Introduction to zeta functions

In many mathematical problems which involves enumeration or asymptotic behaviour $n \mapsto a_n$ it is convenient to pack the sequence $\{a_n\}$ in a generating function, e.g.

$$f(x) = \sum_{n=1}^{\infty} a_n x^n / n!$$

the a_n are then obtained by, e.g. $a_n = f^{(n)}(0)$.

In number theory such a 'generating function' is attached to any number field (= finite extensions of $\mathbb{Q}$), e.g.

$$\zeta_{\mathbb{Q}}(s) = \sum_{(0) \neq I \triangleleft \mathbb{Z}} [\mathbb{Z} : I]^{-s} = \sum_{n=1}^{\infty} n^{-s} \qquad \mathrm{Re}(s) > 1$$

or, more generally if $\mathcal{O} \subset K$ is the ring of integers

$$\zeta_K(s) = \sum_{I \triangleleft \mathcal{O}} [\mathcal{O} : I]^{-s} = \sum_{n=1}^{\infty} r_n \cdot n^{-s}$$

r_n = # ideals in $\mathcal{O}$ of index n

These generating functions are called 'zeta functions' and they have unexpected remarkable properties.

If we take $\zeta_{\mathbb{Q}}$ for example, then we have

$$\sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

$$= \prod_p (1 + p^{-s} + p^{-2s} + \cdots)$$

by the fundamental thm of arithmetic + some minor analysis.

We have:

- $\zeta(1) = \infty \Rightarrow \exists$ infinitely many primes (Euler's proof)

- The rate in which $\zeta(s) \longrightarrow \infty$ as $s \to 1$ gives the equidistribution of primes in $(\mathbb{Z}/N\mathbb{Z})^\times$ (Dirichlet thm)

- $\zeta(1+it) \neq 0 \quad \forall t \in \mathbb{R}$ is equivalent to the prime number throem:

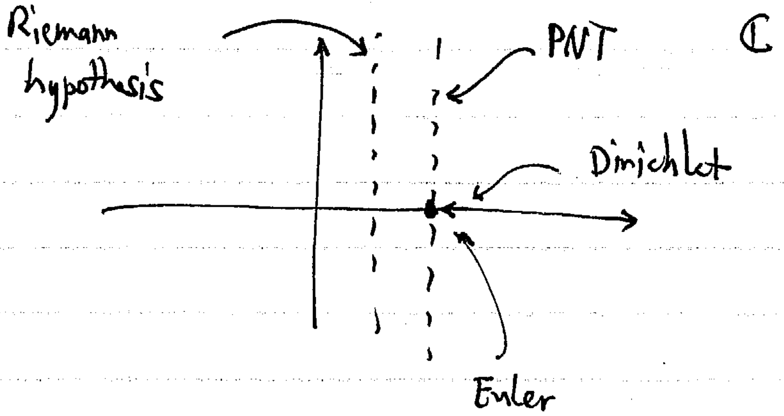
$$\pi(x) \sim \frac{x}{\log x} \sim \int_0^x \frac{dt}{\log t}$$

when $\pi(x) = \#$ of primes less than x.

(*) $\sim$ means $\dfrac{\pi(x)}{x/\log x} \longrightarrow 1$ as $x \to \infty$.

- Finally "$\zeta(s)$" vanishes only on $Re(s) = \frac{1}{2}$ (= Riemann hypothesis) predicts the best possible deviation in (*):

$$\left| \pi(x) - \int_0^x \frac{dt}{\log t} \right| < \frac{1}{8\pi} \sqrt{x} \log x.$$



More precisely, Riemann defined

$$\hat{\zeta}(s) = \pi^{-s/2}\,\Gamma\!\left(\tfrac{s}{2}\right)\prod_p \frac{1}{1-p^{-s}} \qquad \left(\Gamma(z) = \int_0^\infty t^z e^{-t}\,\frac{dt}{t}\right)$$

and showed that $\hat{\zeta}(s)$ is meromorphic with 2 poles at $s=0,1$ and that $\hat{\zeta}(s) = \hat{\zeta}(1-s)$ (= functional equation) and his hypothesis is : $\hat{\zeta}(s)=0$ iff $\mathrm{Re}(s) = \frac{1}{2}$.

Thus zeta functions are 'supposed' to have :

* meromorphic continuation
* Euler product
* functional equation

In 1951 Tate published his thesis and gave a transparent meaning to Dedekind zeta functions : $\zeta_K(s)$.
Let us specialize the discussion to $K=\mathbb{Q}$, the Riemann zeta.

Interestingly, this goes back to Riemann, but in a different aspect of his work - the Riemann sphere.

