A DANZER SET FOR AXIS PARALLEL BOXES

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Abstract. We present concrete constructions of discrete sets in $\mathbb{R}^d$ ($d \geq 2$) that intersect every aligned box of volume 1 in $\mathbb{R}^d$, and which have optimal growth rate $O(T^d)$.

1. Introduction

A set $D \subseteq \mathbb{R}^d$ is called a *Danzer set* if there exists an $s > 0$ such that D intersects every convex set of volume s. The question whether a discrete Danzer set in $\mathbb{R}^d$ of growth rate $O(T^d)$ exists is due to Danzer, see [CFG, Go, GL], and has been open since the sixties.

There are several variants of this question. One is to weaken the Danzer property in the following sense. We say that $Y \subseteq \mathbb{R}^d$ is a *dense forest* if there is a function $\varepsilon = \varepsilon(T) \xrightarrow{T\to\infty} 0$ so that for every $x \in \mathbb{R}^d$ and for every direction $v \in \mathcal{S}^{d-1}$, the distance between Y and the line segment of length T which starts at x and proceeds in direction v is less than $\varepsilon(T)$. Intuitively, as it was presented in [Bi], T is the maximal distance that a man can see when standing in a forest with a trunk of radius ε located at each element of Y. Note that every Danzer set is a dense forest with $\varepsilon(T) = O(T^{-1/(d-1)})$, and a dense forest with $\varepsilon(T) = O(T^{-(d-1)})$ is a Danzer set.[1] A construction of a dense forest of growth rate $O(T^d)$ is given in [SW], and another construction in the plane follows from the proof of [Bi, Lemma 2.4].

One other interesting direction is to look for Danzer sets with faster growth rates. A Danzer set of growth rate $O(T^d(\log T)^{d-1})$ is given in [BW]; this bound was improved recently in [SW] by a probabilistic construction that gives growth rate $O(T^d \log T)$.

[1]The second statement is proven as follows: let D be a dense forest with $\varepsilon(T) = O(T^{-(d-1)})$, and let $R \subseteq \mathbb{R}^d$ be a box (i.e. a parallelotope with adjacent faces orthogonal) with volume s and shortest edge length 2ε. Since the volume of a box is the product of the length of its sides, R has an edge of length at least $T := \left(\frac{s}{2\varepsilon}\right)^{1/(d-1)}$. Let L be the line segment parallel to this edge, passing through the center of R, and of length $T - 2\varepsilon$. If R does not contain any points of D, then the distance from L to D is at least ε, which implies that $\varepsilon \leq O(T^{-(d-1)}) = O(\varepsilon/s)$. For s sufficiently large, this is a contradiction, so every box of sufficiently large volume intersects D. Since every convex set contains a box of volume at least a constant times the volume of the convex set, this shows that D is a Danzer set.

Another approach in trying to weaken the Danzer problem is by hitting a smaller family of sets, instead of all the convex sets. John's theorem [Jo] implies that replacing convex sets by boxes[2] gives an equivalent question. In this note we consider a question that arises naturally from the Danzer problem. We say that $D \subseteq \mathbb{R}^d$ is an *align-Danzer set* if there is an $s > 0$ such that D intersects every aligned box of volume s. In our main results, Theorem 1.1 and Theorem 1.3 below, we present simple constructions for align-Danzer sets in $\mathbb{R}^d$ of growth rate $O(T^d)$. Neither of these constructions is new, but the viewpoint of seeing them as connected with Danzer's problem is new.

We denote by $\{0,1\}^{\mathbb{Z}}_{Fin}$ the subset of $\{0,1\}^{\mathbb{Z}}$ consisting of those bi-infinite sequences that contain only finitely many 1s.

