"Model Theory" (~~of Valued fields~~) 09/06/09.

Language: $\mathcal{L} = \{ R_i, f_j, c_k \}$ $i \in I$, $j \in J$, $k \in K$ (need not to be finite v.s.)

(Vocab.)

relation symbols, function symbols, constant

$\mathcal{L} = \{ 0, 1, +, -, \cdot, ^{-1} \}$

constant, binary, binary relation, unary.

binary relation $^{-1}(x,y) \Leftrightarrow xy = 1$

"functions should be total"

We will always assume that $=$ is part of language.

Given a language L, a structure for L is

(i) A nonempty set M (we always assume infinite)

(ii) (a) For every constant symbol $c_k \in \mathcal{L}$ an element $a \in M$ (which we call the interpretation of c_k in M)

(b) For any function symbol $f_j \in \mathcal{L}$, if f_j is n-ary, a total function $f : M^n \to M$

(c) for every relation $R_i \in L$, if R is m-ary a total function $f : M^m \to \{0,1\}$

equivalent :- for every such R_i we observe a subset of M^m.

Def of truth in a structure :-

Formula :- Def. of a formula is inductive

need to define terms
(i) Every constant symbol is "term"
(ii) Any variable from a given (usually explicitly) list of variables is a term x, y, z, t
(iii) If f_i is an n-ary function in L and $t_1, t_2, \cdots, t_n$ are terms then $f_i(t_1, t_2, \cdots, t_n)$ is a term

mple :- If $\mathcal{L}$ is the language of rings, all polynomials over $\mathbb{Z}$ are in L and thats it.

A formula is defined inductively as follows:

1) If R_i is an m-ary relation in $\mathcal{L}$ and $t_1, t_2, \cdots, t_m$ are terms in $\mathcal{L}$ then
$R_i(t_1, t_2, \cdots, t_m)$ is an atomic formula (need truth value)

) If ϕ_1, ϕ_2 are formulas then so are $-\phi_i$ and $\{\phi_1 \wedge \phi_2\}$ where $\{-, \wedge\}$ are the logical connectives (in the language)

If ϕ_1 is a formula then $(\exists x)\, \phi_1$ and $(\forall x)\, \phi_1$ are formulas.

a formula ϕ we have to define what are free variables of ϕ (or when is a variable free in ϕ)
(x appears in ϕ) is not in the range of any quantifier $(\exists x)$ or $(\forall x)$

$\phi(t)$
$(\phi(y)) \wedge (x = x)$

In the :→ $\forall(x)\,(x = x) \wedge (x < y)$

free variable

In the language of ring

$\forall(x_1, \ldots x_n)\, \exists y\ (x_n X^n + \cdots + x_1 X = 0)$

X - free variable

Thu · 8.00

<u>Def^n</u>:- A formula with no free variables is called statement.

Let $\mathcal{M}$ be a structure for a voc $\mathcal{L}$ and $\phi(x_1, x_2, \ldots, x_k)$ a formula with free variables $x_1, x_2, \ldots, x_k$. A substitut is a function from the set of variables to M and we have to define inductively $s(\phi(x_1, x_2, \ldots, x_k))$

(i) If t is a term x_i so we have $s(x_i) \in M$

(ii) If t is a constant symbol $s(c_k) = c_k(\mathcal{M})$

(iii) If ϕ · $s(f_i(t_1, \ldots, t_m)) = f_i^M(s(t_1), \ldots, s(t_m))$

$\longrightarrow \exists x \, (x^2 = y) \wedge (x + y = z)$

substitution in variables not in range of quantifiers.

<u>Definition of truth value</u> :- If R_i is an m-ary relation, $t_1, t_2, \ldots, t_m$ terms and s is a substitution then $s(R_i(t_1, \ldots, t_m))$ is true (written $M \models s(R_i(t_1, \ldots, t_m))$ if $(s(t_1), \ldots, s(t_m)) \in R_i^M$ (or $R(s(t_1), \ldots, s(t_m)) = 1$)

the integer of R_i in $\mathcal{M}$.

If ϕ_1, ϕ_2 are formulas then $M \models s(\phi_1 \wedge \phi_2)$ if $M \models s(\phi_1)$ and $M \models s(\phi_2)$

$M \models s(\neg\phi)$ if $M \not\models s(\phi)$

$M \models s(\exists x \; \phi(x, y_1, \ldots, y_n))$ if there exists a substitution s' such that $s'(y_i) = s(y_i)$ for all $1 \le i \le k$ and $M \models s'(\phi(x, y_1, \ldots, y_n))$

$M \models s((\forall x) \; \phi(x, y_1, \ldots, y_n))$ for any s's as above.

substitution — elements of $\mathcal{M}$.

For $L = \{ < \}$ it is impossible to say well ordered.