$\mathbb{C}$

$\mathbb{C}[z] \longleftrightarrow \mathbb{Z}$

$f(z) = *\,(z-\alpha_1)^{r_1}\cdots(z-\alpha_k)^{r_k}$ $\qquad$ $f = *\,p_1^{r_1}\cdots p_k^{r_k}$

primes: $\{(z-\alpha) \mid \alpha\in\mathbb{C}\}$ $\qquad$ primes: $\{2,3,5,7,\ldots\}$

$\downarrow$ $\qquad$ $\downarrow$ $\qquad$ $\downarrow$

$\mathbb{C}(z)$ rational functions $\qquad$ $\mathbb{Q}$

$\rightsquigarrow$ global theorems

$\forall \alpha \in \mathbb{C}$, let $|\cdot|_\alpha : \mathbb{C}(z) \to \mathbb{R}^+$ be the function

$$|f(z)|_\alpha = |(z-\alpha)^r \tilde{f}(z)|_\alpha = q^r$$

for some fixed q.

<u>Exercise</u> $|\ |_\alpha$ is a norm :
- $|fg| = |f||g|$
- $|f+g| \leq \max\{|f|,|g|\} \leq |f|+|g|$
- $|\mathbb{C}^\times|_\alpha = 1$

$$|f|_\infty = |f(\tfrac{1}{z})|_0 \quad \text{also a norm}$$

<u>Fact</u>: These are all the norms.
<u>Cor</u>: We can retrieve the space from the field.

Do it for $\mathbb{Q}$: $\quad |f|_p = |p^r \tilde{f}|_p = \frac{1}{p^r} \quad p$ prime

$$|f|_\infty = \text{abs value.}$$

<u>Fact: (Ostrowski)</u>: These are all the norms.

If we take the completion of $\mathbb{Q}$ w.r.t. to $|\ |_\infty$ we get $\mathbb{R}$

<u>Def</u>: $\mathbb{Q}_p = \mathbb{Q}^{|\ |_p}$ = the completion of $\mathbb{Q}$ w.r.t. $|\ |_p$.

<u>example</u>: $p^n \to 0$ as $n \to \infty$.

<u>Exercise</u>: $\mathbb{Q}_p = \left\{ \frac{a_{-N}}{p^N} + \dots + a_0 + a_1 p + \dots \right\}$ and is a field
$a_i \in \{0,1,\dots,p-1\}$

<u>Set:</u> $\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$

$= \{a_0 + a_1 p + a_2 p^2 + \cdots \mid 0 \leq a_i \leq p-1\}$

$= \overline{\mathbb{Z}}^{|\,|_p}$

$= $ max compact subring of $\mathbb{Q}_p$ (because of ultra metric)

$$\begin{array}{ccc} \mathbb{Q}_p & \supset & \mathbb{Z}_p \\ \uparrow & & \uparrow \\ \mathbb{Q} & \supset & \mathbb{Z} \end{array}$$

$p\mathbb{Z}_p = \{a_1 p + \cdots\} = \{x \in \mathbb{Z}_p \mid |x|_p < 1\}$ max ideal (unique)

$p^n\mathbb{Z}_p = $ basis of Nbhd of 0.

$\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p \mid |x|_p = 1\} = $ group of units

$\mathbb{Q}_p$ is locally compact (any point has a compact Nbhd)

$+, \cdot$ are continuous w.r.t. $|\,|_p$.

$\Rightarrow \exists$ Haar measure μ w.r.t. $+$ and μ^* w.r.t. $\cdot$

(in general) if G is locally compact gp $\exists$ unique measure which is invariant under G. $\mu(E) = \mu(aE)$ $\forall a \in E$

E.g. for $G = \mathbb{R}$ we have the Lebesgue measure)

$(\mathbb{Q}_p, +, \mu, \mu(\mathbb{Z}_p) = 1)$

$(\mathbb{Q}_p^\times, \cdot, \mu^*, \mu(\mathbb{Z}_p^\times) = 1)$

In $(\mathbb{R}, +)$ case we have the Fourier transform

$$\hat{f}(x) = \int_{y \in \mathbb{R}} f(y)\, e^{-2\pi i y x}\, dy$$

$$\hat{f}(\psi_x) = \int_{\hat{\mathbb{R}}} f(y)\, \psi_x(y)\, dy$$

$$\psi_x : \mathbb{R} \to \mathbb{C}^\times \qquad x \leftrightarrow \psi_x$$
$$y \mapsto e^{2\pi i x y}$$

as such: $e^{-\pi x^2}$ is a fixed point of $f \mapsto \hat{f}$

For the p-adics:

$$\hat{f}(\psi_x) = \int_{\hat{\mathbb{Q}}_p} f(y)\, \psi_x(y)\, d\mu$$

$\mathbb{1}_{\mathbb{Z}_p}$ is a fixed point

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Similarly there $\exists$ multiplicative Fourier transform ($\doteq$ Melin trans)

Tate:
$$\zeta_p(s) = \int_{\mathbb{Q}_p^\times} \mathbb{1}_{\mathbb{Z}_p}(x)\, |x|_p^s\, d^\times\mu = \frac{1}{1-p^{-s}}$$

$$\zeta_\infty(s) = \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s\, d^*x = \pi^{-s/2}\, \Gamma(s/2)$$

$$\hat{\zeta}(s) = \int_{\mathbb{A}_\mathbb{Q}^\times} \phi(a)\, |a|^s\, d^*a$$