Theorem 1.1. *The set*

$$D \stackrel{\text{def}}{=} \left\{ \left(\pm \sum_{n\in\mathbb{Z}} a_n 2^n, \pm \sum_{n\in\mathbb{Z}} a_n 2^{-n} \right) \in \mathbb{R}^2 : (a_n) \in \{0,1\}^{\mathbb{Z}}_{Fin} \right\}$$

is an align-Danzer set in $\mathbb{R}^2$ of growth rate $O(T^2)$.

The set in Theorem 1.1 is a variant of the binary version of the well-known van der Corput sequence (see e.g. [vdC]).

Although the set D in Theorem 1.1 is given very explicitly, and the proof is by elementary means, it only solves the problem in dimension 2, and no simple higher-dimensional extension comes to mind. To solve the problem in higher dimensions we use a dynamical approach.

For a fixed $d \geq 2$ let $A \subseteq \mathrm{SL}_d(\mathbb{R})$ be the subgroup of diagonal matrices with positive entries, and let Ω be the space of all lattices in $\mathbb{R}^d$.

Definition 1.2 ([Sk, p.6])**.** A lattice $\Lambda \in \Omega$ is *admissible* if its orbit under A is precompact in Ω.

Theorem 1.3 (Corollary of [Sk, Theorem 1.2])**.** *For every $d \geq 2$ there exists an admissible lattice in $\mathbb{R}^d$, and every admissible lattice is an align-Danzer set.*

Although Theorem 1.3 is a direct consequence of [Sk, Theorem 1.2], we provide the proof since it is elementary. We also refer to the discussions in [GL, p. 24-31] for additional reading.

As a direct consequence we reprove a result in computational geometry, that follows from a result of Halton on low discrepancy sequences, see [Ha]. We remark that Corollary 1.4 is not stated in [Ha], but it is well known in the computational geometry and combinatorics communities that Halton's construction satisfies it.

[2]A *box* in $\mathbb{R}^d$ is the image of an aligned box $[a_1, b_1] \times \cdots \times [a_d, b_d]$ under an orthogonal matrix.

Corollary 1.4. *For every $\varepsilon > 0$ there are ε-nets of optimal sizes $O(1/\varepsilon)$ for the range space $(X, \mathcal{R})$, where $X = [0,1]^d$ and $\mathcal{R} = \{$aligned boxes$\}$.*

This Corollary follows directly from the above Theorems by restricting to a bounded cube and rescaling to $[0,1]^d$. We refer to [AS, Ma] for a more comprehensive reading about the notions in Corollary 1.4.

Remark 1.5. Align Danzer sets in $\mathbb{R}^d$ of growth rate $O(T^d)$ can also be constructed by modifying the proof of [SW, Theorem 1.4] to work for aligned boxes and then combining with the result of [Ha] or [vdC] in the unit cube. Nonetheless, our constructions here are simple and the proofs are straightforward.

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2. Proof of Theorem 1.1

Proof of Theorem 1.1. We first show that D intersects every aligned box of volume 64. It suffices to show that

$$D_+ \stackrel{\text{def}}{=} \left\{ \left(\sum_{n\in\mathbb{Z}} a_n 2^n, \sum_{n\in\mathbb{Z}} a_n 2^{-n} \right) \in \mathbb{R}^2 : (a_n) \in \{0,1\}^{\mathbb{Z}}_{Fin} \right\}$$

intersects every aligned box of volume 16 that sits in $\mathbb{R}^2_+ \stackrel{\text{def}}{=} [0,\infty)^2$.

Let $R \subseteq \mathbb{R}^2_+$ be an aligned box of volume 16, and denote its lower left vertex by (x, y). Let $t > 0$ be such that the lower right and the upper left vertices of R are $(x+t, y)$ and $(x, y + \frac{16}{t})$ respectively. We define a sequence $(a_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}_{Fin}$ so that $\left(\sum_{n\in\mathbb{Z}} a_n 2^n, \sum_{n\in\mathbb{Z}} a_n 2^{-n}\right) \in R$.

