdot$ are continuous w.r.t. $|\ |_p$ and each point has a compact neighbourhood.

In general, if G is a locally compact abelian group then G has a G-invariant measure:

$$\forall E \text{ a borel set in } G \qquad \mu(gE) = \mu(E) \quad \forall g \in G$$

(for example $\mathbb{R}$ has Lebesgue measure which is invariant under $+$).

We have two measures here:

$$(\mathbb{Q}_p, +, \mu) \qquad \mu(\mathbb{Z}_p) = 1$$

$$(\mathbb{Q}_p^\times, \cdot, \mu^\times) \qquad \mu^\times(\mathbb{Z}_p^\times) = 1$$

<u>Exercise</u>: (i) show that $\mu(p^n \mathbb{Z}_p) = \frac{1}{p^n}$
(ii) show that $\mu(\mathbb{Z}_p^\times) = 1 - \frac{1}{p}$

For locally compact abelian groups it can be shown that

$$\hat{G} := \mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{C}^\times)$$

is also locally compact and that $\hat{\hat{G}} = G$. For example, the characters of $\mathbb{R}$ are given by

$$\forall x \in \mathbb{R}, \qquad \psi_x(y) = e^{2\pi i x y}$$

thus we have even $\hat{\mathbb{R}} \cong \mathbb{R}$.

This holds also for $(\mathbb{Q}_p, +)$: We fix one character of $\mathbb{Q}_p$

$$\psi : \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \xleftrightarrow{\sim} \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\exp(2\pi i \cdot)} \mathbb{C}^\times$$

Now, $\forall a \in \mathbb{Q}_p^\times$ it is easily verified that

$$\psi_a(x) := \psi(ax)$$

is also a character (continuous!) of $\mathbb{Q}_p$, and $\psi_0(x)$ is the trivial character.

Thus, at least as sets, and even as groups:

$$\mathbb{Q}_p \simeq \hat{\mathbb{Q}}_p$$

(one can show that this is isom of topological gps).

<u>Side remark</u>: $\mathbb{Z}_p$ can be identified (as a compact topological ring) with

$$\underset{\substack{\longleftarrow \\ k}}{\lim} \mathbb{Z}/p^k \subset \prod_k \mathbb{Z}/p^k$$

closed subset

product topology

with the obvious map:

$x \in \mathbb{Z}_p$ maps to $(x \,(\mathrm{mod}\, p), x\,(\mathrm{mod}\, p^2), x\,(\mathrm{mod}\, p^3), \dots)$

$$\mathbb{Z}_p \longrightarrow \varprojlim \mathbb{Z}/p^k$$

<u>Exercise</u>: Check that this is indeed isom of topological gps.

We have: $0 \hookrightarrow \mathbb{Z}_p \hookrightarrow \mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p \to 0$
which gives after applying the functor $\mathrm{Hom}_{\mathrm{cont}}(\cdot, \mathbb{C}^*)$

$$0 \hookrightarrow \widehat{\mathbb{Q}_p/\mathbb{Z}_p} \hookrightarrow \hat{\mathbb{Q}}_p \twoheadrightarrow \hat{\mathbb{Z}}_p \to 0$$

thus $\hat{\mathbb{Z}}_p \simeq \mathbb{Q}_p/\mathbb{Z}_p$.

Alternatively $\hat{\mathbb{Z}}_p = \varinjlim \frac{1}{p^k}\mathbb{Z}/\mathbb{Z}$ = p-power roots of 1.

We can also define $\widehat{\mathbb{Q}_p^\times}$ and study 'multiplicative' harmonic analysis on $\mathbb{Q}_p$. It is the interaction between the additive and the multiplicative structures which gives the desired functional equation.

<u>Exercise</u>: (·) show that $e^{-\pi x^2}$ is a fixed function of the Fourier transform over $\mathbb{R}$

$$\hat{f}(x) = \int_{\mathbb{R}} f(y)\, \psi_x(y)\, dy$$

(··) show that $\mathbb{1}_{\mathbb{Z}_p}$ is a fixed point of the Fourier transform over $\mathbb{Q}_p$

$$\hat{f}(x) = \int_{\mathbb{Q}_p} f(y)\, \psi_x(y)\, d\mu$$

<u>Now</u>:

$$\zeta_p(s) = \frac{1}{1-p^{-s}} = \int_{\mathbb{Q}_p^\times} \mathbb{1}_{\mathbb{Z}_p}(x)\, |x|_p^s\, d\mu^*(x)$$

$$\zeta_\infty(s) = \pi^{-s/2}\, \Gamma\left(\frac{s}{2}\right) = \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s\, d^*x$$

These are multiplicative Fourier transforms of ϕ_p, where ϕ_p is a fixed point of the local additive Fourier trans.