For each integer k, we denote by $\{0,1\}^{\geq k}_{Fin}$ and $\{0,1\}^{<k}_{Fin}$ the subsets of $\{0,1\}^{\geq k}$ and $\{0,1\}^{<k}$, respectively, consisting of those sequences that contain only finitely many 1s. Here $\{0,1\}^{\geq k}$ is the set of all sequences in $\{0,1\}$ of the form $(a_k, a_{k+1}, \ldots)$, and $\{0,1\}^{<k}$ is the set of all sequences in $\{0,1\}$ of the form $(\ldots, a_{k-2}, a_{k-1})$.

Let $k \in \mathbb{Z}$ be such that $2^k \leq \frac{t}{2} < 2^{k+1}$. Observe that $\sum_{n<k} a_n 2^n < 2^k \leq \frac{t}{2}$ for any sequence (a_n) in $\{0,1\}^{<k}_{Fin}$, and that the interval $(x, x+\frac{t}{2})$ intersects the set

$$2^k\mathbb{N} = \left\{ \sum_{n\geq k} a_n 2^n : (a_n) \in \{0,1\}^{\geq k}_{Fin} \right\}.$$

Then we may choose the a_ns for $n \geq k$ so that $\sum_{n\geq k} a_n 2^n \in (x, x+\frac{t}{2})$, and thus for any choice of the a_ns for $n < k$ (and in particular for the choice described below) we have $\sum_{n\in\mathbb{Z}} a_n 2^n \in (x, x+t)$.

The analysis of the y coordinate is similar. Here $2^{-k-1} < \frac{2}{t} \leq 2^{-k}$, and therefore $2^{-k+1} < \frac{8}{t} \leq 2^{-k+2}$. We have $\sum_{n\geq k} a_n 2^{-n} < 2^{-k+1} < \frac{8}{t}$ for any sequence (a_n) in $\{0,1\}^{\geq k}_{Fin}$, and the interval $(y, y+\frac{8}{t})$ intersects the set

$$2^{-k+1}\mathbb{N} = \left\{ \sum_{n<k} a_n 2^{-n} : (a_n) \in \{0,1\}^{<k}_{Fin} \right\}.$$

Then we may choose the a_ns for $n < k$ so that $\sum_{n<k} a_n 2^{-n} \in (y, y+\frac{8}{t})$, and thus for any choice of the a_ns for $n \geq k$ (and in particular for the choice described above) we have $\sum_{n\in\mathbb{Z}} a_n 2^{-n} \in (y, y+\frac{16}{t})$.

It is left to show that D (or D_+) is of growth rate $O(T^2)$. To see that, consider the set

$$B \stackrel{\text{def}}{=} \left\{ \left(\sum_{n\geq 0} a_n 2^n, \sum_{n<0} a_n 2^{-n} \right) \in \mathbb{R}^2 : (a_n) \in \{0,1\}^{\mathbb{Z}}_{Fin} \right\}.$$

Observe that the mapping $g : D_+ \to B$ which is defined in the obvious way by

$$\left(\sum_{n\in\mathbb{Z}} a_n 2^n, \sum_{n\in\mathbb{Z}} a_n 2^{-n} \right) \stackrel{g}{\mapsto} \left(\sum_{n\geq 0} a_n 2^n, \sum_{n<0} a_n 2^{-n} \right)$$

is a bijection, and for any $(x,y) \in D_+$ we have $\|(x,y) - g(x,y)\|_2 \leq \sqrt{5}$ (where $\|\cdot\|_2$ denotes the Euclidean norm). But since $B = \mathbb{N} \times 2\mathbb{N}$, the assertion follows. ☐

Remark 2.1. We want to stress that D is not a Danzer set in $\mathbb{R}^2$ and not even a dense forest. To see it, observe that symmetric sequences (a_n) correspond to points on the line $y = x$. On the other hand, non-symmetric sequences correspond to points (x,y) with $|x-y| > 1$, and in particular D misses a neighborhood of the line $y = x + \frac{1}{4}$.

3. Proof of Theorem 1.3

Fix $d \geq 2$. Let $V = \{\mathbf{t} \in \mathbb{R}^d : \sum_{i=1}^d t_i = 0\}$, and for each $\mathbf{t} \in V$ let $g_{\mathbf{t}} \in \mathrm{SL}_d(\mathbb{R})$ be the diagonal matrix whose entries are e^{t_i}. Then $\mathbf{t} \mapsto g_{\mathbf{t}}$ is a homomorphism.

Proof of Theorem 1.3. Let K be a totally real number field of degree d, and let $\mathcal{O}_K$ be its ring of integers. Let $\phi_1, \ldots, \phi_d : K \to \mathbb{R}$ be the Galois embeddings of K into $\mathbb{R}$, and let $\Phi : K \to \mathbb{R}^d$ be their direct sum. Then $\Lambda \stackrel{\text{def}}{=} \Phi(\mathcal{O}_K)$ is a lattice in $\mathbb{R}^d$. To see that Λ is admissible, fix $\mathbf{x} = \Phi(\alpha) \in \Lambda$, and observe that if $\mathbf{x} \neq 0$,

$$\prod_{i=1}^{d} |x_i| = \prod_{i=1}^{d} |\phi_i(\alpha)| = |N(\alpha)| \in \mathbb{Z} \smallsetminus \{0\}.$$

Here N denotes the norm in the field K. In particular, $\prod_{i=1}^{d} |x_i| \geq 1$ and thus $\prod_{i=1}^{d} |e^{t_i} x_i| \geq 1$ for all $\mathbf{t} \in V$. It follows that $|e^{t_i} x_i| \geq 1$ for some $i = 1, \ldots, d$ and thus $\|g_{\mathbf{t}}\mathbf{x}\| \geq 1$. Since $\mathbf{t}$, $\mathbf{x}$ were arbitrary, Mahler's compactness criterion shows that Λ is admissible.

For the second part of the proof, let Λ be an admissible lattice in $\mathbb{R}^d$. Let R be an aligned box disjoint from Λ. Then there exists $\mathbf{t} \in V$ such that $g_{\mathbf{t}}R$ is a cube. By assumption $g_{\mathbf{t}}\Lambda$ is in a compact subset $K \subseteq \Omega$, hence the codiameter[3] of $g_{\mathbf{t}}\Lambda$ is bounded above by a constant independent of $\mathbf{t}$. But since $g_{\mathbf{t}}R$ is disjoint from $g_{\mathbf{t}}\Lambda$, the distance from the center of $g_{\mathbf{t}}R$ to the complement of $g_{\mathbf{t}}R$, i.e. half the edge length of the cube $g_{\mathbf{t}}R$, is bounded above by the distance from the center of $g_{\mathbf{t}}R$ to $g_{\mathbf{t}}\Lambda$, which is in turn bounded above by the codiameter of $g_{\mathbf{t}}\Lambda$. Thus both the diameter and the volume of $g_{\mathbf{t}}R$ are bounded above by a constant independent of $\mathbf{t}$. Since $\mathrm{Vol}(R) = \mathrm{Vol}(g_{\mathbf{t}}R)$, the proof is complete. ☐

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[3]The *codiameter* of a lattice $\Gamma \subseteq \mathbb{R}^d$ is the diameter of the quotient space $\mathbb{R}^d/\Gamma$ (with respect to the quotient metric $d([\mathbf{x}], [\mathbf{y}]) = \min\{\|\mathbf{y} - \mathbf{x}\| : \mathbf{x}, \mathbf{y}$ representatives of $[\mathbf{x}], [\mathbf{y}]\}$), or equivalently the maximum of the function $\mathbb{R}^d \ni \mathbf{x} \mapsto d(\mathbf{x}, \Gamma)$. The codiameter is continuous as a function of the lattice.

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